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Équations aux dérivées partielles/*Partial Differential Equations*

Nonlinear elliptic equations with critical Sobolev exponent in nearly starshaped domains

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Abstract Under suitable assumptions on Ω, we show that, for $\varepsilon > 0$ small and k large enough, problem (1) below has solutions which concentrate and blow-up as $\varepsilon \to 0$ at exactly k points; the blowing-up points approach $\partial\Omega$ as $k \to \infty$; the number of solutions tends to infinity as $\varepsilon \to 0$. These assumptions allow Ω to be contractible and even arbitrarily close to starshaped domains. ***To cite this article: R. Molle, D. Passaseo, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1029–1032.***

Équations elliptiques non linéaire avec non-linéarité critique en ouverts presque étoilés

Résumé On montre que, si Ω satisfait certaines conditions, le problème (1) ci-dessous, pour $\varepsilon > 0$ suffisamment petit et k grand, admet des solutions qui pour $\varepsilon \to 0$ se concentrent et explosent exactement en k points ; les points de concentration s'approchent du bord de Ω quand $k \to \infty$; le nombre de solutions est arbitrairement grand pourvu que ε soit suffisamment petit. Parmi les ouverts bornés Ω qui satisfont ces conditions il y en a aussi de contractibles, qui peuvent même être arbitrairement proches de ouverts étoilés. ***Pour citer cet article : R. Molle, D. Passaseo, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1029–1032.***

Let us consider the problem

$$\begin{cases} -\Delta u = u^{(n+2)/(n-2)} - \varepsilon u & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 3$, and ε is a real parameter. It is well known that, as a consequence of the Pohozaev's identity (see [15]), there exists no solution if Ω is starshaped and $\varepsilon \geqslant 0$.

For $\varepsilon = 0$, the existence of solutions is proved (see [1]) in domains with nontrivial topology (in the sense that suitable homology groups are nontrivial). Notice that this nontriviality condition is only sufficient for

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the existence of solutions but not necessary since existence results hold also in some contractible domains (see [5,7,12]).

The case $\varepsilon < 0$ has been firstly considered in [3]; if $n \geqslant 4$, for any Ω (even starshaped) it is proved the existence of solutions for all $\varepsilon \in]-\lambda_1, 0[$, where λ_1 denotes the first eigenvalue of $-\Delta$ in $\mathrm{H}_0^1(\Omega)$ (if $n = 3$ the problem is more complex). When $\varepsilon \to 0$, these solutions tend to concentrate as Dirac masses at special points of Ω (see [4,8,17]). Exploiting this concentration phenomena, it is possible to relate the number of solutions to the topology of Ω, when $\varepsilon < 0$ is small enough. For example, if $n \geqslant 5$, the existence of at least as many solutions as the Ljusternik–Schnirelmann category of Ω is proved in [16] (an improved multiplicity result, which holds also if $n = 4$, is obtained in [13]).

In this Note we are concerned with the case $\varepsilon > 0$. We give sufficient conditions on Ω, which guarantee that the following property holds: for k large and $\varepsilon > 0$ small enough, problem (1) has solutions which concentrate and blow-up at exactly k points as $\varepsilon \to 0$. Thus, in domains satisfying these conditions, the number of geometrically distinct solutions tends to infinity as $\varepsilon \to 0$ from above (while the problem may have no solution for $\varepsilon = 0$). Let us point out that these results hold also in bounded contractible domains, which (unlike the case considered in [5,7,12]) are not required to be close to nontrivial domains; indeed they may be even arbitrarily close to starshaped domains in the sense specified below.

For any smooth bounded domain Ω of $\mathbb{R}^n$, let us set

$$\sigma(\Omega) = \sup_{x_0 \in \Omega} \inf \left\{ \nu(x) \cdot \frac{x - x_0}{|x - x_0|} : x \in \partial\Omega \right\}, \tag{2}$$

where $\nu(x)$ denotes the outward normal to $\partial\Omega$. It is natural to say that Ω is a “nearly starshaped” domain if $\sigma(\Omega)^- = \max\{0, -\sigma(\Omega)\}$ is small (a different definition of nearly starshaped domain is used in [6]).

The results we present in this Note prove, in particular, the following proposition (see Example 1).

PROPOSITION 1. – *For any $\mu > 0$ there exists a smooth bounded domain Ω such that $\sigma(\Omega) \in]-\mu, 0[$ and problem* (1) *has solutions for $\varepsilon > 0$ small enough. Moreover, the number of geometrically distinct solutions tends to infinity as $\varepsilon \to 0$.*

In order to prove this proposition, we consider domains satisfying the following conditions

$$(x_1, x_2, \ldots, x_n) \in \Omega \iff \left(\sqrt{x_1^2 + x_2^2}, 0, x_3, \ldots, x_n\right) \in \Omega, \tag{3}$$

$$(x_1, \ldots, x_i, \ldots, x_n) \in \Omega \iff (x_1, \ldots, -x_i, \ldots, x_n) \in \Omega \quad \text{for } i = 3, \ldots, n-1 \tag{4}$$

and, exploiting these symmetry properties, we look for solutions of the form

$$u_{k,\varepsilon}(x) = \left[n(n-2)\right]^{(n-2)/4} \sum_{i=1}^{k} \frac{\mu_{k,\varepsilon}^{(n-2)/2}}{(\mu_{k,\varepsilon}^2 + |x - \xi_{i,k,\varepsilon}|^2)^{(n-2)/2}} + \theta_{k,\varepsilon}(x), \tag{5}$$

where $\theta_{k,\varepsilon} \to 0$ as $\varepsilon \to 0$, $\mu_{k,\varepsilon} > 0$ is a concentration parameter and the concentration points $\xi_{i,k,\varepsilon}$ have the form

$$\xi_{i,k,\varepsilon} = \left(\rho_{k,\varepsilon} \cos(2\pi/k)i, \rho_{k,\varepsilon} \sin(2\pi/k)i, 0, \ldots, 0, \tau_{k,\varepsilon}\right) \quad \text{for } i = 1, \ldots, k. \tag{6}$$

The following theorems are proved in [10].

THEOREM 1. – *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geqslant 5$, satisfying conditions* (3) *and* (4). *Assume that there exist ρ_1, ρ_2, ρ_3 and τ_1, τ_2, τ_3 in $\mathbb{R}$ such that $\tau_1 < \tau_2 < \tau_3$, $\max\{\rho_1, \rho_3\} < \rho_2$, Ω contains $(\rho_1, 0, \ldots, 0, \tau_1)$ and $(\rho_3, 0, \ldots, 0, \tau_3)$ while $(\rho_2, 0, \ldots, 0, \tau_2) \notin \Omega$. Also assume that there exists a continuous function $\gamma : [\tau_1, \tau_3] \to \mathbb{R}^+$ such that $\gamma(\tau_1) = \rho_1$, $\gamma(\tau_3) = \rho_3$, $\gamma(\tau_2) > \rho_2$ and $(\gamma(\tau), 0, \ldots, 0, \tau) \in \Omega$ $\forall \tau \in [\tau_1, \tau_3]$. Then there exist $\bar{k} \in \mathbb{N}$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0$ for all $k \geqslant \bar{k}$, such that, for all $k \geqslant \bar{k}$ and $\varepsilon \in]0, \varepsilon_k]$, problem* (1) *has at least one solution of the form* (5). *As $\varepsilon \to 0$*

and $k \to \infty$, this solution behaves as follows: $\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \operatorname{dist}(\xi_{i,k,\varepsilon}, \partial\Omega) = 0$ *for* $i = 1, \ldots, k$ *and* $\lim_{\varepsilon\to 0} \mu_{k,\varepsilon}\varepsilon^{1/(4-n)} = \lambda_k > 0$ $\forall k \geqslant \bar{k}$, *with* $\lim_{k\to\infty} \lambda_k = 0$.

THEOREM 2. – *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geqslant 5$, satisfying conditions* (3) *and* (4). *Let us set*

$$S(\Omega) = \left\{(\rho, \tau) \in \mathbb{R}^2 : \rho > 0, (\rho, 0, \ldots, 0, \tau) \in \Omega\right\} \tag{7}$$

and consider the function $\Pi_\Omega : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Pi_\Omega(\rho, \tau) = \rho \quad \text{if } (\rho, \tau) \in \overline{S(\Omega)}, \qquad \Pi_\Omega(\rho, \tau) = +\infty \quad \text{otherwise}. \tag{8}$$

Assume that there exists an open subset $A \subset \mathbb{R}^2$ such that $0 < \inf_A \Pi_\Omega < \inf_{\partial A} \Pi_\Omega$. Then the same conclusion of Theorem 1 *holds*; *moreover,* $\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \operatorname{dist}((\rho_{k,\varepsilon}, \tau_{k,\varepsilon}), M_A) = 0$, *where M_A is the set of the minimum points for Π_Ω constrained on A (notice that $M_A \subset \partial S(\Omega) \cap A$).*

The proof of Theorems 1 and 2 is based on a finite dimensional reduction method introduced in [2] and [17] (see also [9,11] and references therein).

Let us consider the function $\Psi_k(\rho, \tau, \lambda) = \Phi_k(\rho, \tau)\lambda^{n-2} + k\lambda^2$, with

$$\Phi_k(\rho, \tau) = \sum_{i=1}^{k} H(\xi_{i,k}, \xi_{i,k}) - 2 \sum_{1\leqslant i<j\leqslant k} G(\xi_{i,k}, \xi_{j,k}), \tag{9}$$

where $\xi_{i,k} = (\rho\cos(2\pi/k)i, \rho\sin(2\pi/k)i, 0, \ldots, 0, \tau) \in \Omega$, G denotes the Green's function of $-\Delta$ in $\mathrm{H}^1_0(\Omega)$ and H its regular part. Taking into account the symmetry properties (3) and (4), the problem reduces to finding critical points (ρ, τ, λ) for Ψ_k, with $\lambda > 0$, which persist with respect to small C^1 perturbations. Clearly, it is equivalent to finding critical points (ρ, τ) for Φ_k, with $\Phi_k(\rho, \tau) < 0$, which are stable with respect to C^1 perturbations.

The following lemma (see [9]) plays a crucial role in the proof of Theorems 1 and 2.

LEMMA 1. – *There exists a sequence $(c_k)_k$ in $\mathbb{R}$, $c_k \to +\infty$, such that*

$$\frac{1}{c_k}\Phi_k(\rho, \tau) \geqslant -\rho^{2-n} \quad \forall(\rho, \tau) \in S(\Omega),\ \forall k \in \mathbb{N} \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{c_k}\Phi_k(\rho, \tau) = -\rho^{2-n} \quad \forall(\rho, \tau) \in S(\Omega).$$

Moreover, $\Phi_k(\rho, \tau) \to +\infty$ as $(\rho, \tau) \to (\hat{\rho}, \hat{\tau})$, for all $(\hat{\rho}, \hat{\tau}) \in \partial S(\Omega)$ such that $\hat{\rho} > 0$.

These properties of the function Φ_k allow us to say that, if the assumptions of Theorem 2 are satisfied, for k large enough there exists a minimum point for Φ_k constrained on $A \cap S(\Omega)$ while, under the assumptions of Theorem 1, a critical point for Φ_k can be obtained by a mini-max argument. In both cases the critical points (ρ_k, τ_k) we get for Φ_k persist with respect to small C^1 perturbations; moreover, for k large enough, they correspond to negative critical values (indeed, $\lim_{k\to\infty} \Phi_k(\rho_k, \tau_k) = -\infty$). So they give rise to solutions of the form (5) with $\mu_{k,\varepsilon}$ satisfying $\lim_{\varepsilon\to 0} \mu_{k,\varepsilon}\varepsilon^{1/(4-n)} = a_n[-\frac{1}{k}\Phi_k(\rho_k, \tau_k)]^{1/(4-n)}$, where a_n is a positive constant depending only on the dimension n.

Remark 1. – The proof shows also that for the solution obtained under the assumptions of Theorem 2 we have $\lim_{k\to\infty} \frac{1}{c_k}\Phi_k(\rho_k, \tau_k) = -[\min_A \Pi_\Omega]^{2-n}$, while the solution given by Theorem 1 satisfies $\lim_{k\to\infty} \frac{1}{c_k}\Phi_k(\rho_k, \tau_k) \geqslant -\rho_2^{2-n}$.

Example 1. – For all $r > 1$, $s > 0$ and $\delta > 0$, let us consider the domain $\Omega^\delta_{r,s} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega_{r,s}) < \delta\}$, where $\Omega_{r,s} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\ :\ 1 < |x| < r,\ (\sum_{i=1}^{n-1} x_i^2)^{1/2} > sx_n\}$. If $\delta < s(1+s^2)^{-1/2}$, then $\Omega^\delta_{r,s}$ is a contractible smooth bounded domain of $\mathbb{R}^n$; moreover one can verify that $\lim_{r,s\to\infty} \sigma(\Omega^\delta_{r,s}) = 0$ for any $\delta \in]0, 1[$ (note that $\delta < s(1+s^2)^{-1/2}$ for s large enough). Thus, in order to prove Proposition 1, it suffices to observe that, if $n \geqslant 5$, both Theorems 1 and 2 apply when $\Omega = \Omega^\delta_{r,s}$ and guarantee the existence of two k-spike solutions of problem (1) for k large and $\varepsilon > 0$ small enough. Notice that we have indeed two distinct k-spike solutions in $\Omega^{\delta}_{r,s}$ because (see Remark 1) the solution given by Theorem 2 satisfies $\lim_{k\to\infty} \frac{1}{c_k}\Phi(\rho_k, \tau_k) = -[s(1+s^2)^{-1/2} - \delta]^{2-n}$, while for the solution given by Theorem 1 we have $\lim_{k\to\infty} \frac{1}{c_k}\Phi(\rho_k, \tau_k) = -[1-\delta]^{2-n}$.

Remark 2. – Solutions which blow-up as $\varepsilon \to 0$ can be obtained also if $n = 4$; in this case the concentration parameter μ_ε satisfies $\lim_{\varepsilon\to 0} \mu_\varepsilon \exp(a/\varepsilon) = b$, where a and b are suitable positive constants. On the contrary, for $n = 3$ similar concentration phenomena do not occur (at least not when $\varepsilon \to 0$).

Remark 3. – Notice that condition (4) is not really necessary for the construction of multispike solutions of this type. In fact, if we assume only condition (3) and set $\Sigma(\Omega) = \{(\rho, \tau_1, \ldots, \tau_{n-2}) \in \mathbb{R}^{n-1} : \rho > 0, (0, \rho, \tau_1, \ldots, \tau_{n-2}) \in \Omega\}$, then a general result (reported in [10]) relates the existence of k-spike solutions, for k large and $\varepsilon > 0$ small enough, to the presence of suitable critical points of the function $\mathcal{E}(\rho, \tau_1, \ldots, \tau_{n-2}) = -\rho^{n-2}$ constrained on $\overline{\Sigma(\Omega)}$.

Remark 4. – If we replace the parameter ε in problem (1) with a variable coefficient $a(x)$, then Pohozaev's identity does not give contradiction and the problem may have solutions even if Ω is a starshaped domain and $a(x) \geqslant 0$ for all $x \in \Omega$. In [14] it is proved that, if $a(x)$ concentrates at a finite number of points of Ω, then (independently of the shape of Ω) there exist solutions which concentrate and blow-up at the same points.

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