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Finite extensions and unipotent shadows of affine crystallographic groups

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Abstract Let Γ be a virtually polycyclic group so that the Fitting subgroup is torsion-free and contains its centralizer. We prove that an effective extension of Γ by a finite group μ is isomorphic to an affine crystallographic group if and only if there exists a fixed point for the action of μ on the deformation space of affine crystallographic actions of Γ. We associate to Γ a finitely generated torsion-free nilpotent group Θ which is called the unipotent shadow of Γ, and we relate the deformation space of Γ to the deformation space of Θ. As an application, we show that Γ is isomorphic to an affine crystallographic group if, e.g., Θ has nilpotency class $\leqslant 3$, or if the polycylic rank of Γ is $\leqslant 5$, and also in some other cases. ***To cite this article: O. Baues, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 785–788.***

Extensions finies et ombres unipotentes des groupes affines cristallographiques

Résumé Soit Γ un groupe virtuellement polycyclique tel que le sous-groupe de Fitting soit sans torsion et contienne son centralisateur. Nous montrons qu'une extension effective de Γ par un groupe fini μ est isomorphe à un groupe affine cristallographique si et seulement si μ laisse fixe un point dans l'espace des déformations des actions affines cristallographiques de Γ. Nous associons à Γ un groupe nilpotent sans torsion et de type fini Θ que nous appelons l'ombre unipotente de Γ. Ensuite nous relions l'espace des déformations de Γ à l'espace des déformations de Θ. Comme application nous montrons que Γ est isomorphe à un groupe affine cristallographique si, par exemple, Θ est de classe de nilpotence $\leqslant 3$, ou si le rang polycyclique de Γ est $\leqslant 5$, ainsi que dans certains autres cas. ***Pour citer cet article : O. Baues, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 785–788.***

1. Introduction

Let V be a finite dimensional real vectorspace, and $\mathrm{Aff}(V) = V \cdot \mathrm{GL}(V)$ the group of affine transformations of V. A group $\Gamma \leqslant \mathrm{Aff}(V)$ is called an *affine crystallographic group* (ACG) if (i) for all compact subsets $K \subset V$, $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite, and (ii) the quotient space $M = {}_{\Gamma}\backslash V$ is compact. If Γ is a torsion-free ACG then the quotient M is a compact *complete affinely flat manifold*. If $A \leqslant \mathrm{Aff}(V)$ is a subgroup and $\Gamma \leqslant A$ then Γ is called an ACG of type A. If $A = E(n)$, the group of Euclidean motions of $\mathbb{R}^n$,

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then one obtains the concepts of Euclidean crystallographic group and compact flat Riemannian manifold, respectively. It is conjectured that an affine crystallographic group is a virtually polycyclic (v.p.-) group. The present note is concerned with some aspects of the question which abstract virtually polycyclic groups Γ are isomorphic to an affine crystallographic group.

Bieberbach [4] showed that Euclidean crystallographic groups have a torsion-free maximal Abelian normal subgroup of finite index. Later Burckhardt and Zassenhaus [5,13] showed that every abstract group with these properties is isomorphic to an Euclidean crystallographic group. In general, a necessary condition for a v.p.-group Γ being isomorphic to an ACG is that the Fitting subgroup $\mathrm{Fitt}(\Gamma)$, i.e., the maximal nilpotent normal subgroup of Γ, is torsion-free (see [7]) and contains its centralizer. We assume from now on that Γ satisfies this condition. (An equivalent condition is that Γ does not contain any non-trivial finite normal subgroups.) If $\Gamma \leqslant \mathrm{Aff}(V)$ is an ACG it satisfies $\operatorname{rank}\Gamma = \dim V$, where $\operatorname{rank}\Gamma$ denotes the cyclic rank of Γ. If $\operatorname{rank}\Gamma \leqslant 3$ then Γ acts as an ACG (as follows from [6]). However, there do exist finitely generated torsion-free nilpotent (f.t.n.-) groups (with rank $\geqslant 10$) which are not isomorphic to an ACG, see [3]. On the other hand it is known [11,12] that every f.t.n.-group Γ which is nilpotent of class $\leqslant 3$ is affine crystallographic. This result was extended in [8] to the case that Γ is a torsion-free finite extension group of some f.t.n.-group of nilpotency class $\leqslant 3$.

As to a general approach on classifying virtually polycyclic ACGs it seems tempting to try to reduce the problem to a question on the Fitting subgroup of Γ. For example, one can ask: [1] *Is Γ isomorphic to an ACG if and only if* $\mathrm{Fitt}(\Gamma)$ *is?* In a similar vein is the following problem: *Let Γ be an affine crystallographic group and Δ a finite extension group of Γ. Is Δ isomorphic to an ACG?* We explain here how the answer to such kind of questions is determined by the appropiate 'Teichmüller theory' for affine crystallographic actions, and give some applications. Complete proofs will be provided in a separate publication.

2. Deformation spaces

A homomorphism $\rho : \Gamma \to A \leqslant \mathrm{Aff}(V)$ will be called a *crystallographic homomorphism* if ρ is injective and the image $\rho(\Gamma) \leqslant \mathrm{Aff}(V)$ is a crystallographic subgroup. We define the *space of crystallographic homomorphisms* as

$$\mathrm{Hom}_c(\Gamma, A) = \{\rho : \Gamma \to A \mid \rho \text{ is crystallographic}\}.$$

The group A acts by conjugation on the space of homomorphisms and the associated quotient space

$$\mathcal{D}_c(\Gamma, A) = \mathrm{Hom}_c(\Gamma, A)/A$$

is called the *deformation space*. $\mathcal{D}_c(\Gamma, A)$ is a topological space with the quotient topology inherited from $\mathrm{Hom}_c(\Gamma, A)$, where $\mathrm{Hom}_c(\Gamma, A)$ carries the subspace topology from $\mathrm{Hom}(\Gamma, A)$.

An *extension* of Γ is an exact sequence $1 \to \Gamma \to \Delta \to \mu \to 1$. The extension is called finite if μ is finite. In this case, the group Δ is called a finite extension group of Γ. The extension induces a homomorphism $\alpha : \mu \to \mathrm{Out}(\Gamma)$, and the extension is called *effective* if α is injective. The group $\mathrm{Out}(\Gamma)$ naturally acts on $\mathcal{D}_c(\Gamma, A)$, and therefore also μ acts on $\mathcal{D}_c(\Gamma, A)$. Let $\mathcal{D}_c(\Gamma, A)^{\mu}$ denote the set of fixed points for the action of μ. The natural restriction map

$$\mathit{res} : \mathrm{Hom}_c(\Delta, A) \longrightarrow \mathrm{Hom}_c(\Gamma, A)$$

gives rise to a restriction map $\overline{\mathit{res}}$ on deformation spaces. Let Δ be a finite effective extension group of Γ. We have

THEOREM 2.1. – *The restriction map res induces a continuous bijection*

$$\overline{\mathit{res}} : \mathcal{D}_c(\Delta, A) \to \mathcal{D}_c(\Gamma, A)^{\mu}.$$

COROLLARY 2.2. – *Δ is isomorphic to an affine crystallographic group of type A if and only if the induced action of $\mu = \Delta / \Gamma$ on $\mathcal{D}_c(\Gamma, A)$ has a fixed point.*

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3. Algebraic hulls and the unipotent shadow

We need some terminology. A group $\mathbf{G}$ is called a $\mathbb{Q}$-defined linear algebraic group if it is a Zariski-closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is defined by polynomials with rational coefficients. Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{G}$. We say that $\mathbf{G}$ has a *strong unipotent radical* if the centralizer $C_{\mathbf{G}}(\mathbf{U})$ is contained in $\mathbf{U}$.

THEOREM 3.1. – *There exists a $\mathbb{Q}$-defined linear algebraic group* $\mathbf{H}$ *and an embedding* $\psi : \Gamma \to \mathbf{H}_{\mathbb{Q}}$ *so that,*

(i) $\psi(\Gamma)$ *is Zariski-dense in* $\mathbf{H}$,
(ii) $\mathbf{H}$ *has a strong unipotent radical* $\mathbf{U}$,
(iii) $\dim \mathbf{U} = \operatorname{rank} \Gamma$.

We remark that the group $\mathbf{H}$ is determined by the conditions (i)–(iii) up to $\mathbb{Q}$-isomorphism of algebraic groups. We call the $\mathbb{Q}$-defined linear algebraic group $\mathbf{H}$ the *algebraic hull for* Γ. If Γ is finitely generated torsion-free nilpotent then $\mathbf{H}$ is unipotent and Theorem 3.1 is essentially due to Malcev. If Γ is torsion-free polycyclic, Theorem 3.1 is due to Mostow [9] (see also [10, §IV]).

We explain next the role the algebraic hull $\mathbf{H}$ plays for affine crystallographic actions of Γ. Let $H = \mathbf{H}_{\mathbb{R}}$ denote the group of real points of $\mathbf{H}$. A homomorphism of H into $\mathrm{Aff}(V)$ will be called *u-simply transitive* if the unipotent radical $U = \mathbf{U}_{\mathbb{R}} \leqslant H$ acts simply transitively on V.

THEOREM 3.2. – *A homomorphism $\rho : \Gamma \to A$ is crystallographic if and only if ρ extends to a u-simply-transitive embedding of real algebraic groups $\bar{\rho} : H \to A$.*

Note that, by the properties of the algebraic hull, the algebraic extension $\bar{\rho}$ is uniquely determined by ρ. Auslander [1] observed that simply transitive groups on affine space contain a unipotent simply transitive group in their algebraic closure. The interplay between solvable simply transitive Lie groups and virtually polycyclic ACGs was studied extensively in [6] and [7].

3.1. The unipotent shadow

The existence and uniqueness of the algebraic hull $\mathbf{H}$ for Γ allows to associate to Γ in a canonical way a f.t.n.-group $\Theta \leqslant \mathbf{U}_{\mathbb{Q}}$ with $\operatorname{rank} \Theta = \operatorname{rank} \Gamma$. As an abstract group, Θ is then defined uniquely up to commensurability. We call Θ a *unipotent shadow* for Γ. Let $\Theta \leqslant \mathbf{U}_{\mathbb{Q}}$ be a fixed unipotent shadow for Γ. Θ is a lattice in the real Lie group U. Hence, by Theorem 3.2, there exists a *shadow map*

$$s : \mathrm{Hom}_c(\Gamma, A) \longrightarrow \mathrm{Hom}_c(\Theta, A)$$

which induces a corresponding map $\bar{s}$ on deformation spaces. The automorphism group $\mathrm{Aut}(H)$ acts by left composition on the space of u-simply transitive representations of H. By Theorem 3.2, $\mathrm{Aut}(H)$ acts on $\mathrm{Hom}_c(\Gamma, A)$, and $\mathrm{Out}(H)$ acts on $\mathcal{D}_c(\Gamma, A)$. Let $\mathbf{T} = \mathbf{H}_\Gamma / \mathrm{u}(\mathbf{H}_\Gamma)$, $T = T_{\mathbb{R}}$. Via the induced homomorphism $\alpha : T \to \mathrm{Out}(U)$ there is a natural action of T on $\mathcal{D}_c(\Theta, A)$. Let $\mathcal{D}_c(\Theta, A)^T$ denote the set of fixed points of T in $\mathcal{D}_c(\Theta, A)$. We have the following 'shadow theorem':

THEOREM 3.3. – *The shadow map induces a continuous bijection*

$$\bar{s} : \mathcal{D}_c(\Gamma, A) \longrightarrow \mathcal{D}_c(\Theta, A)^T.$$

Note that in the case that Γ is virtually nilpotent, Theorem 3.3 is in fact a special case of Theorem 2.1. In the general case, Theorem 2.1 will be implied by the shadow theorem.

COROLLARY 3.4. – *Γ is isomorphic to an affine crystallographic group of type A if and only if the induced action of T on $\mathcal{D}_c(\Theta, A)$ has a fixed point.*

4. Properties of deformation spaces

Let Θ be a f.t.n.-group and $U = U_\Theta$ the real Malcev hull of Θ. Let $V = \mathbb{R}^{\operatorname{rank}\Theta}$. Let L be a Levi (maximal reductive-) subgroup of $\operatorname{Aut}(U)$.

PROPOSITION 4.1. – *If Θ is of nilpotency class $\leqslant 2$ then there exists a fixed point for* $\operatorname{Aut}(U)$ *in* $\mathcal{D}_c(\Theta, \operatorname{Aff}(V))$. *If Θ is of of nilpotency class* 3 *then L has a fixed point in* $\mathcal{D}_c(\Theta, \operatorname{Aff}(V))$.

Let $\mathfrak{g}$ be a Lie algebra. A positive grading $\mathfrak{g} = \bigoplus_{l=1,\dots,k} \mathfrak{g}_l$ is called invariant by a subgroup $G \leqslant \operatorname{Aut}(\mathfrak{g})$ if the corresponding filtration of $\mathfrak{g}$ is invariant, i.e., for all $g \in G$, $g \bigoplus_{l=i,\dots,k} \mathfrak{g}_l \subseteq \bigoplus_{l=i,\dots,k} \mathfrak{g}_l$, $i = 1, \dots, k$. Let $\mathfrak{u}$ denote the Lie algebra of U.

PROPOSITION 4.2. – *If $\mathfrak{u}$ admits a L-invariant positive grading then L has a fixed point in* $\mathcal{D}_c(\Theta, \operatorname{Aff}(V))$.

Let Θ be the unipotent shadow of Γ, $\alpha : T \to \operatorname{Out}(U_\Theta)$ the associated semisimple kernel. We consider the two following conditions:

1. Θ is of of nilpotency class $\leqslant 3$.
2. U_Θ has a T-invariant positive grading.

COROLLARY 4.3. – *If Γ satisfies one of the two conditions above then Γ is isomorphic to an affine crystallographic group.*

COROLLARY 4.4. – *If Γ satisfies* $\operatorname{rank}\Gamma \leqslant 5$ *then Γ is isomorphic to an affine crystallographic group.*

In particular, in both cases, every finite effective extension of Γ is isomorphic to an affine crystallographic group.

4.1. Some examples

The deformation space $\mathcal{D}_c(\mathbb{Z}^n, E(n))$ of Euclidean crystallographic actions of $\mathbb{Z}^n$ is GL_n/O_n, and the action of $\mathrm{GL}(n,\mathbb{Z}) = \operatorname{Out}(\mathbb{Z}^n)$ on $\mathcal{D}_c(\mathbb{Z}^n, E(n))$ is represented by left multiplication on GL_n/O_n. In particular, any finite subgroup $\mu \subset \operatorname{Out}(\mathbb{Z}^n)$ has a fixed point in $\mathcal{D}_c(\mathbb{Z}^n, E(n))$. More generally, it is possible to show that the spaces $\mathcal{D}_c(\mathbb{Z}^n, A)$ are Hausdorff spaces which are homeomorphic to a real algebraic set. It was proved in [2] that the space $\mathcal{D}_c(\mathbb{Z}^2, \operatorname{Aff}(2))$ is homeomorphic to $\mathbb{R}^2$ with $\mathrm{GL}(2,\mathbb{Z}) = \operatorname{Out}(\mathbb{Z}^2)$ acting on $\mathbb{R}^2$ in the standard linear way. Let $\pi_1(K)$ be the fundamental group of the Klein-bottle. We deduce that $\mathcal{D}_c(\pi_1(K), \operatorname{Aff}(2))$ is homeomorphic to the diagonal in $\mathbb{R}^2$.

[1] This question was suggested by Fritz Grunewald.

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