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On the convergence at infinity of the Leray solution of the two-dimensional Navier–Stokes equations to the prescribed asymptotic value

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Abstract

In this Note we prove that $\underset{\sim}{v}_L$, the Leray velocity solution to the steady incompressible, two-dimensional Navier–Stokes equations, tends at infinity to the prescribed vector $\underset{\sim}{v}_\infty$. We show also that the sequence $(\underset{\sim}{v}_{R_i}, p_{R_i})$ of Leray solutions to the same boundary value problem in the bounded domains $\Omega_{R_i},\ i \in \mathbb{N}$, converges quasi-uniformly in $\overline{\Omega}$ to $(\underset{\sim}{v}_L, p_L)$. ***To cite this article: D. Socolescu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Sur la convergence à l'infini de la solution de Leray des équations bidimensionnelles de Navier–Stokes vers la valeur asymptotique imposée. Dans cette Note on prouve que $\underset{\sim}{v}_L$, la solution vitesse de Leray des équations stationnaires, incompressibles, bidimensionnelles de Navier–Stokes, tend à l'infini vers le vecteur imposé $\underset{\sim}{v}_\infty$. On montre aussi que la suite $(\underset{\sim}{v}_{R_i}, p_{R_i})$ de solutions de Leray du même problème aux limites dans les domaines bornés $\Omega_{R_i},\ i \in \mathbb{N}$, converge quasi-uniformément dans $\overline{\Omega}$ vers $(\underset{\sim}{v}_L, p_L)$. ***Pour citer cet article : D. Socolescu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

L'écoulement stationnaire bidimensionnel d'un fluide visqueux incompressible autour d'un obstacle D de frontière Γ conduit au problème de Dirichlet (1), (2) et (3) dans $\Omega = \mathbb{R}^2 \setminus D$.

En 1933 Leray [5] construisit une solution $(\underset{\sim}{v}_L, p_L)$ de (1) et (2) et ayant une intégrale de Dirichlet de la vitesse bornée (4).

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Le comportement asymptotique de $(\underset{\sim}{v}_L, p_L)$ et d'une solution arbitraire $(\underset{\sim}{v}_D, p_D)$ de (1), (2) et (4) fut étudié en 1974 et respectivement en 1978 par Gilbarg et Weinberger [3,4] et ensuite par Amick en 1988 [1] et par Socolescu en 2000 et 2002 [8,9]. En fait ils prouvent que : (i) chaque solution $(\underset{\sim}{v}_D, p_D)$ et en particulier $(\underset{\sim}{v}_L, p_L)$ est bornée [1,3,4] ; (ii) $(\underset{\sim}{v}_D, p_D)$ converge à l'infini vers $(\underset{\sim}{v}_0, p_\infty)$, ou $|\underset{\sim}{v}_0| = \lim_{r\to\infty} \max_{[0,2\pi]} |\underset{\sim}{v}(r,\theta)|$, [1,4,8] ; (iii) la solution $(\underset{\sim}{v}_D, p_D)$ convergeant à l'infini vers $(\underset{\sim}{v}_0, p_\infty)$ est unique, à condition que $\nu \geqslant \nu_{\mathrm{cr}}$, [9].

Dans la présente Note nous obtenons le résultat suivant :

Théorème 1. *La solution de Leray* $(\underset{\sim}{v}_L, p_L)$ *de* (1), (2) *et* (3), *tend ponctuellement à l'infini vers* $(\underset{\sim}{v}_\infty, p_\infty)$. *La suite* $(\underset{\sim}{v}_{R_i}, p_{R_i})$ *de solutions de Leray de* (1) *et* (5) *converge quasi-uniformément dans* $\overline{\Omega}$ *vers* $(\underset{\sim}{v}_L, p_L)$.

1. Introduction

Let us consider the exterior Dirichlet problem for the steady, two-dimensional, incompressible Navier–Stokes equations:

$$\Omega : \nu \triangle \underset{\sim}{v} - (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v} - \nabla p = \underset{\sim}{0}, \qquad \nabla\cdot\underset{\sim}{v} = 0, \tag{1}$$

$$\Gamma : \underset{\sim}{v} = \underset{\sim}{0}, \tag{2}$$

$$[0, 2\pi) : \lim_{r\to\infty} \underset{\sim}{v}(r,\theta) = \underset{\sim}{v}_\infty, \tag{3}$$

where $\Omega = \mathbb{R}^2 \setminus D$, D is a compact set with smooth boundary Γ, r and θ are polar coordinates taken with respect to an origin interior to D.

In 1933 Leray [5] constructed a certain solution $(\underset{\sim}{v}_L, p_L)$ satisfying (1) and (2) and having a velocity with finite Dirichlet integral

$$\iint_\Omega |\nabla \underset{\sim}{v}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant C\big(1+\nu^{-1}\big), \tag{4}$$

where C depends on Γ and $\underset{\sim}{v}_\infty$, but not on ν. Leray's construction went as follows. Let Ω_R be the set of points in Ω of radius $r < R$. He first proved that for every $R > \max_D r =: r_D \geqslant 1$ and every constant vector $\underset{\sim}{v}_\infty$ there is at least one solution $(\underset{\sim}{v}_R, p_R)$ of (1) in Ω_R satisfying the following boundary conditions

$$\underset{\sim}{v}_R = \begin{cases} \underset{\sim}{0} & \text{on } \partial D = \Gamma, \\ \underset{\sim}{v}_\infty & \text{on } r = R. \end{cases} \tag{5}$$

Concerning all such (velocity) solutions $\underset{\sim}{v}_R$ Leray proved the existence of a uniform bound for the Dirichlet integral (4), where C depends on Γ and $\underset{\sim}{v}_\infty$, but is independent of R and ν. He then showed that a sequence $R_i \to \infty$ exists, such that the solutions $(\underset{\sim}{v}_{R_i}, p_{R_i})$ of (1) and (5) in Ω_{R_i} converge uniformly together with all their first order derivatives in any compact subset of $\overline{\Omega}$ to a solution $(\underset{\sim}{v}_L, p_L)$ satisfying (1), (2) and (4). The further behaviour of $\underset{\sim}{v}_L$ and p_L as $r \to \infty$ was left unsettled.

The asymptotic behaviour of $(\underset{\sim}{v}_L, p_L)$ and of an arbitrary solution $(\underset{\sim}{v}_D, p_D)$ with finite Dirichlet integral of (1) and (2) was investigated in 1974 and 1978 by Gilbarg and Weinberger [3,4], by Amick in 1988 [1], and by Socolescu in 2000 and 2002 [8,9]. In fact they proved that (i) every solution $(\underset{\sim}{v}_D, p_D)$ and particularly $(\underset{\sim}{v}_L, p_L)$ is bounded, [1,3,4]; (ii) $(\underset{\sim}{v}_D, p_D)$ converges at infinity to $(\underset{\sim}{v}_0, p_\infty)$, where $|\underset{\sim}{v}_0| = \lim_{r\to\infty} \max_{[0,2\pi]} |\underset{\sim}{v}(r,\theta)|$, [1,4,8]; (iii) the solution $(\underset{\sim}{v}_D, p_D)$ tending at infinity to $(\underset{\sim}{v}_0, p_\infty)$ is a unique one, provided $\nu \geqslant \nu_{\mathrm{cr}}$, [9].

The present Note is concerned with the convergence at infinity of $\underset{\sim}{v}_L$ to the prescribed vector $\underset{\sim}{v}_\infty$.

2. Convergence at infinity of the Leray solution

Theorem 1. *The Leray solution* $(\underset{\sim}{v}_L, p_L)$ *of* (1), (2) *and* (3) *tends pointwise at infinity to* $(\underset{\sim}{v}_\infty, p_\infty)$. *The Leray sequence* $(\underset{\sim}{v}_{R_i}, p_{R_i})$ *of solutions of* (1) *and* (5), *converges quasi-uniformly in* $\overline{\Omega}$ *to* $(\underset{\sim}{v}_L, p_L)$.

For the proof we need the following results:

Definition of quasi-uniform convergence [7, p. 66]. Let X and Y be metric spaces and let $f_n,\ n \in \mathbb{N}$, map X into Y. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to converge quasi-uniformly on X to $f : X \to Y$ if (i) $\{f_n\}$ converges pointwise to f; (ii) for every $\varepsilon > 0$ there exists a finite or infinite, increasing sequence $\{n_p\}_{p\in\mathbb{N}} \subset N$ and a sequence $\{D_p\}_{p\in\mathbb{N}}$ of open sets $D_p \subset X,\ X = \bigcup_{p=1}^{\infty} D_p$, such that $\operatorname{dist}_Y(f(x), f_{n_p}(x)) < \varepsilon,\ p \in \mathbb{N},\ x \in D_p$.

Theorem of Amick [1, p. 114]. *The velocity solution* $\underset{\sim}{v}_L$ *is non-trivial iff* $\lim_{R_i\to\infty} \inf \iint_{\Omega_{R_i}} |\nabla \underset{\sim}{v}_{R_i}|^2 \,\mathrm{d}x\,\mathrm{d}y > 0$.

Theorem of Arzelá, Gagaeff and Alexandrov [7, p. 68]. *Let* X, Y *be metric spaces and let* $f_n,\ n \in \mathbb{N}$, *map* X *into* Y *continuously. The sequence* $\{f_n\}$ *converges on* X *to a continuous map* $f : X \to Y$, *iff the convergence is quasi-uniform.*

Proof of Theorem 1. Let $\underset{\sim}{v}_\infty \neq \underset{\sim}{0}$. We have first to show that $\underset{\sim}{v}_0 = \underset{\sim}{v}_\infty$. To this end we note that (1) can be given the equivalent form

$$\Omega : \nabla H = -\nu\, \nabla^\perp \omega + \omega\, \underset{\sim}{v}^\perp, \qquad \nabla \cdot \underset{\sim}{v} = 0, \tag{6}$$

where $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$, $H := \frac{1}{2}|\underset{\sim}{v}|^2 + p$ is the Helmholtz–Bernoulli function, $\omega = -\nabla^\perp \cdot \underset{\sim}{v}$ is the vorticity function, $\underset{\sim}{v}^\perp := (-v, u)$. By multiplying (1), written for $\underset{\sim}{v}_R$, by $\underset{\sim}{v}_R$, integrating over Ω_R, and using the Gauss–Green theorem and (5) we get on one hand

$$\nu \iint_{\Omega_R} |\nabla \underset{\sim}{v}_R|^2 \,\mathrm{d}x\,\mathrm{d}y = \nu \iint_{\Omega_R} \left[\triangle\left(\frac{\underset{\sim}{v}_R^{\,2}}{2}\right) - \frac{1}{\nu}\nabla \cdot (H_R\, \underset{\sim}{v}_R)\right] \mathrm{d}x\,\mathrm{d}y = 2\pi\nu R\, \underset{\sim}{v}_\infty \cdot \left[\frac{\mathrm{d}\,\hat{\underset{\sim}{v}}_R}{\mathrm{d}R} - \frac{1}{\nu}\big(\widehat{H}_R^{c1}, \widehat{H}_R^{s1}\big)\right], \tag{7}$$

where by

$$(r_D, \infty) : \hat{f}(r)^{cn(sn)} = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) \cos n\theta (\sin n\theta)\,\mathrm{d}\theta, \quad n \in \mathbb{N} \cup \{0\},\ \hat{f} = \hat{f}^{c0}, \tag{8}$$

we denote the Fourier coeficients of the periodic, smooth function $f(r,\theta)$. On the other hand $(6)_1$ can be given in polar coordinates the equivalent form

$$\Omega : \begin{cases} \dfrac{\partial H}{\partial r} = \dfrac{\nu}{r}\dfrac{\partial \omega}{\partial \theta} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 F) + \dfrac{1}{r}\dfrac{\partial G}{\partial \theta}, \\[2mm] \dfrac{1}{r}\dfrac{\partial H}{\partial \theta} = -\nu\dfrac{\partial \omega}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 G) - \dfrac{1}{r}\dfrac{\partial F}{\partial \theta}, \end{cases} \tag{9}$$

where $F(r,\theta) = \frac{1}{2}[(u\sin\theta - v\cos\theta)^2 - (u\cos\theta + v\sin\theta)^2]$, $G(r,\theta) = (u\sin\theta - v\cos\theta)(u\cos\theta + v\sin\theta)$.

By multiplying (9) by $\cos\theta(\sin\theta)$ and $\sin\theta(\cos\theta)$ and integrating we get

$$(r_D,\infty):\begin{cases}\widehat{H}^{c1}(r)=\widehat{F}^{c1}(r)-\widehat{G}^{s1}(r)-\dfrac{C_1}{2\pi r}+\nu\widehat{\omega}^{s1}(r),\\ \widehat{H}^{s1}(r)=\widehat{F}^{s1}(r)+\widehat{G}^{c1}(r)-\dfrac{C_2}{2\pi r}-\nu\widehat{\omega}^{c1}(r),\end{cases}\tag{10}$$

where $\widehat{\omega}^{s1}(r)=\frac{\mathrm{d}\hat{u}}{\mathrm{d}r}$, $\widehat{\omega}^{c1}(r)=-\frac{\mathrm{d}\hat{v}}{\mathrm{d}r}$. Using (10), from (7) it follows then

$$\nu\iint\limits_{\Omega_R}|\nabla \underset{\sim}{v}_R|^2\,\mathrm{d}x\,\mathrm{d}y=(C_1,C_2)\cdot\underset{\sim}{v}_\infty>0,\tag{11}$$

where C_1 and C_2 are independent of R. Letting $R\to\infty$, from (11) we obtain

$$\lim\inf_{R_i\to\infty}\iint\limits_{\Omega_{R_i}}|\nabla \underset{\sim}{v}_{R_i}|^2\,\mathrm{d}x\,\mathrm{d}y=(C_1,C_2)\cdot\underset{\sim}{v}_\infty>0,\tag{12}$$

and the theorem of Amick and the Liouville principle – see the Appendix – imply that $\underset{\sim}{v}_L$ is non-trivial and that $\underset{\sim}{v}_0\neq\underset{\sim}{0}$. Let now $r\in\,]r_D,\infty[$. Using the representation (21) for w_L in $\Omega\setminus\Omega_r$ and for w_{R_i} in $\Omega_{R_i}\setminus\Omega_r$ we get

$$\begin{aligned}\Omega_{R_i}\setminus\Omega_r:(w_L-w_{R_i})(z,\bar z)&=w_0-\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta|=r}\frac{w_L(\zeta,\bar\zeta)-w_{R_i}(\zeta,\bar\zeta)}{\zeta-z}\,\mathrm{d}\zeta\\&\quad-w_\infty+\frac{1}{2\pi\mathrm{i}}\left[PV\iint\limits_{\Omega\setminus\Omega_r}\frac{\omega_L(\zeta,\bar\zeta)}{\zeta-z}\,\mathrm{d}\xi\,\mathrm{d}\eta-PV\iint\limits_{\Omega_{R_i}\setminus\Omega_r}\frac{\omega_{R_i}(\zeta,\bar\zeta)}{\zeta-z}\,\mathrm{d}\xi\,\mathrm{d}\eta\right].\end{aligned}\tag{13}$$

Choosing now, for $\varepsilon>0$, r and R_i appropriately, i.e., big enough, we get

$$\Omega_{R_i}\setminus\Omega_r:w_\infty-w_0=0(\varepsilon),\tag{14}$$

and hence, the first assertion of the theorem follows.

For the sake of completeness let us note that (12) still holds for $\underset{\sim}{v}_L$. To this end we use the same argument as for (11), by taking into account, in the formula analogous to (7), the asymptotic behaviour of $(\underset{\sim}{v}_L,p_L)$ [2] as well as the fact that (10) still holds for $(\underset{\sim}{v}_L,p_L)$.

It remains only to show the quasi-uniform convergence. To this end we define

$$\overline{\Omega}:\underset{\sim}{\overset{e}{v}}_{R_i}=\begin{cases}\underset{\sim}{v}_{R_i} & \text{in }\overline{\Omega}_{R_i},\\ \underset{\sim}{v}_\infty & \text{in }\overline{\Omega}\setminus\overline{\Omega}_{R_i},\end{cases}\qquad \underset{\sim}{\overset{e}{v}}_L=\begin{cases}\underset{\sim}{v}_L & \text{in }\Omega\cup\Gamma,\\ \underset{\sim}{v}_\infty & \text{at infinity.}\end{cases}\tag{15}$$

We note on one hand that $\underset{\sim}{\overset{e}{v}}_{R_i}$, and $\underset{\sim}{\overset{e}{v}}_L$ are continuous on $\overline{\Omega}$. On the other hand, by using the stereographic projection the extended plane $\overline{\mathbb{R}}^2$ becomes a metric space. By the definition of quasi-uniform convergence and the theorem of Arzelà, Gagaeff and Alexandrov the last assertion of the theorem then follows.

Appendix: The Pompeiu type representation formula and the Liouville principle

Theorem 2 [9]. *Every velocity solution $\underset{\sim}{v}_D$ of* (1), (2) *and* (4) *admits a Pompeiu type representation formula. Moreover $\underset{\sim}{v}_D$ satisfies the Liouville principle, that is, if $\underset{\sim}{v}_0=\underset{\sim}{0}$, then $\underset{\sim}{v}_D=\underset{\sim}{0}$ in Ω.*

Proof. We note first that $\underset{\sim}{v}_D$ solves also the Poincaré–Stekloff problem [10]

$$\begin{cases} \Omega : \nabla \cdot v = 0, \quad \nabla^{\perp} \cdot \underset{\sim}{v} = -\omega, \\ \Gamma : \underset{\sim}{v} = \underset{\sim}{0}, \quad [0, 2\pi) : \lim\limits_{r\to\infty} \underset{\sim}{v}(r,\theta) = \underset{\sim}{v}_0 . \end{cases} \tag{16}$$

We introduce next the complex velocity $w = u - \mathrm{i}v$, the complex variables z, $\bar{z}$ and the complex differential operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$. (16) becomes then

$$\Omega : \frac{\partial w}{\partial \bar{z}} = \frac{\mathrm{i}}{2}\omega, \qquad \Gamma : w = 0, \qquad [0, 2\pi) : \lim_{r\to\infty} w(r,\theta) = w_0, \tag{17}$$

where, according to [4], $\omega \in C^2(\Omega \cap \Gamma) \cap L^2(\Omega)$ and has at infinity the behaviour

$$\lim_{r\to\infty} r^{3/4}\omega(r,\theta) = 0, \quad \text{uniformly in } \theta. \tag{18}$$

Denote next by $\Omega_R = \Omega \cap \{(r,\theta) \in \mathbb{R}^2 \mid r < R\}$. As known [11] the solution of (17) is given by the Pompeiu formula [6],

$$\Omega_R : w(z,\bar{z}) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Sigma_R} \frac{w(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} PV \iint\limits_{\Omega_R} \frac{\omega(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{19}$$

where $\zeta = \xi + \mathrm{i}\eta$, $\Sigma_R = \partial\Omega_R \setminus \Gamma$, and PV stays for the Cauchy principal value. We note that the left-hand side of (19) is independent of R, and that the first integral on the right-hand side, is analytic in Ω_R. Letting $R \to \infty$ and using the boundedness of $\underset{\sim}{v}_D$, (18) and the estimate

$$\left| PV \iint\limits_{\Omega_R} \frac{\omega(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta \right| \leqslant PV \iint\limits_{\mathbb{R}^2} \left| \frac{\widetilde{\omega}(\zeta,\bar{\zeta})}{\zeta - z} \right| \mathrm{d}\xi\,\mathrm{d}\eta = \mathrm{O}\big(R^{1/4}\big) \quad \text{for } R \to \infty, \tag{20}$$

where $\widetilde{\omega}(\zeta,\bar{\zeta}) = \omega(\zeta,\bar{\zeta})$ in Ω and is equal to zero in $\mathbb{R}^2 \setminus \Omega$, from (19) we obtain the Pompeiu type representation formula

$$\Omega : w(z,\bar{z}) = w_0 + \tfrac{1}{2\pi \mathrm{i}} PV \iint\limits_{\Omega} \frac{\omega(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{21}$$

We show next the Liouville principle. Assume to this end $w_0 = 0$ and write the representation formula for $z \in \Gamma$. Repeating the above argument we obtain

$$\Gamma : 0 = \frac{1}{2\pi \mathrm{i}} PV \iint\limits_{\Omega} \frac{\omega(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{22}$$

Let $D(z;\varepsilon)$ be the disc of center $z \in \Gamma$ and radius ε, and denote by $\Gamma_\varepsilon := \Gamma \cap D(z;\varepsilon)$, $\Sigma_\varepsilon := \partial D(z;\varepsilon) \cap \Omega$. Using Pompeiu type formulas from (22) we obtain by decomposition of the right-hand side integral

$$\begin{aligned} \Gamma : 0 = {} & \frac{1}{2\pi \mathrm{i}} \Bigg[\int\limits_{\Sigma_\varepsilon} \frac{w(\zeta,\bar{\zeta})}{\zeta - z}\,\mathrm{d}\zeta + PV \iint\limits_{D(z;\varepsilon)\cap\Omega} \frac{\omega(\zeta,\bar{\zeta}) - \omega(z,\bar{z})}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta \Bigg] \\ & + \frac{\mathrm{i}}{2}\omega(z,\bar{z}) \Bigg\{ \frac{\bar{z}}{2} - \frac{1}{2\pi \mathrm{i}} \Bigg[\int\limits_{\Sigma_\varepsilon} \frac{\bar{\zeta}\,\mathrm{d}\zeta}{\zeta - z} - PV \int\limits_{\Gamma_\varepsilon} \frac{\bar{\zeta}\,\mathrm{d}\zeta}{\zeta - z} \Bigg] \Bigg\}. \end{aligned} \tag{23}$$

The first and the third integrals in (23) are analytic functions, while the second and the fourth ones are non-analytic. Using these facts as well as the estimate

$$\Gamma : \left| PV \iint_{D(z;\varepsilon)\cap\Omega} \frac{\omega(\zeta,\bar{\zeta}) - \omega(z,\bar{z})}{\zeta - z}\, \mathrm{d}\xi\, \mathrm{d}\eta \right| \leqslant C_1 \sup_{\Omega} \left\{ \left|\frac{\partial\omega}{\partial\zeta}\right|, \left|\frac{\partial\omega}{\partial\bar{\zeta}}\right| \right\} \varepsilon^2 \tag{24}$$

from (23) it follows then

$$\Gamma : \omega = 0 \tag{25}$$

provided Γ is different from a circle, i.e., $|z| \neq c,\ c > 0$ and ω depends also on θ. From (6) and (25) we obtain that ω is solution of

$$\Omega : \nu \,\triangle\, \omega - (\underset{\sim}{v} \cdot \nabla)\omega = 0, \qquad \Gamma : \omega = 0, \qquad [0, 2\pi) : \lim_{r\to\infty} \omega(r,\theta) = 0. \tag{26}$$

By the Hopf maximum-minimum principle and (21), where $w_0 = 0$ we get

$$\Omega : \omega = 0, \quad \underset{\sim}{v} = \underset{\sim}{0}. \tag{27}$$

It remains now to investigate the case where Γ is a circle, and ω is a radial function. Using polar coordinates, from $(1)_2$ and (6) it follows

$$\Omega : \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\omega}{\mathrm{d}r}\right) - (u\cos\theta + v\sin\theta)\frac{\mathrm{d}\omega}{\mathrm{d}r} = 0, \tag{28}$$

$$\Omega : \frac{\partial}{\partial r}\big[r(u\cos\theta + v\sin\theta)\big] - \frac{\partial}{\partial\theta}(u\sin\theta - v\cos\theta) = 0. \tag{29}$$

Since ω is a radial function and using the periodicity of u and $v,$ we get then

$$\Omega : r(u\cos\theta + v\sin\theta) = C, \tag{30}$$

where C is the constant flux through $\Gamma,$ and, hence by integrating and using (2) and $(26)_3$ we obtain (27) again.

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