C. R. Acad. Sci. Paris, Ser. I 335 (2002) 801–804

Équations aux dérivées partielles/*Partial Differential Equations*

Sharp Sobolev type inequalities for higher fractional derivatives

Athanase Cotsiolis, Nikolaos Con. Tavoularis [1]

Department of Mathematics, University of Patras, Patras 26110, Greece

Received and accepted 1 October 2002

Note presented by Thierry Aubin.

Abstract On $\mathbb{R}^n$, $n \geqslant 1$ and $n \neq 2$, we prove the existence of a sharp constant for Sobolev inequalities with higher fractional derivatives. Let s be a positive real number. For $n > 2s$ and $q = \frac{2n}{n-2s}$ any function $f \in \mathrm{H}^s(\mathbb{R}^n)$ satisfies

$$\|f\|_q^2 \leqslant S_{n,s} \left\|(-\Delta)^{s/2} f\right\|_2^2,$$

where the operator $(-\Delta)^s$ in Fourier spaces is defined by $\widehat{(-\Delta)^s f}(k) := (2\pi|k|)^{2s} \widehat{f}(k)$. ***To cite this article: A. Cotsiolis, N.C. Tavoularis, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 801–804.***
© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Inégalités optimales de type Sobolev pour les dérivées fractionelles d'ordre supérieur

Résumé Sur $\mathbb{R}^n$, $n \geqslant 1$ et $n \neq 2$, on établit l'existence de meilleurs constantes dans les inégalités de Sobolev pour les dérivées fractionelles d'ordre supérieur. Soit s un reel positif. Pour $n > 2s$ et $q = \frac{2n}{n-2s}$ toute fonction $f \in \mathrm{H}^s(\mathbb{R}^n)$ vérifie l'inégalité suivante

$$\|f\|_q^2 \leqslant S_{n,s} \|(-\Delta)^{s/2} f\|_2^2,$$

où $S_{n,s}$ est la meilleure constante. L'opérateur $(-\Delta)^s$ est defini dans les espaces de Fourier par $\widehat{(-\Delta)^s f}(k) := (2\pi|k|)^{2s} \widehat{f}(k)$. ***Pour citer cet article : A. Cotsiolis, N.C. Tavoularis, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 801–804.***
© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Preliminaries

The Sobolev space $\mathrm{H}^l(\mathbb{R}^n)$ is endowed with the norm $\|f\|^2_{\mathrm{H}^l(\mathbb{R}^n)} = \sum_{0 \leqslant \alpha \leqslant l} \|D^\alpha f\|^2_{\mathrm{L}^2(\mathbb{R}^n)}$ for $f : \mathbb{R}^n \to \mathbb{C}$ with l a positive integer. This norm is equivalent to the norm $\|f\|'^2_{\mathrm{H}^l(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\widehat{f}(k)|^2 (1 + (2\pi|k|)^{2l})\, \mathrm{d}k$ (thanks to the Plancherel formula) where $|k| = (\sum_{i=1}^n k_i^2)^{1/2}$ and $\widehat{f}(k) := \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} kx} f(x)\, \mathrm{d}x$ is the Fourier transform of the function $f(kx := \sum_{i=1}^n k_i x_i)$. We set $(f)^\wedge(-x) = \widehat{f}(-x) = f^\vee(x)$. So $f = (\widehat{f})^\vee$.

E-mail addresses: cotsioli@math.upatras.gr (A. Cotsiolis); niktav@master.math.upatras.gr (N.C. Tavoularis).

© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés
S1631-073X(02)02576-1/FLA

DEFINITION 1.1. – A function $f \in \mathrm{L}^2(\mathbb{R}^n)$ is said to be in $\mathrm{H}^s(\mathbb{R}^n)$ if and only if

$$\|f\|^2_{\mathrm{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \left|\widehat{f}(k)\right|^2 \left(1 + \left(2\pi|k|\right)^{2s}\right) \mathrm{d}k < \infty.$$

The space $\mathrm{H}^s(\mathbb{R}^n)$ is endowed with the inner product

$$(f, g)_{\mathrm{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \overline{\widehat{f}(k)}\widehat{g}(k)\left(1 + \left(2\pi|k|\right)^{2s}\right) \mathrm{d}k.$$

DEFINITION 1.2. – The operator $(-\Delta)^s$ is defined in Fourier spaces (i.e., in spaces with functions which have Fourier transform such as $\mathrm{L}^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant 2$, see [5]) as multiplication by $(2\pi|k|)^{2s}$, i.e.,

$$\widehat{(-\Delta)^s} f(k) := \left(2\pi|k|\right)^{2s} \widehat{f}(k).$$

If f and g are in $\mathrm{H}^s(\mathbb{R}^n)$ then the sesquilinear form

$$\left(g, (-\Delta)^s f\right) = \int_{\mathbb{R}^n} \overline{\widehat{g}(k)}\widehat{f}(k)\left(1 + \left(2\pi|k|\right)^{2s}\right) \mathrm{d}k$$

makes sense by Hölder inequality. Also $(f, (-\Delta)^s f) = \|(-\Delta)^{s/2} f\|_2^2$.

PROPOSITION 1.1. – (i) *We have* $\mathrm{H}^{[s]+1}(\mathbb{R}^n) \subset \mathrm{H}^s(\mathbb{R}^n) \subseteq \mathrm{H}^{[s]}(\mathbb{R}^n)$ *where* $[s]$ *is the integer part of* s.
(ii) *For* $0 < \delta < \gamma < 1$, $\mathrm{H}^\gamma(\mathbb{R}^n) \subset \mathrm{H}^\delta(\mathbb{R}^n)$.

PROPOSITION 1.2. – *If* f *is in* $\mathrm{H}^l(\mathbb{R}^n)$ *then* $\widehat{D^\alpha f}(k) = (2\pi \mathrm{i}k)^\alpha \widehat{f}(k)$ *with* $|\alpha| \leqslant l$, l *an integer.*

PROPOSITION 1.3. – *If* $f \in \mathrm{H}^s(\mathbb{R}^n)$ *then* $|f| \in \mathrm{H}^s(\mathbb{R}^n)$.

THEOREM 1.1. – (i) *If f is in* $\mathrm{H}^l(\mathbb{R}^n)$, *then there exists a sequence of functions in* $\mathrm{C}_c^\infty(\mathbb{R}^n)$ *such that* $\|f^m - f\|_{\mathrm{H}^l(\mathbb{R}^n)} \to 0$ *as* $m \to \infty$.
(ii) *If f is in* $\mathrm{H}^s(\mathbb{R}^n)$ *then there exists a sequence of functions in* $\mathrm{C}_c^\infty(\mathbb{R}^n)$ *such that* $\|f^m - f\|_{\mathrm{H}^s(\mathbb{R}^n)} \to 0$ *as* $m \to \infty$.

Remark 1.1. – These results are in [5] for the spaces $\mathrm{H}^1(\mathbb{R}^n)$ and $\mathrm{H}^{1/2}(\mathbb{R}^n)$.

DEFINITION 1.3. – Set $F_s(k) = \mathrm{e}^{-t(2\pi|k|)^{2s}}$. When $t > 0$, we define the operator $\mathrm{e}^{-t(-\Delta)^s}$ on functions f in $\mathrm{L}^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant 2)$ by

$$\left(\mathrm{e}^{-t(-\Delta)^s} f\right)^\wedge(k) = \mathrm{e}^{-t(2\pi|k|)^{2s}} \widehat{f}(k).$$

Remark 1.2. – The function $F_s(k) = \mathrm{e}^{-t(2\pi|k|)^{2s}}$ is in $\mathrm{L}^p(\mathbb{R}^n)$ for $p \geqslant 1$. By Hausdorff–Young inequality (see [5]) we have that the function $\widehat{F_s}$ is in $\mathrm{L}^{p'}(\mathbb{R}^n)$ with $1/p + 1/p' = 1$, so we can define its convolution $\widehat{F_s} * f$ with $f \in \mathrm{L}^p(\mathbb{R}^n)$.

THEOREM 1.2. – *A function* f *is in* $\mathrm{H}^s(\mathbb{R}^n)$ *if and only if it is in* $\mathrm{L}^2(\mathbb{R}^n)$ *and*

$$I_s^t(f) = \frac{1}{t}\left[(f, f) - \left(f, \mathrm{e}^{-t(-\Delta)^s} f\right)\right]$$

is uniformly bounded and we have in which case

$$\sup_{t>0} I_s^t(f) = \lim_{t\to 0} I_s^t(f) = \left(f, (-\Delta)^s f\right).$$

To cite this article: A. Cotsiolis, N.C. Tavoularis, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 801–804

Remark 1.3. – For the cases $s = 1$ and $s = 1/2$ Theorem 1.2 is given by [5].

2. Sharp Sobolev type inequalities (see [3])

The following inequality ($n > 2s$, s a positive real)

$$\left|(f, g)\right|^2 \leqslant 2^{-2s}\pi^{-n/2}\frac{\Gamma((n-2s)/2)}{\Gamma(s)}\left(f, (-\Delta)^s f\right)\left(g, |x|^{2s-n} * g\right)$$

is valid for $f \in \mathrm{H}^s(\mathbb{R}^n)$ and $g \in \mathrm{L}^p(\mathbb{R}^n)$

THEOREM 2.1. – *For $n > 2s$, let $f \in \mathrm{H}^s(\mathbb{R}^n)$ and $q = \frac{2n}{n-2s}$. Then the following inequality holds*:

$$\|f\|_q^2 \leqslant S_{n,s}\left\|(-\Delta)^{s/2} f\right\|_2^2,$$

where

$$S_{n,s} = 2^{-2s/n}\pi^{-s(n+1)/n}\frac{\Gamma((n-2s)/2)}{\Gamma((n+2s)/2)}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^{2s/n}$$

and Γ denotes the Gamma function. There is equality in the inequality above if and only if $f(x)$ is a multiple of the function $(\mu^2 + (x-\alpha)^2)^{-(n-2s)/2}$ with $\mu > 0$ and $\alpha \in \mathbb{R}^n$.

Remark 2.1. – (i) For $s = 1$ the best constant of Theorem 2.1 is given in [1,2] and [7].
(ii) For $s = 1/2$ the best constant of Theorem 2.1 is given in [5].
(iii) For $s = 2$ the best constant of Theorem 2.1 is given in [4] and [8].

Finally we give an inequality for the space $\mathrm{H}^s(\mathbb{R})$.

THEOREM 2.2. – *For $f \in \mathrm{H}^s(\mathbb{R})$ the inequality*

$$\left\|(-\Delta)^{s/2} f\right\|_2^2 + \|f\|_2^2 \geqslant S_{1,q,s}\|f\|_q^2$$

holds for all $2 < q < \infty$ and $2\frac{q}{q-2}s > 1$ with a constant $S_{1,q,s}$ that satisfies

$$S_{1,q,s} > (q-1)^{1-1/q}q^{-1+2/q}\left[\frac{\Gamma(1+1/(2s))\Gamma(-1/(2s)+q/(q-2))}{\pi\Gamma(q/(q-2))}\right]^{-(q-2)/q}.$$

Remark 2.2. – For the spaces $\mathrm{H}^{1/2}(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$ Lieb's results are in [5] and [6] respectively.

---

[1] Partially supported by State Scholarship's Foundation.

References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer, 1998.
[2] T. Aubin, Problèmes isopèrimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.
[3] A. Cotsiolis, N. Tavoularis, Best constant for the embedding of the space $\mathrm{H}^s(\mathbb{R}^n)$ into $\mathrm{L}^{2n/(n-2s)}(\mathbb{R}^n)$ and applications, in preparation.
[4] Z. Djadli, E. Hebey, M. Ledoux, Paneitz type operators and applications, Duke Math. J. 104 (1) (2000) 129–169.
[5] E. Lieb, M. Loss, Analysis, 2nd edition, American Mathematical Society, 2001.
[6] E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. Math. 118 (1983) 349–374.

[7] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[8] R.C.A.M. van der Vorst, Best constant for the embedding of the space $\mathrm{H}^2 \cap \mathrm{H}_0^1(\Omega)$ into $\mathrm{L}^{2n/(n-4)}(\Omega)$, Differential Integral Equations 2 (2) (1993) 259–270.