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Stochastic calculus of variations and Harnack inequality on Riemannian path spaces

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Abstract We describe the tangent space of Riemannian path space as a space of tangent processes localized on Brownian sheets; the bundle of adapted frames above a Riemannian path space and its structural equation are given. The stochastic calculus of variations allows us to derive Harnack–Bismut inequality for the Norris semigroup. ***To cite this article: A.-B. Cruzeiro, P. Malliavin, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 817–820.***

Calcul de variations stochastiques et l'inéqualité de Harnack sur l'espace de chemins riemanniens

Résumé On décrit l'espace tangent à l'espace de chemins riemanniens comme un espace de processus tangents localisé sur des fueuilles browniennes ; le fibré de repères adaptés sur l'espace de chemins riemanniens et son équation de structure sont donnés. Le calcul de variations stochastiques permet de dériver l'inégalité de Harnack–Bismut pour le semigroupe de Norris. ***Pour citer cet article : A.-B. Cruzeiro, P. Malliavin, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 817–820.***

1. Structural equations

Given a d-dimensional compact Riemannian M we fix a point $m_0 \in M$ and we denote $P_{m_0}(M)$ the path space that is the space of continuous maps $p : [0,1] \mapsto M$ such $p(0) = m_0$ [12,14]. In [1–3] it has been emphasized that the differential geometry on the path space has to be compatible with the underlyig Itô filtration $\mathcal{N}_*$; in particular the natural unitary group associated to the change of frame is the subgroup of all unitary transformations of the Cameron–Martin space which commute with the orthogonal projections defined by the conditional expectations $E^{\mathcal{N}_*}$. This group can be realized as $P_e(\mathrm{SO}(d))$, the path group over the d-dimensional orthogonal group; in [4] the orthonormal frame bundle $O(P_{m_0}(M))$ has been defined so that its structural group is $P_e(\mathrm{SO}(d))$; the Levi-Civita parallel transport on M transport gives a canonic section $\sigma : P_{m_0}(M) \mapsto O(P_{M_0}(M))$.

As a substitute of the Levi-Civita connection for which the adaptness criterium to the Itô filtration fails, the *Markovian connection* has been introduced in [3]. This connnection induces on $O(P_{m_0}(M))$ a parallelism given by two differential forms $\pi = (\dot{\pi}, \ddot{\pi})$ where $\ddot{\pi}$ takes its values in $P_0(\mathrm{so}(d))$; a novelty of the present Note is to consider that the range of $\dot{\pi}$ is a *space $\mathcal{P}$ of localized tangent processes*. In [3] the tangent processes on the whole path space have been defined as the family of $\mathbb{R}^d$ valued semi-martingales

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$\mathrm{d}\zeta^\alpha = a^\alpha_\beta\,\mathrm{d}x^\beta + c^\alpha\,\mathrm{d}\tau$ such that $a^\alpha_\beta + a^\beta_\alpha = 0$; as this notion of tangent process involves the concept of semi-martingale, it is a global notion on the whole space $P_{m_0}(M)$. A differential form π is said to take its values in the space of tangent processes if and only if $\langle Z, \dot\pi\rangle$ is a tangent process for every adapted vector field Z defined on $O(P_{m_0}(M))$: constant vector fields in the parallelism are adapted vector fields and generate the space of all adapted vector fields. To obtain a Harnack inequality one needs to localize this global notion of tangent process: we shall proceed by parametrizing $P_{p_0}(P_{m_0}(M))$ by a Brownian sheet on which we shall use the global notion of tangent process.

A key fact established in [3,4,9] is the structural equations of the parallelism. In order to shorten this Note we shall assume all around that the Ricci curvature of M vanishes; when this hypothesis is not fulfilled the situation can be mastered by a suitable modification of the formalism, replacing the usual gradient on $P_{m_0}(M)$ by the *damped gradient* introduced in [10] (see [1]).

THEOREM. – *Assume that the Ricci tensor of M vanishes; denote by Z_* constant vector fields on $O(P_{m_0}(M))$; then*

$$\big(\langle Z_1 \wedge Z_2, \mathrm{d}\dot\pi\rangle\big)_{\sigma(p)} = \big(\ddot\pi(Z_2)\dot\pi(Z_1) - \ddot\pi(Z_1)\dot\pi(Z_2)\big)_{\sigma(p)} + \mathcal{T}\big(\dot\pi(Z_1), \dot\pi(Z_2)\big);$$

here $\mathcal{T}$ is the torsion defined by the following Itô stochastic integral:

$$\big(\mathcal{T}(z_1, z_2)\big)_\tau := \int_0^\tau \Omega(z_1, z_2)\,\mathrm{d}x,$$

where Ω is the Riemann curvature tensor of M and where x denotes the antideveloppement of $p \in P_{m_0}(M)$; the functor $\mathcal{T}$ is a bilinear map of $\mathcal{P}\times\mathcal{P} \mapsto \mathcal{P}$. In the same way introduce the functor $\mathcal{C}$ which associates to $\zeta, \zeta' \in \mathcal{P}$ the endomorphism of $\mathcal{P}$ defined by

$$\mathcal{C}(\zeta, \zeta') : \eta \mapsto \int_0^* \Omega(\zeta, \zeta')\,\mathrm{d}\eta;$$

then $\mathcal{C}$ is the curvature of the parallelism in the sense that the following structural equation holds true:

$$\big(\langle Z_1 \wedge Z_2, \mathrm{d}\ddot\pi\rangle\big)_{\sigma(p)} - \big[\ddot\pi(Z_1), \ddot\pi(Z_2)\big]_{\sigma(p)} = \mathcal{C}.$$

Remark. – The torsion can be obtained by saturating one index of the curvature by the stochastic differential along the path $\mathrm{d}x : \mathcal{T} = \mathcal{C}(\mathrm{d}x)$; this type of contraction is a key building stone in the theory of iterated path integrals.

2. Harnack inequality

Denote μ the Wiener measure on $P_{m_0}(M)$; then the Cameron–Martin type gradient on $P_{m_0}(M)$ defines a Dirichlet form; the corresponding process has been constructed in [7]; under our hypothesis of vanishing Ricci this process coincides with the process defined in [13] and therefore is compatible with the Itô filtration. Its infinitesimal generator $\mathcal{L}$, using [11], has been written in [4] on $O(P_{m_0}(M))$ using the covariant derivative associated to the parallelism $\widehat\nabla$ as

$$2\mathcal{L} = \sum_\alpha \int_0^1 \big(\widehat\nabla^2_{\tau,\alpha}\,\mathrm{d}\tau - \widehat\nabla_{\tau,\alpha}\,\mathrm{d}x^\alpha(\tau)\big).$$

We denote by $\Pi_t(p_0, \mathrm{d}p)$ the heat kernel associated to the semi-group $\exp(t\mathcal{L})$ which is defined in [7] for p_0 outside a set of null capacity; the strong machinery of [13] makes possible to define it for all p_0; in this work we have for objective to prove inequalities which can be reached through uniform estimates of finite dimensional approximations in the spirit of [5] and in this context we shall not emphasize the problem of the domain of definition of $\Pi_t(*, \mathrm{d}p)$. For Harnack–Bismut type derivative formulas see [8] and references therein.

The Cameron–Martin space of the Wiener space of the R^d valued brownian motion is the Hilbert space $\mathrm{H}^1(R^d)$ of R^d valued paths z such that $\dot z \in \mathrm{L}^2$; we denote $T^1_{p_0}(P_{m_0}(M))$ its image by parallel transport along p_0 of $\mathrm{H}^1(T_{m_0}(M))$. For any vector $z \in T^1_{p_0}(P_{m_0}(M))$ we define the logarithmic derivative $\partial_z \log \Pi_t(p_0, p)$ in the spirit of [6] by the following identity holding true for every bounded test function ϕ:

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$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\big(\Pi_t(p_0+\varepsilon z,\mathrm{d}p)-\Pi_t(p_0,\mathrm{d}p)\big)\phi(p)=\int\Pi_t(p_0,\mathrm{d}p)\partial_z\log\Pi_t(p_0,p)\,\phi(p).$$

HARNACK THEOREM. – *Let M be a d-dimensional Riemmanian manifold with vanishing Ricci curvature and with curvature tensor as well as its first three covariant derivatives bounded; then for all $r\in\,]1,\infty[$ there exists a constant C_r depending only on the previous bounds such that for all vector field Z such that the Cameron–Martin norm $\|Z_p\|_{T^1_{p_0}(P_{m_0}(M))}\leqslant 1$, $\forall p$ we have*

$$\int_{P_{m_0}(M)\otimes P_{m_0}(M)}\big|\partial_Z\log\Pi_t(p_0,p)\big|^r\mu(\mathrm{d}p_0)\otimes\mu(\mathrm{d}p)\leqslant C_r\exp(-rt).$$

This theorem will be approached by the stochastic calculus of variations which consists in looking at the propagation of z along the time evolution of the stochastic flow associated to $\mathcal{L}$; the structural equations imply that a vector $z\in T^1_{p_0}(P_{m_0}(M))$ propagates as a *local tangent processes* along the time evolution: one novelty of this Note is to consider tangent processes on an auxiliary probability space which allows us to fix the starting point p_0 without loosing their properties.

3. Stochastic calculus of variations along the Brownian sheet

We denote $w^*(t,\tau)$ the R^d valued Brownian sheet defined for $t>0$, $\tau\in[0,1]$ such that $w(0,*)=w(*,0)=0$. We denote y_t the R^d-valued Brownian curve defined as $y_t(\tau)=w(t,\tau)$. We consider for initial σ field the data of a starting path together with a vector in the Cameron–Martin tangent space at this starting point; then we denote $\mathcal{N}_{t,\tau}$ the σ field obtained by adding at the initial σ field the information given by the knowledge of $x^\alpha(t',\tau')$ for $t'<t$, $\tau'<\tau$. We denote $\mathcal{I}_t^{t'}$, $t<t'$, the innovation $\mathcal{N}_{t',1}-\mathcal{N}_{t,1}$. We define a vector field

$$Z_t(p)=E^{p_{y_t}(t)=p}\big(\zeta_1(t)\big).$$

Considering a trajectory p_t of the process associated to $\mathcal{L}$ starting from p_0, we denote x_t its antidevelopment. Then we shall use the parametrization

$$d_tx_t=\mathrm{d}y_t-x_t\,\mathrm{d}t,\quad x_0\neq 0,\quad y_0=0. \tag{1}$$

THEOREM. – *Denote $\zeta(t,\tau)$ a variation of the Brownian sheet*; *denote ζ_1 the corresponding variation of p_t looked upon the parallelism then we have*:

$$d_t\zeta=d_t\zeta_1+\zeta_1\,\mathrm{d}t-\rho\,o\,d_tx_t+\mathcal{T}(\zeta_1,o\,d_tx_t),\qquad d_t\rho=\mathcal{C}(\zeta_1,o\,d_tx_t). \tag{2}$$

The stochastic contraction involving ρ appearing (2) takes the shape

$$\big(\mathcal{C}(\zeta_1,d_tx)\,d_tx_t\big)_\tau=\int_0^\tau\mathrm{Ricci}^M\big(\zeta_1(s)\big)\,\mathrm{d}s$$

the right-hand side can be interpreted as defining $\mathrm{Ricci}^{P_{m_0}(M)}$; this interpretation is coherent with the interpretation given [3], p. 165, formula (9.7.1). In our setting where $\mathrm{Ricci}^M=0$ this contraction disappears.

We can choose the variations ζ, ζ_1 as we like as soon that (2) is fullfiled; we make the following choice determining first ζ_1 and subsequently ζ:

$$d_t\zeta_1(t)=-\zeta_1(t)\,\mathrm{d}t-\mathcal{T}\big(\zeta_1(t),o\,d_tx_t\big),\quad\zeta_1(0)=z\in T^1\big(P_{m_0}(M)\big), \tag{3}$$

$$d_t\zeta(t)=-\rho\,d_tx_t,\quad d_t\rho=\mathcal{C}(\zeta_1,o\,d_tx_t),\quad\zeta(0)=0,\quad\rho(0)=0. \tag{4}$$

Using the change of variable (1) Eqs. (3), (4) can be transfered on the Brownian sheet; we call a *local tangent process* a map which for each t defines a semi-martingale on the Brownian space $w(t,*)$ with the antisymmetry of its martingale part; then ζ_1 is a local tangent process. We denote μ_t the Wiener measure on $P_0(R^d)$ defined by $w(t,*)$. We fix a finite mass Borelian measure θ on R^+ and we consider the finite mass measure defined on $P_0(R^d)$ by $\nu_\theta=\int_0^\infty\mu_t\theta(\mathrm{d}t)$; then

THEOREM. – *The infinitesimal transformation associated to $\zeta(*)$ preserves the measures ν_θ.*

4. Integration by parts procedure

A key step in the proof of Harnack–Bismut formula in finite dimensions is to realize an integration by parts by using a Girsanov formula. In our setting Girsanov formula is not available; worst, the coefficient of this "tentative Girsanov formula" will not be a function in a Cameron–Martin space relatively to the variable τ but a tangent process. Trotter–Kato formula will supply for us the missing Girsanov formula; the annoying contribution of tangent processes will then disappear. We consider the OU process associated to $\mathcal{L}$ and its lift to the frame bundle, associated to $\widetilde{\mathcal{L}}$, a crucial object constructed in [5]; we denote for $t_1 < t_2$ the corresponding stochastic flow $U^w_{t_2\leftarrow t_1}$, $\widetilde{U}^w_{t_2\leftarrow t_1}$. The measures μ_t being mutually singular, we shall realize the semi-group associated to $\mathcal{L}$ on space time introducing for $s < t$ a map $P_{s\leftarrow t} : L^2_{\mu_t} \mapsto L^2_{\mu_s}$ defined by $P_{s\leftarrow t} = E^{\mathcal{F}_s}((U^w_{t\leftarrow s})^* f)$. We denote by $\mathcal{P}_t$ the space of tangent process on the Wiener space $w(t,*)$ and we define a flow $(Q_{t\leftarrow s}(\zeta_s))_y = E^{w(t,*)=y}(\eta(t))$, where $t > s$, and where η satisfies Eq. (3) with the initial value $\eta(s) = \zeta$. We have the intertwinning formula

$$\langle \zeta_s, dP_{s\leftarrow t} f\rangle = P_{s\leftarrow t}\big(\langle Q_{t\leftarrow s}\zeta_s, df\rangle\big).$$

Then we have the formula of integration by parts

$$tE\langle \zeta_0, dP_{0\leftarrow t} f\rangle_{p_0} = E\left(\left(\int_0^t \delta_s(Q_{s\leftarrow 0}\zeta_0)\,\mathrm{d}s\right) f\big(U^w_{t\leftarrow 0}(p_0)\big)\right), \tag{5}$$

where δ_s denotes the divergence relatively to the infinitesimal measure generated by $\mathcal{F}_{s+\varepsilon} - \mathcal{F}_s$.

From a differential geometer point of view our strategy can be summarized as follows: the pionneering work of Norris [13] has shown the fittness of Brownian sheet stochastic calculus for the study of path space; Norris used a delicate two parameters M-valued Stratonovitch stochastic calculus; lifting the situation to frame bundle only a milded R^d valued two parameters stochastic calculus is needed here; frame bundle technology needs firstly a computable knowledge of frame bundle structural equations recently obtained in [4,9] and secondly the realization of the lift of the OU process to frame bundle: this realization depends upon finite dimensional approximation realized in [5].

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