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On rigid displacements and their relation to the infinitesimal rigid displacement lemma in three-dimensional elasticity

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Abstract

Let Ω be an open connected subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta}$ be an immersion from Ω into $\mathbb{R}^3$. It is established that the set formed by all rigid displacements of the open set $\boldsymbol{\Theta}(\Omega)$ is a submanifold of dimension 6 and of class $\mathcal{C}^\infty$ of the space $\boldsymbol{H}^1(\Omega)$. It is also shown that the infinitesimal rigid displacements of the same set $\boldsymbol{\Theta}(\Omega)$ span the tangent space at the origin to this submanifold. ***To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Déplacements rigides et leur relation au lemme du déplacement rigide infinitésimal en élasticité tri-dimensionnelle. Soit Ω un ouvert connexe de $\mathbb{R}^3$ et $\boldsymbol{\Theta}$ une immersion de Ω dans $\mathbb{R}^3$. On établit que l'ensemble formé par les déplacements rigides de l'ouvert $\boldsymbol{\Theta}(\Omega)$ est une sous-variété de dimension 6 et de classe $\mathcal{C}^\infty$ de l'espace $\boldsymbol{H}^1(\Omega)$. On montre aussi que les déplacements rigides infinitésimaux du même ouvert $\boldsymbol{\Theta}(\Omega)$ engendrent le plan tangent à l'origine à cette sous-variété. ***Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Les notations sont définies dans la version anglaise.

Le *lemme du déplacement rigide infinitésimal en coordonnées curvilignes*, qui joue un rôle important en *élasticité linéarisée tri-dimensionnelle en coordonnées curvilignes* (cf. [2, Chapitre 1]), s'énonce ainsi : Soit Ω un ouvert connexe de $\mathbb{R}^3$, soit $\boldsymbol{\Theta}$ une immersion suffisamment régulière de Ω dans un espace euclidien tri-dimensionnel $\mathbb{E}^3$, et soit $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ un champ de vecteurs vérifiant

$$e_{i\|j}(\tilde{\boldsymbol{v}}) = 0 \quad \text{dans } \Omega,$$

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où

$$e_{i\|j}(\tilde{\boldsymbol{v}}) = \frac{1}{2}(\partial_i\tilde{\boldsymbol{v}}\cdot\boldsymbol{g}_j + \partial_j\tilde{\boldsymbol{v}}\cdot\boldsymbol{g}_i) \quad \text{et} \quad \boldsymbol{g}_i = \partial_i\boldsymbol{\Theta}.$$

Alors il existe des vecteurs $\boldsymbol{c} \in \mathbb{E}^3$ et $\boldsymbol{d} \in \mathbb{E}^3$ tels que (cf. [2, Théorème 1.7-3])

$$\tilde{\boldsymbol{v}}(x) = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\Theta}(x), \quad x \in \Omega.$$

En élasticité tri-dimensionnelle en coordonnées curvilignes, l'ensemble $\boldsymbol{\Theta}(\Omega) \subset \mathbb{E}^3$ est la *configuration de référence* d'un corps élastique et le champ $\tilde{\boldsymbol{v}}$ est un *champ de déplacements* de l'ensemble $\boldsymbol{\Theta}(\Omega)$. Les fonctions $e_{i\|j}(\tilde{\boldsymbol{v}})$ sont les composantes covariantes du *tenseur linéarisé de changement de métrique* associé au champ $\tilde{\boldsymbol{v}}$, et un déplacement de la forme ci-dessus $\tilde{\boldsymbol{v}} = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\Theta}$ est appelé un *déplacement rigide infinitésimal* de l'ensemble $\boldsymbol{\Theta}(\Omega)$.

L'objet de cette Note est d'établir que le lemme du mouvement rigide infinitésimal en coordonnées curvilignes n'est autre que la version linéarisée (dans un sens précisé au Théorème 4.1) du *théorème de rigidité* bien connu de la géométrie différentielle, une fois celui-ci convenablement étendu à l'espace de Sobolev $\boldsymbol{H}^1(\Omega)$.

Cette extension (Théorème 2.1) repose elle-même sur une généralisation du théorème de Liouville, qui est due à Reshetnyak [7] (voir [6] pour une démonstration particulièrement courte et élégante).

On établit ensuite (Théorème 3.1 et son corollaire) que l'ensemble $\boldsymbol{\mathcal{M}}_{\mathrm{rig}}$ formé des *déplacements rigides* (au sens du Théorème 2.1) de l'ensemble $\boldsymbol{\Theta}(\Omega)$ est une *sous-variété de dimension* 6 *et de classe* $\mathcal{C}^\infty$ *de l'espace* $\boldsymbol{H}^1(\Omega)$.

On montre enfin (Théorème 4.1) que l'espace vectoriel engendré par les déplacements rigides infinitésimaux de l'ensemble $\boldsymbol{\Theta}(\Omega)$ n'est autre que *l'espace tangent à l'origine à la variété* $\boldsymbol{\mathcal{M}}_{\mathrm{rig}}$.

Les énoncés des Théorèmes 2.1, 3.1, et 4.1 mentionnés ci-dessus, accompagnés d'esquisses de démonstrations, se trouvent dans la version anglaise. On trouvera les démonstrations complètes de ces théorèmes dans [4].

Les résultats de cette Note sont étendus aux *déplacements rigides et rigides infinitésimaux sur une surface* dans [5].

1. Preliminaries

Complete proofs of Theorems 2.1, 3.1, and 4.1 are found in [4].

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^3$, $\mathbb{O}^3$, $\mathbb{O}^3_+$, and $\mathbb{A}^3$ respectively designate the sets of all square matrices of order 3, of all orthogonal matrices of order 3, of all matrices $\boldsymbol{Q} \in \mathbb{O}^3$ with $\det \boldsymbol{Q} = 1$, and of all antisymmetric matrices of order 3.

Latin indices range over the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is used in conjunction with this rule. The identity mapping of a set X is denoted $\mathbf{id}_X$.

The notation $\mathbb{E}^3$ designates a three-dimensional Euclidean space and $\boldsymbol{a}\cdot\boldsymbol{b}$, $\boldsymbol{a}\wedge\boldsymbol{b}$, and $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ respectively designate the Euclidean inner product, the exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$, and the Euclidean norm of $\boldsymbol{a} \in \mathbb{E}^3$.

2. The rigidity theorem in Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^3$, let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, and let $\partial_i := \partial/\partial x_i$.

Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{E}^3)$ be an *immersion*. The *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ of the set $\boldsymbol{\Theta}(\Omega)$ (which is open in $\mathbb{E}^3$ since $\boldsymbol{\Theta}$ is an immersion) is then defined by means of its covariant components

$$g_{ij}(x) := \partial_i\boldsymbol{\Theta}(x)\cdot\partial_j\boldsymbol{\Theta}(x), \quad x \in \Omega.$$

The classical *rigidity theorem for an open set* (cf., e.g., [3, Theorem 3]) asserts that, if two immersions $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{\mathcal{C}}^1(\Omega) := \mathcal{C}^1(\Omega; \mathbb{E}^3)$ and $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ have the same metric tensor fields, i.e., if $\tilde{g}_{ij} = g_{ij}$ in Ω (with self-explanatory notations) and Ω is connected, then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \quad \text{for all } x \in \Omega.$$

The following result shows that *a similar result holds under the assumption that* $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega) := H^1(\Omega; \mathbb{E}^3)$. The matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) \in \mathbb{M}^3$ is that whose i-th column is $\partial_i \boldsymbol{\Theta}(x)$.

Theorem 2.1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ be a mapping that satisfies* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in Ω. Assume that there exists a vector field $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$ that satisfies*

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \tilde{g}_{ij} = g_{ij} \quad \text{a.e. in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in \Omega.$$

Sketch of proof. The Euclidean space $\mathbb{E}^3$ is identified with the space $\mathbb{R}^3$ throughout the proof.

(i) To begin with, consider the special case where $\boldsymbol{\Theta} = \mathbf{id}_\Omega$. In other words, we are given a mapping $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$ that satisfies $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$. Then, by a result of Reshetnyak [7] (for a modern proof, see Friesecke, James and Müller [6]), there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{o}\boldsymbol{x} \quad \text{for almost all } x \in \Omega.$$

Note that this result is a generalization of the classical *Liouville theorem*.

(ii) Consider next the general case. Let $x_0 \in \Omega$ be given. Since $\boldsymbol{\Theta}$ is an immersion, the local inversion theorem can be applied; there thus exist bounded open neighborhoods U of x_0 and $\widehat{U}$ of $\boldsymbol{\Theta}(x_0)$ satisfying $\overline{U} \subset \Omega$ and $\{\widehat{U}\}^- \subset \boldsymbol{\Theta}(\Omega)$, such that the restriction $\boldsymbol{\Theta}_U$ of $\boldsymbol{\Theta}$ to U can be extended to a $\boldsymbol{\mathcal{C}}^1$-diffeomorphism from $\overline{U}$ onto $\{\widehat{U}\}^-$.

Then the composite mapping $\widehat{\boldsymbol{\Phi}} := \widetilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta}_U^{-1}$ belongs to $\boldsymbol{H}^1(\widehat{U})$ and satisfies $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\hat{x}) \in \mathbb{O}^3_+$ for almost all $\hat{x} \in \widehat{U}$. By (i), there thus exist $\boldsymbol{c} \in \mathbb{R}^3$ and $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that $\widehat{\boldsymbol{\Phi}}(\hat{x}) = \boldsymbol{c} + \boldsymbol{Q}\,\boldsymbol{o}\hat{\boldsymbol{x}}$ for almost all $\hat{x} \in \widehat{U}$, hence such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in U.$$

(iii) The assumed connectedness of the open set Ω then implies that the last relation holds in fact for almost all $x \in \Omega$. □

3. The submanifold of rigid displacements

All the results needed below about submanifolds in infinite-dimensional Banach spaces are found in [1]. If $\mathcal{M}$ is a submanifold, the tangent space to $\mathcal{M}$ at $m \in \mathcal{M}$ is denoted $T_m\mathcal{M}$.

The next theorem shows that the set $\boldsymbol{\mathcal{M}}$ formed by all the mappings $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$ that satisfy the assumptions of the rigidity theorem for an open set (Theorem 2.1) is a *finite-dimensional submanifold of the space $\boldsymbol{H}^1(\Omega)$*. The *tangent space to $\boldsymbol{\mathcal{M}}$ at $\boldsymbol{\Theta}$* is also identified. Another equally important characterization of the same tangent space, involving this time the linearized change of metric tensor, will be given in Theorem 4.1.

Theorem 3.1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) \cap \boldsymbol{H}^1(\Omega)$ be a mapping that satisfies* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in Ω. Then the set*

$$\boldsymbol{\mathcal{M}} := \left\{\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega);\ \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ and } \tilde{g}_{ij} = g_{ij} \text{ a.e. in } \Omega\right\}$$

is a submanifold of class $\mathcal{C}^\infty$ *and of dimension* 6 *of the space* $\boldsymbol{H}^1(\Omega)$ *and its tangent space at* $\boldsymbol{\Theta}$ *is given by*

$$T_{\boldsymbol{\Theta}}\boldsymbol{\mathcal{M}} = \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{v}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta} \text{ a.e. in } \Omega\right\}.$$

Sketch of proof. It is easily seen that the linear mapping $\boldsymbol{f}$ defined by

$$\boldsymbol{f} : (\boldsymbol{c}, \boldsymbol{F}) \in \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{F}) := \boldsymbol{c} + \boldsymbol{F}\boldsymbol{\Theta} \in \boldsymbol{H}^1(\Omega)$$

is injective. Consequently, the image $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ is a linear subspace of dimension 12 of $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{f}$ is a $\mathcal{C}^\infty$-diffeomorphism between $\mathbb{E}^3 \times \mathbb{M}^3$ and $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$.

By the rigidity theorem (Theorem 2.1), the set $\boldsymbol{\mathcal{M}}$ may be equivalently defined as

$$\boldsymbol{\mathcal{M}} = \boldsymbol{f}\left(\mathbb{E}^3 \times \mathbb{O}^3_+\right).$$

Since $\mathbb{E} \times \mathbb{O}^3_+$ is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\mathbb{E}^3 \times \mathbb{M}^3$, $\boldsymbol{\mathcal{M}}$ is thus a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$. Besides, the closed subspace $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ is "split" in $\boldsymbol{H}^1(\Omega)$. Hence $\boldsymbol{\mathcal{M}}$ is also a *submanifold of class* $\mathcal{C}^\infty$ *and of dimension* 6 *of* $\boldsymbol{H}^1(\Omega)$.

Since $\boldsymbol{f}$ is linear and $T_{\boldsymbol{I}}\mathbb{O}^3_+ = \mathbb{A}^3$, the tangent space to $\boldsymbol{\mathcal{M}}$ at $\boldsymbol{\Theta}$ is given by

$$\begin{aligned} T_{\boldsymbol{\Theta}}\boldsymbol{\mathcal{M}} &= T_{\boldsymbol{f}(\boldsymbol{0},\boldsymbol{I})}\boldsymbol{f}\left(\mathbb{E}^3 \times \mathbb{O}^3_+\right) = \boldsymbol{f}\left(T_{(\boldsymbol{0},\boldsymbol{I})}\left(\mathbb{E}^3 \times \mathbb{O}^3_+\right)\right) = \boldsymbol{f}\left(\mathbb{E}^3 \times \mathbb{A}^3\right) \\ &= \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{v}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta} \text{ a.e. in } \Omega\right\}. \qquad \square \end{aligned}$$

In three-dimensional elasticity in curvilinear coordinates, the set $\boldsymbol{\Theta}(\Omega)$ is the *reference configuration* of a three-dimensional elastic body (under the additional assumption that the immersion $\boldsymbol{\Theta}$ is injective, but this assumption is irrelevant for our present purposes). Then, for each $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$, the set $\widetilde{\boldsymbol{\Theta}}(\Omega)$ is a *deformed configuration* and the field $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ defined by $\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta} + \tilde{\boldsymbol{v}}$ is a *displacement field* of the reference configuration $\boldsymbol{\Theta}(\Omega)$. If in particular $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{\mathcal{M}}$, the field $\tilde{\boldsymbol{v}}$ defined in this fashion is called a *rigid displacement*, and the subset $\boldsymbol{\mathcal{M}}_{\text{rig}}$ of $\boldsymbol{H}^1(\Omega)$ defined by

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\Theta} + \boldsymbol{\mathcal{M}}_{\text{rig}}$$

is accordingly called the *manifold of rigid displacements*. We now recast Theorem 3.1 in terms of the manifold $\boldsymbol{\mathcal{M}}_{\text{rig}}$.

Corollary to Theorem 3.1. *Let* Ω *be a connected open subset of* $\mathbb{R}^3$, *and let* $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) \cap \boldsymbol{H}^1(\Omega)$ *be a mapping that satisfies* $\det \nabla\boldsymbol{\Theta} > 0$ *in* Ω. *Then the manifold of rigid displacements of the set* $\boldsymbol{\Theta}(\Omega)$, *viz.*,

$$\boldsymbol{\mathcal{M}}_{\text{rig}} := \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \det(\nabla\boldsymbol{\Theta} + \nabla\tilde{\boldsymbol{v}}) > 0 \text{ and } \tilde{g}_{ij} = g_{ij} \text{ a.e. in } \Omega\right\},$$

is a submanifold of class $\mathcal{C}^\infty$ *and of dimension* 6 *of the space* $\boldsymbol{H}^1(\Omega)$ *and its tangent space at* $\boldsymbol{0}$ *is given by*

$$\begin{aligned} T_{\boldsymbol{0}}\boldsymbol{\mathcal{M}}_{\text{rig}} &= T_{\boldsymbol{\Theta}}\boldsymbol{\mathcal{M}} \\ &= \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{v}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta} \text{ a.e. in } \Omega\right\}. \qquad \square \end{aligned}$$

4. The infinitesimal rigid displacement lemma in curvilinear coordinates revisited

The covariant components of the *linearized change of metric tensor* associated with a displacement field $\tilde{\boldsymbol{v}}$ of the set $\boldsymbol{\Theta}(\Omega)$ are defined by

$$e_{i\|j}(\tilde{\boldsymbol{v}}) := \frac{1}{2}[\tilde{g}_{ij} - g_{ij}]^{\text{lin}},$$

where g_{ij} and $\tilde{g}_{ij}$ are the covariant components of the metric tensors of the sets $\boldsymbol{\Theta}(\Omega)$ and $\widetilde{\boldsymbol{\Theta}}(\Omega)$ where $\widetilde{\boldsymbol{\Theta}} := \boldsymbol{\Theta} + \tilde{\boldsymbol{v}}$, and $[\cdot]^{\text{lin}}$ denotes the *linear part with respect to* $\tilde{\boldsymbol{v}}$ in the expression $[\cdot]$. A formal computation immediately gives

$$e_{i\|j}(\tilde{\boldsymbol{v}}) = \frac{1}{2}(\partial_i \tilde{\boldsymbol{v}} \cdot \boldsymbol{g}_j + \partial_j \tilde{\boldsymbol{v}} \cdot \boldsymbol{g}_i), \quad \text{where } \boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}.$$

This expression thus shows that

$$e_{i\|j}(\tilde{\boldsymbol{v}}) \in L^2_{\text{loc}}(\Omega) \quad \text{if } \tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega) \text{ and } \boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega).$$

Under this assumption on the mapping $\boldsymbol{\Theta}$, a displacement field $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ that satisfies $e_{i\|j}(\tilde{\boldsymbol{v}}) = 0$ a.e. in Ω is called an *infinitesimal rigid displacement* of the set $\boldsymbol{\Theta}(\Omega)$. Accordingly, the *infinitesimal rigid displacement lemma in curvilinear coordinates* consists in identifying the vector space $\boldsymbol{\mathcal{V}}^{\text{lin}}_{\text{rig}}$ formed by such displacements. This is the object of the next theorem, which shows that the space $\boldsymbol{\mathcal{V}}^{\text{lin}}_{\text{rig}}$ has a *remarkably simple interpretation* in terms of the manifold $\boldsymbol{\mathcal{M}}_{\text{rig}}$ of rigid displacements introduced at the end of Section 3.

Theorem 4.1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) \cap \boldsymbol{H}^1(\Omega)$ be a mapping that satisfies* $\det \nabla\boldsymbol{\Theta} > 0$ *in Ω. Then the space of infinitesimal rigid displacements of the set $\boldsymbol{\Theta}(\Omega)$, viz.,*

$$\boldsymbol{\mathcal{V}}^{\text{lin}}_{\text{rig}} := \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ [\tilde{g}_{ij} - g_{ij}]^{\text{lin}} = 0 \textit{ a.e. in } \Omega\right\},$$

is given by

$$\boldsymbol{\mathcal{V}}^{\text{lin}}_{\text{rig}} = T_{\boldsymbol{0}}\boldsymbol{\mathcal{M}}_{\text{rig}},$$

where the tangent space $T_{\boldsymbol{0}}\boldsymbol{\mathcal{M}}_{\text{rig}}$ has been identified in the Corollary to Theorem 3.1.

Sketch of proof. Let $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ be such that $e_{i\|j}(\tilde{\boldsymbol{v}}) = 0$ in Ω. An extension of the proof in [2, Theorem 1.7-3] then shows that, given any $x_0 \in \Omega$, there exist an open neighborhood $U \subset \Omega$ of x_0, a vector $\boldsymbol{c} \in \mathbb{E}^3$, and a matrix $\boldsymbol{A} \in \mathbb{M}^3$ such that

$$\tilde{\boldsymbol{v}}(x) = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in U.$$

Since the point x_0 is arbitrary and Ω is connected, this relation holds in fact for almost all $x \in \Omega$ (given any $x_1 \in \Omega$, cover any path joining x_0 to x_1 by a finite number of neighborhoods similar to U and repeat the previous argument). In other words,

$$\boldsymbol{\mathcal{V}}^{\text{lin}}_{\text{rig}} = \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{v}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta} \text{ a.e. in } \Omega\right\}.$$

The conclusion then follows from the Corollary to Theorem 3.1. □

The manifold of rigid displacements can be equivalently written as

$$\boldsymbol{\mathcal{M}}_{\text{rig}} = \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ \det(\nabla\boldsymbol{\Theta} + \nabla\tilde{\boldsymbol{v}}) > 0 \text{ a.e. in } \Omega,\ \mathcal{F}_{ij}(\tilde{\boldsymbol{v}}) = \boldsymbol{0} \text{ a.e. in } \Omega\right\},$$

where the mappings $\mathcal{F}_{ij} : \boldsymbol{H}^1(\Omega) \to L^1(\Omega)$ are defined by

$$\mathcal{F}_{ij}(\tilde{\boldsymbol{v}}) := \partial_i(\boldsymbol{\Theta} + \tilde{\boldsymbol{v}}) \cdot \partial_j(\boldsymbol{\Theta} + \tilde{\boldsymbol{v}}) - \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta}, \quad \tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega).$$

Such mappings are Fréchet-differentiable and their Gâteaux derivatives at $\boldsymbol{0}$ are given by $D\mathcal{F}_{ij}(\boldsymbol{0})\tilde{\boldsymbol{v}} = 2e_{i\|j}(\tilde{\boldsymbol{v}})$ for all $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$, by definition of the functions $e_{i\|j}(\tilde{\boldsymbol{v}})$. Theorem 4.1 thus shows that

$$T_{\boldsymbol{0}}\boldsymbol{\mathcal{M}}_{\text{rig}} = \left\{\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega);\ D\mathcal{F}_{ij}(\boldsymbol{0})\tilde{\boldsymbol{v}} = 0 \text{ a.e. in } \Omega\right\}.$$

In other words, the space $T_{\boldsymbol{0}}\boldsymbol{\mathcal{M}}_{\text{rig}}$ has the expression that is naturally expected, yet is often delicate to establish, of the tangent space to a submanifold of an infinite-dimensional Banach space defined by means of equations; see in this respect [1, Chapter 3].

The results of this Note can be extended to *rigid and infinitesimal rigid displacements on a surface* (cf. [5]).

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