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Bianchi–Euler system for relativistic fluids and Bel–Robinson type energy

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Abstract We write a first order symmetric hyperbolic system coupling the Riemann tensor with the dynamical acceleration of a prefect relativistic fluid. We determine the associated, coupled, Bel–Robinson type energy, and the integral equality that it satisfies. ***To cite this article: Y. Choquet-Bruhat, J.W. York, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 711–716.***
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Système de Bianchi–Euler pour un fluide relativiste, et énergie de type Bel–Robinson

Résumé On écrit un système symétrique hyperbolique satisfait par le tenseur de Riemann de l'espace temps et l'accélération dynamique d'un fluide parfait relativiste. On détermine l'énergie du type Bel–Robinson correspondante, et l'égalité intégrale qu'elle satisfait. ***Pour citer cet article : Y. Choquet-Bruhat, J.W. York, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 711–716.***
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La gravitation relativiste est représentée par le tenseur de Riemann de la métrique de l'espace temps. Son évolution est régie par les « équations d'ordre supérieur », déduites des identités de Bianchi (cf. [2,11]). Ce système, comme le système analogue construit avec le tenseur de Weyl (cf. [6]) peut être écrit sous la forme FOSH (First Order Symmetric Hyperbolic), linéaire, avec contraintes, homogène dans le vide (cf. [4,5], dans un repère ou la métrique s'écrit sous la forme $3+1$ usuelle :

$$ds^2 = -N^2\,dt^2 + g_{ij}\left(dx^i + \beta^i\,dt\right)\left(dx^j + \beta^j\,dt\right). \tag{1}$$

En présence de sources le système est non homogène, il admet un second membre, linéaire dans la dérivée covariante du tenseur d'impulsion énergie. Dans le cas fluide parfait, considéré ici, ce tenseur est :

$$T_{\alpha\beta} \equiv (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta}, \tag{2}$$

où u_α est la quadrivitesse cinématique, $u_\alpha u^\alpha = -1$; μ et p sont l'énergie et la pression spécifiques, positives, liées à l'entropie spécifique S par une équation d'état, $\mu = \mu(p, S)$.

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On introduit l'indice f du fluide donné par (χ est une constante)

$$f(p,S) \equiv \chi \exp F(p,S), \quad F(p,S) \equiv \int_{p_0}^{p} \frac{\mathrm{d}p}{\mu(p,S)+p}, \tag{3}$$

et la quadrivitesse dynamique $C^\alpha \equiv f u^\alpha$, telle que $C^\alpha C_\alpha = -f^2$. L'accélération dynamique est $\nabla_\beta C_\alpha$. On démontre que :

THÉORÈME 0.1. – *Le tenseur de Riemann et l'accélération dynamique satisfont un système FOSH si* $\mu'_p \geqslant 1$.

On écrit l'égalité energétique correspondante.

1. Introduction

The effective strength of the gravitational field lies in the Riemann tensor of the spacetime metric. Its evolution is governed by the so-called higher order equations [2,11], deduced from the Bianchi identities. The system satisfied by the trace free part of the Riemann tensor, the Weyl tensor, was some time ago recognized as a linear, first order symmetric hyperbolic system (FOSH), with constraints, homogeneous in vacuum; see [6] and references therein. The evolution equations for the Riemann tensor itself have also been written [4] as a FOSH system. In the presence of sources these equations are no longer homogeneous; their right-hand sides are linear in the covariant derivative of the stress energy tensor of the sources.

The Einstein equations with fluid sources have long ago [9,3] been proved to be a well posed Leray-hyperbolic system (with constraints). The fluid equations have also been written as a FOSH system (in special relativity [8]; in general relativity [13,1,12]). It seems interesting to have a system of equations which would be a FOSH system both for the gravitational field, namely the Riemann tensor of space time, and the fluid variables. Such a system has been written in Lagrangian variables (that is, in a frame whose timelike axis is tangent to the fluid flow lines) by Friedrich [7], who used the Weyl tensor and by Choquet-Bruhat and York [5], using directly the Riemann tensor. In this paper we use eulerian variables, that is a Cauchy adapted frame, with time axis orthogonal to the space slices. We obtain a FOSH system for the Riemann tensor and the dynamical fluid acceleration. The corresponding energy is the sum of the usual Bel–Robinson energy of the gravitational field, and the dynamical acceleration energy of the fluid. It is not conserved in general – no more than the Bel–Robinson energy in vacuum – but its evolution can be controlled through the equations we obtain.

2. Einstein equations with fluid sources

The Einstein equations with source a perfect fluid are, with an equation of state $\mu = \mu(p,S)$,

$$R_{\alpha\beta} = \rho_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T = (\mu+p) u_\alpha u_\beta + \frac{1}{2} g_{\alpha\beta}(\mu - p). \tag{4}$$

The enthalpy index f of the fluid is given by the identity (3), where χ is a constant such that f^2 has the dimensions of an energy density.

The dynamical velocity C^α incorporates information on the kinematic velocity u^α and the thermodynamic quantities. It is defined by

$$C^\alpha \equiv f u^\alpha, \quad \text{hence} \quad C^\alpha C_\alpha = -f^2. \tag{5}$$

In terms of the dynamical velocity the equations of a perfect fluid in an arbitrary frame are

$$C^\alpha \nabla_\alpha C_\beta + f \partial_\alpha f - f^2 F'_S \partial_\alpha S \equiv C^\alpha (\nabla_\alpha C_\beta - \nabla_\beta C_\alpha + C_\alpha F'_S \partial_\beta S) = 0, \tag{6}$$

$$\nabla_\alpha C^\alpha + (\mu'_p - 1)\frac{C^\alpha C^\beta}{C^\lambda C_\lambda}\nabla_\alpha C_\beta = 0 \tag{7}$$

with $F'_S(p,S) \equiv \partial F(p,S)/\partial S$ and $\mu'_p(p,S) \equiv \partial\mu(p,S)/\partial p$, given functions of p and S, and

$$C^\alpha \nabla_\alpha S = 0. \tag{8}$$

In these equations the unknowns are the four components of the vector C^α, and the scalar S. The specific pressure p is a known function of f (i.e., of C^α) and of S.

Eqs. (6) are not independent, they satisfy the following identity:

$$C^\alpha C^\beta(\nabla_\alpha C_\beta - \nabla_\beta C_\alpha) \equiv 0.$$

Eq. (8) says that S is constant along the flow lines, hence constant in spacetime if constant initially. *We suppose*, to simplify what follows below, *that S is constant*. Removing this hypothesis introduces no essential difficulty, but one has to add as new unknowns the derivatives $\partial_\alpha S$, and the equations obtained from (8) by taking a derivative ∇_α. Our results are valid as they stand for a barotropic fluid, because then $\mu \equiv \mu(p)$.

3. Bianchi equations

In a Cauchy adapted frame the spacetime metric takes the usual $3+1$ form (1). The derivatives ∂_α are the Pfaff derivatives in the coframe $\theta^0 = \mathrm{d}t$, $\theta^i = \mathrm{d}x^i + \beta^i\,\mathrm{d}t$, that is $\partial_0 = \frac{\partial}{\partial t} - \beta^i\frac{\partial}{\partial x^i}$, $\partial_i = \frac{\partial}{\partial x^i}$.

We have written in [4] the Bianchi equations satisfied by the Riemann tensor as a FOSH system

$$\nabla_0 R_{hk,\lambda\mu} + \nabla_k R_{0h,\lambda\mu} - \nabla_h R_{0k,\lambda\mu} = 0, \tag{9}$$

$$\nabla_0 R^0_{:::i,\lambda\mu} + \nabla_h R^h_{:::i,\lambda\mu} = J_{i,\lambda\mu} \equiv \nabla_\lambda\rho_{\mu i} - \nabla_\mu\rho_{\lambda i}. \tag{10}$$

Indeed Eqs. (9) and (10) are for each given pair $(\lambda\mu, \lambda < \mu)$ a first order symmetric system, hyperbolic relative to the space sections for the components $R_{hk,\lambda\mu}$ and $R_{0h,\lambda\mu}$.

To this Bianchi system is associated its Bel–Robinson energy density on a space slice, namely

$$\mathcal{B} \equiv \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2\right),$$

where **E**, **H**, **D**, **B** are the electric and magnetic fields and inductions space 2-tensors associated with the Riemann tensor: [1]

$$E_{ij} \equiv R^0{}_{i,0j}, \qquad D_{ij} \equiv \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hk,lm},$$

$$H_{ij} \equiv \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}{}_{,oj}, \qquad B_{ji} \equiv \frac{1}{2}N^{-1}\eta_{ihk}R_{0j,}{}^{hk}.$$

This energy satisfies the equality

$$\partial_0\mathcal{B} + \overline{\nabla}_h\left\{N\eta^{lh}{}_i\left(E^{ij}H_{lj} - B^{ij}D_{lj}\right)\right\} = Q(\mathbf{E},\mathbf{H},\mathbf{D},\mathbf{B}) + \mathcal{S},$$

where Q is a quadratic form in $\mathbf{E},\ldots,\mathbf{B}$ with coefficients $\overline{\nabla}N$ and $N\mathbf{K}$, $\mathbf{K}$ extrinsic curvature of the spaceslices. The source term $\mathcal{S}$, zero in vacuum, is linear in $\mathbf{E},\ldots,\mathbf{B}$ with coefficients $\mathbf{J} \equiv (J_{i,\lambda\mu})$.

The tensor $\rho_{\alpha\beta}$ for a perfect fluid is given in terms of C_α by

$$\rho_{\alpha\beta} \equiv (\mu + p)f^{-2}C_\alpha C_\beta + \frac{1}{2}g_{\alpha\beta}(\mu - p). \tag{11}$$

The sources of the Bianchi equations are therefore linear in the derivatives $\nabla_\alpha C_\beta$.

4. Equations for ∇C

The dynamical acceleration $C_{\gamma\beta} \equiv \nabla_\gamma C_\beta$ satisfies the following equations obtained by covariant differentiation of (6) and (7), and use of the Ricci identities:

$$M_{\gamma\beta} \equiv C^\alpha(\nabla_\alpha C_{\gamma\beta} - \nabla_\beta C_{\gamma\alpha}) + a_{\gamma\beta} = 0, \tag{12}$$

$$g^{\alpha\beta}\nabla_\alpha C_{\gamma\beta} + (\mu'_p - 1)\frac{C^\alpha C^\beta}{C^\lambda C_\lambda}\nabla_\alpha C_{\gamma\beta} + b_\gamma = 0 \tag{13}$$

with

$$a_{\gamma\beta} \equiv C_\gamma{}^\alpha(C_{\alpha\beta} - C_{\beta\alpha}) + C^\alpha C_\lambda R_{\gamma\alpha,\beta}{}^{\lambda}, \tag{14}$$

$$b_\gamma \equiv -R_{\gamma\lambda}C^\lambda + \nabla_\gamma\left\{(\mu'_p - 1)\frac{C^\alpha C^\beta}{C^\lambda C_\lambda}\right\}C_{\alpha\beta}. \tag{15}$$

The last term in b_γ is a quadratic form in $C_{\alpha\beta}$ whose coefficients are functions of the C^α, which can be computed by using the identity

$$\nabla_\gamma \mu'_p \equiv \mu''_{p^2}\partial_\gamma p, \quad \text{with } \mu''_{p^2} \equiv \frac{\mathrm{d}\mu'}{\mathrm{d}p},$$

where, by the definition of f it holds that

$$\partial_\gamma p = (\mu + p) f^{-1}\partial_\gamma f = -(\mu + p)\big(C^\lambda C_\lambda\big)^{-1} C^\alpha C_{\gamma\alpha}.$$

Eqs. (12) are not independent, because they satisfy the identities

$$C^\beta M_{\gamma\beta} \equiv 0. \tag{16}$$

Eqs. (12) and (13) are not a well posed system. Instead of the 4×4 Eqs. (12) we consider[2] the 4×3 ones:

$$\widetilde{M}_{\gamma i} \equiv M_{\gamma i} - \frac{C_i}{C_0}M_{\gamma 0} = 0. \tag{17}$$

The terms in derivatives of $C_{\gamma\lambda}$ in these equations can be written in the following form:

$$C^\alpha\partial_\alpha\left(C_{\gamma i} - \frac{C_i}{C_0}C_{\gamma 0}\right) - \left(\partial_i - \frac{C_i}{C_0}\partial_0\right)\big(C^\alpha C_{\gamma\alpha}\big). \tag{18}$$

LEMMA 4.1. – *The system* (13), (17) *is equivalent to a FOS* (*First Order Symmetric*) *system for* $C_{\gamma\alpha}$ *with coefficients functions of the Riemann tensor, the connection and the dynamical velocity* C_λ, *but not of their derivatives.*

Proof. – The system is quasidiagonal by blocks, each block corresponding to a given value of the index γ. We will write the principal operator of a block by omitting this index. We set

$$U_i \equiv C_{\gamma i} - \frac{C_i}{C_0}C_{\gamma 0}, \quad U_0 \equiv C^\alpha C_{\gamma\alpha}, \tag{19}$$

and we define the differential operators $\tilde\partial_\alpha$ as follows:

$$\tilde\partial_0 \equiv C^\alpha\partial_\alpha, \qquad \tilde\partial_i = \partial_i - \frac{C_i}{C_0}\partial_0. \tag{20}$$

The principal terms (derivatives of $C_{\gamma\alpha}$) in Eqs. (17) with index γ are

$$\tilde\partial_0 U_i - \tilde\partial_i U_0. \tag{21}$$

We have by inverting (19):

$$C_{\gamma 0} \equiv \frac{C_0(U_0 - C^i U_i)}{C^\lambda C_\lambda}, \qquad C_{\gamma i} \equiv U_i + \frac{C_i(U_0 - C^j U_j)}{C^\lambda C_\lambda}.$$

The principal terms of (13) read, using the above formulae

$$\frac{\mu'_p C^\alpha \partial_\alpha U_0}{C^\lambda C_\lambda} + \left(g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda}\right)\partial_i U_j - \frac{C^0 C^i}{C^\lambda C_\lambda}\partial_0 U_i. \tag{22}$$

We introduce the positive definite (if C is timelike) quadratic form $\tilde{g}^{ij} \equiv g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda}$. Then we find that

$$\tilde{g}^{ij}\frac{C_j}{C_0} \equiv \frac{C^0 C^i}{C^\lambda C_\lambda}. \tag{23}$$

The principal terms (22) are therefore

$$\frac{\mu'_p \tilde{\partial}_0 U_0}{C^\lambda C_\lambda} + \tilde{g}^{ij}\tilde{\partial}_i U_j. \tag{24}$$

The matrix of the coefficients of the derivatives $\tilde{\partial}_\alpha$ in the equations deduced from the system (13), (17) is

$$\begin{pmatrix} -\frac{\mu'_p}{C^\lambda C_\lambda}\tilde{\partial}_0 & -\tilde{\partial}^1 & -\tilde{\partial}^2 & -\tilde{\partial}^3 \\ -\tilde{\partial}_1 & \tilde{\partial}_0 & 0 & 0 \\ -\tilde{\partial}_2 & 0 & \tilde{\partial}_0 & 0 \\ -\tilde{\partial}_3 & 0 & 0 & \tilde{\partial}_0 \end{pmatrix},$$

symmetrized by product with the 4×4 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \tilde{g}_{ij} \end{smallmatrix}\right)$. □

4.1. Hyperbolicity

A FOS system is hyperbolic with respect to the space slices $x^0 =$ constant if the matrix M^t of the coefficients of the derivatives $\partial/\partial t$ is positive definite. It admits then an energy equality for a positive definite energy relative to the space slices.

In our case $\widetilde{M}^0$ of the coefficients of the derivatives $\tilde{\partial}_0$ is diagonal, with positive elements if $\mu'_p > 0$ and C^α is timelike. The matrix M^t is not diagonal, and it is not obvious that it is positive definite. We will prove:

THEOREM 4.2. – *The system* (13), (17) *is FOSH if* $\mu'_p \geqslant 1$ *and* C *is timelike.*

Proof. – It is simpler to compute directly the energy inequality for the considered system: its positivity is equivalent to the positivity of the matrix M^t. Multiplying (13) and (17) respectively by U_0 and $\tilde{g}^{ij}U_j$ and using the expressions (24), (21) gives equations of the form

$$\frac{1}{2}\frac{\mu'_p \tilde{\partial}_0 (U_0)^2}{f^2} - \tilde{g}^{ij}U_0\tilde{\partial}_i U_j = U_0 F_0, \tag{25}$$

$$\tilde{g}^{ij}U_j\tilde{\partial}_0 U_i - \tilde{g}^{ij}U_j\tilde{\partial}_i U_0 = \tilde{g}^{ij}U_j F_i, \tag{26}$$

where the F_α contain only nondifferentiated terms. We add these two equations, replace the operators $\tilde{\partial}$ by the operators ∂ and carry out some manipulations using the expression for $\tilde{g}^{ij}$ and the Leibniz rule. We obtain that

$$\partial_0 \mathcal{F} + \overline{\nabla}_i \mathcal{H}^i = Q_F.$$

The function Q_F is a quadratic form in $C_{\gamma\alpha}$ and the Riemann tensor, while $\mathcal{H}^i$ and $\mathcal{F}$ are given by:

$$\mathcal{H}^i = \frac{1}{C^0}\left\{\frac{1}{2}C^i\left(\frac{\mu'_p}{f^2}U_0^2 + \tilde{g}^{ij}U_iU_j\right) - \tilde{g}^{ij}U_0U_j\right\},$$
$$\mathcal{F} \equiv f^{-2}\left[(\mu'_p - 1)U_0^2 + \left(U_0 - C^iU_i\right)^2\right] + g^{ij}U_iU_j.$$

The energy of the dynamical acceleration ∇C relative to the space slices is the quadratic form $\mathcal{F}$. It is positive definite if $\mu'_p \geqslant 1$ and C^α is timelike. □

Remark 4.3. – The system is hyperbolic in the sense of Leray if $\mu'_p > 0$, but the submanifolds $x^0 =$ constant are 'spacelike' with respect to the fluid wave cone only if the fluid sound speed is less than the speed of light, i.e., $\mu'_p \geqslant 1$.

5. Bel–Robinson type energy of the system

The Bianchi system together with the system satisfied by the dynamical acceleration constitute a FOSH system if $\mu'_p \geqslant 1$. Its Bel–Robinson type energy (superenergy) density on a space slice is the sum of the Bel–Robinson energy density of the gravitational field, and the energy density of the dynamical acceleration:

$$\mathcal{E} \equiv \mathcal{B} + \mathcal{F}.$$

Using the expression of ∂_0 and the mean extrinsic curvature $\tau \equiv g^{ij}K_{ij}$ of the space slices S_t, whose volume element we denote by $\mu_{\bar{g}_t}$, we obtain an integral equality whose right-hand side couples gravitational and fluid superenergies:

$$\int_{S_t} \mathcal{E}\mu_{\bar{g}} = \int_{S_t} \mathcal{E}\mu_{\bar{g}} + \int_{t_0}^{t}\int_{S_\theta} \{-N\tau\mathcal{E} + Q_G + \mathcal{S} + Q_F\}\mu_{\bar{g}}\,\mathrm{d}\theta.$$

The scalars $Q_G, \mathcal{S}$ and Q_F can be estimated in terms of $\mathcal{E}$ and thus lead to a linear inequality for $\mathcal{E}$, permitting the estimate of its growth with time.

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[1] η_{ijk} the volume form, $\|$ and $\overline{\nabla}$, the norm and covariant derivative, are defined by the space metric g_{ij}.

[2] An analogous procedure is used for the symmetrization of the Euler equations in [8] and [12].

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