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Homological Algebra

Mixed characteristic homological theorems in low degrees

Théorèmes homologiques en petits degrés dans le cas de caractéristique mixte

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Abstract

Let R be a locally finitely generated algebra over a discrete valuation ring V of mixed characteristic. For any of the homological properties, the Direct Summand Theorem, the Monomial Theorem, the Improved New Intersection Theorem, the Vanishing of Maps of Tors and the Hochster–Roberts Theorem, we show that it holds for R and possibly some other data defined over R , provided the residual characteristic of V is sufficiently large in terms of the complexity of the data, where the complexity is primarily given in terms of the degrees of the polynomials over V that define the data, but possibly also by some additional invariants. **To cite this article:** *H. Schoutens, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Résumé

Soit \mathcal{w} un ensemble fini d'objets algébriques (comme des anneaux, des modules, etc.) de type fini sur un anneau de valuation discrète V en caractéristique mixte. Pour plusieurs propriétés homologiques, nous montrons que la propriété est satisfaite par \mathcal{w} , pourvu que la caractéristique résiduelle de V soit supérieure à une borne qui ne dépend que de la complexité de \mathcal{w} , où la complexité est déterminée notamment par les degrés des polynômes définissant \mathcal{w} . **Pour citer cet article :** *H. Schoutens, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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1. The results

Let V be a mixed characteristic discrete valuation ring with uniformizing parameter π and residue field κ of characteristic p . We say that R is a *local V -affine algebra* of V -complexity at most c , if it is of the form $(V[X]/I)_{\mathfrak{m}}$, with X a tuple of at most c variables, I and \mathfrak{m} ideals generated by polynomials of degree at most c , and \mathfrak{m} a prime

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ideal containing I and π . Similarly, we say that an element a in R (respectively, a tuple \mathbf{x} in R ; a matrix Γ defined over R ; an ideal I in R ; a finitely generated R -module M ; or, a V -algebra homomorphism $R \rightarrow S$) has V -complexity at most c , if R has V -complexity at most c and a is the image in R of a fraction f/g with f and g polynomials of degree at most c and $g \notin \mathfrak{m}$ (respectively, the length of \mathbf{x} is at most c and each of its entries has V -complexity at most c ; the dimensions of Γ are at most c and each of its entries has V -complexity at most c ; the ideal I is generated by elements of V -complexity at most c ; the module M can be realized as the cokernel of a matrix of V -complexity at most c ; and, $R \rightarrow S$ is given by sending generators of R to elements in S of V -complexity at most c).

Theorem 1.1 (Asymptotic Homological \mathcal{P} -Theorem). *Let \mathcal{P} be one of the five homological properties about algebraic data ϖ listed below. For each $c \in \mathbb{N}$, there exists a bound $c' \in \mathbb{N}$, with the property that if V is a mixed characteristic discrete valuation ring such that the data ϖ have V -complexity at most c , and if the residual characteristic of V is at least c' , then property \mathcal{P} holds for ϖ .*

Direct Summand Theorem. *The data ϖ consists of a finite, injective V -algebra homomorphism $R \rightarrow S$ with R regular.*

Under these assumptions, \mathcal{P} proclaims that $R \rightarrow S$ splits as an R -module morphism.

Monomial Theorem. *The data ϖ consist of a system of parameters \mathbf{x} in a V -algebra R , such that $\mathbf{x}R \cap V$ has V -adic valuation at most c , and of c monomials Y^{μ_i} in at most c variables Y such that Y^{μ_0} does not belong to the ideal in $\mathbb{Z}[Y]$ generated by the remaining monomials Y^{μ_i} .*

Under these assumptions, \mathcal{P} proclaims that \mathbf{x}^{μ_0} does not belong to the ideal in R generated by the remaining \mathbf{x}^{μ_i} .

Improved New Intersection Theorem. *The data ϖ consist of a finite free complex*

$$0 \rightarrow R^{a_s} \xrightarrow{\Gamma_s} R^{a_{s-1}} \xrightarrow{\Gamma_{s-1}} \dots \xrightarrow{\Gamma_2} R^{a_1} \xrightarrow{\Gamma_1} R^{a_0} \rightarrow 0 \quad (F_\bullet)$$

over a d -dimensional V -algebra R with $s, a_i \leq c$, and of a minimal generator τ of $H_0(F_\bullet)$ generating a module of length at most c , such that each $R/I_{r_i}(\Gamma_i)$ has dimension at most $d - i$ and parameter degree² at most c , where $r_i = a_i - a_{i+1} + \dots + (-1)^{s-i} a_s$ is the expected rank and $I_n(\Gamma)$, in general, denotes the ideal generated by all $n \times n$ -minors of a matrix Γ .

Under these assumptions, \mathcal{P} proclaims that F_\bullet has length at least d .

Vanishing for Maps of Tors. *The data ϖ consist of V -algebra homomorphisms $R \rightarrow S \rightarrow T$, and of a finitely generated R -module M , such that R and T are regular and $R \rightarrow S$ is integral and injective.*

Under these assumptions, \mathcal{P} proclaims that the natural map $\mathrm{Tor}_n^R(S, M) \rightarrow \mathrm{Tor}_n^R(T, M)$ is zero, for all $n \geq 1$.

Hochster–Roberts Theorem. *The data ϖ consist of a cyclically pure³ V -algebra homomorphism $R \rightarrow S$, such that S is regular.*

Under these assumptions, \mathcal{P} proclaims that R is Cohen–Macaulay.

² The parameter degree of a Noetherian local ring S is defined as the minimal possible length of a residue ring $S/\mathbf{x}S$, where \mathbf{x} runs over all systems of parameters of S (note that homological multiplicity is an upper bound for parameter degree by [15, §4]).

³ A homomorphism $R \rightarrow S$ is cyclically pure if $I = IS \cap R$, for every ideal I in R .

2. The method

If V is equicharacteristic, then each of these homological properties holds unconditionally, that is to say, without any bound on the complexity [3,8,19]. We will use the Ax–Kochen–Ershov Principle to deduce Theorem 1.1 from this. Let me sketch the idea before I give more details. After a faithfully flat extension, we may assume that V is moreover complete. Towards a contradiction, suppose for some c , no such bound exists. This means that for each p , we can find a complete discrete valuation ring V_p of characteristic zero and residual characteristic p , and some data ϖ_p of V_p -complexity at most c satisfying the above conditions, but for which \mathcal{P} fails. Let κ_p be the residue field of V_p . Define $V_p^{\text{eq}} := \kappa_p[[t]]$, for t a single variable. Using the Ax–Kochen–Ershov Principle, we can construct for each p , similar data ϖ_p^{eq} defined over the discrete valuation rings V_p^{eq} , so that for infinitely many p , property \mathcal{P} does not hold for ϖ_p^{eq} , leading to the desired contradiction.

I will now explain this in more detail. The relation between the discrete valuation rings V_p and V_p^{eq} is given by the following result due to Ax and Kochen [2] and Ershov [4,5] (to get an isomorphism for arbitrary non-principal ultrafilters, one needs to assume the continuum hypothesis; without this assumption, one might need to take further ultrapowers, that is to say, take a larger index set with a (non- ω -complete) non-principal ultrafilter).

Theorem 2.1 (Ax–Kochen–Ershov). *For a fixed choice of a non-principal ultrafilter on the set of prime numbers, the ultraproduct of all V_p is isomorphic to the ultraproduct of all V_p^{eq} .*

For a quick review on ultraproducts, including Łos’ Theorem, see [16, §2]; for a more detailed treatment, see [11]. Fix a non-principal ultrafilter on the set of prime numbers. Identify both ultraproducts via a fixed isomorphism and denote the common object by \mathfrak{D} . By Łos’ Theorem, \mathfrak{D} is an equicharacteristic zero Henselian (non-discrete, non-Noetherian) valuation ring with maximal ideal generated by a single element π . Fix a tuple of variables X . It is no longer true that the ultraproduct A_∞^{mix} of the $V_p[X]$ is isomorphic to the ultraproduct A_∞^{eq} of the $V_p^{\text{eq}}[X]$. Nonetheless, both ultraproducts contain $\mathfrak{D}[X]$ as a subring. More precisely, if $f_p \in V_p[X]$ have degree at most c , for some c independent from p , then their ultraproduct f_∞ in A_∞^{mix} is an element of the subring $\mathfrak{D}[X]$, and every element in $\mathfrak{D}[X]$ is realized in this manner. In particular, f_∞ can also be viewed as an element in A_∞^{eq} , that is to say, as the ultraproduct of elements $f_p^{\text{eq}} \in V_p^{\text{eq}}[X]$. In this way, we can associate to a sequence of elements f_p of uniformly bounded V_p -complexity, a sequence of elements f_p^{eq} of uniformly bounded V_p^{eq} -complexity. Although this assignment is not unique, any two choices will be the same almost everywhere (in the sense of the ultrafilter). Similarly, we can associate to a sequence of local V_p -affine algebras R_p of uniformly bounded complexity (or any other object defined in finite terms over V_p), a sequence of local V_p^{eq} -affine algebras R_p^{eq} ; the latter are called an equicharacteristic *approximation* of the former.

Let R_∞^{mix} and R_∞^{eq} be the respective ultraproducts of R_p and R_p^{eq} . These rings have a common local subring $(\mathfrak{R}, \mathfrak{m})$, essentially of finite presentation over \mathfrak{D} , consisting precisely of ultraproducts of elements of uniformly bounded complexity. Using for instance the result in [1] regarding uniform bounds on the complexity of modules of syzygies, one shows that both extensions $\mathfrak{R} \rightarrow R_\infty^{\text{mix}}$ and $\mathfrak{R} \rightarrow R_\infty^{\text{eq}}$ are faithfully flat (alternatively, this can be deduced from the fact that \mathfrak{R} is coherent). Moreover, using results from [12–14], every finitely generated prime ideal of \mathfrak{R} remains prime when extended to either R_∞^{mix} or R_∞^{eq} . It follows that almost all R_p are domains if, and only if, \mathfrak{R} is a domain if, and only if, almost all R_p^{eq} are domains.

The idea is to view \mathfrak{R} as an equicharacteristic zero version of the R_p (or, for that matter, of the R_p^{eq}), so that we are led to prove an analogue of the homological property \mathcal{P} for \mathfrak{R} (and whatever other data required, arising in a similar fashion from data of uniformly bounded V_p -complexity). However, in carrying out this project, we are faced with a serious obstruction: \mathfrak{R} is in general not Noetherian but only coherent. This prompts for a non-Noetherian version of the local algebra required for discussing homological properties. To this end, we define the *pseudo-dimension* of \mathfrak{R} to be the smallest length of a tuple generating an \mathfrak{m} -primary ideal (note that the Krull dimension is infinite and hence of no use). We say that \mathfrak{R} is *pseudo-regular* if its pseudo-dimension equals its embedding dimension (= the minimal number of generators of \mathfrak{m}), and *pseudo-Cohen–Macaulay*, if its pseudo-dimension is

equal to its depth (in the sense of [6]). To derive for instance the asymptotic Hochster–Roberts Theorem, we can now use the fact that almost all R_p are regular (respectively, Cohen–Macaulay) if, and only if, \mathfrak{R} is pseudo-regular (respectively, pseudo-Cohen–Macaulay) if, and only if, almost all R_p^{eq} are regular (respectively, Cohen–Macaulay).

The main tool in establishing a variant of each \mathcal{P} over \mathfrak{D} is via an \mathfrak{D} -analogue of a big Cohen–Macaulay algebra. Hochster has demonstrated (see, for instance, [7,8]) how efficiently big Cohen–Macaulay modules can be used to prove homological theorems. More recently, Hochster and Huneke have given various strengthenings and generalizations using big Cohen–Macaulay *algebras*. Big Cohen–Macaulay algebras in equicharacteristic zero are obtained by reduction modulo p from their existence in positive characteristic via absolute integral closures [9,10]. In [18], I gave an alternative construction for local \mathbb{C} -affine domains, using ultraproducts, and it is this approach we will adopt here. Namely, for \mathfrak{R} a local \mathfrak{D} -affine domain, let $\mathcal{B}(\mathfrak{R})$ be the ultraproduct of the absolute integral closures $(R_p^{\text{eq}})^+$.

Theorem 2.2 (Big Cohen–Macaulay Algebra). *Let $(\mathfrak{R}, \mathfrak{m})$ be a local \mathfrak{D} -affine domain. Every tuple in \mathfrak{R} of length equal to the pseudo-dimension of \mathfrak{R} and generating an \mathfrak{m} -primary ideal, is $\mathcal{B}(\mathfrak{R})$ -regular.*

Proof. Let \mathbf{x} be a tuple of length equal to the pseudo-dimension d of \mathfrak{R} so that $\mathbf{x}\mathfrak{R}$ is \mathfrak{m} -primary. Choose d -tuples \mathbf{x}_p^{eq} in R_p^{eq} whose ultraproduct is \mathbf{x} . One can show that almost all R_p^{eq} have dimension d . By Łos’ Theorem, almost all $\mathbf{x}_p^{\text{eq}} R_p^{\text{eq}}$ are primary to the maximal ideal. Hence almost all \mathbf{x}_p^{eq} are systems of parameters, whence $(R_p^{\text{eq}})^+$ -regular by [9]. By another application of Łos’ Theorem, \mathbf{x} is $\mathcal{B}(\mathfrak{R})$ -regular. \square

Details can be found in the forthcoming [17].

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