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Group Theory

Invariant theory and eigenspaces for unitary reflection groups

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Abstract

We prove some variations of formulas of Orlik and Solomon in the invariant theory of finite unitary reflection groups, and use them to give elementary and case-free proofs of some results of Lehrer and Springer, in particular that an integer is regular for a reflection group G if and only if it divides the same number of degrees and codegrees. ***To cite this article: G.I. Lehrer, J. Michel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Invariants et espaces propres des groupes de réflexion complexes. En utilisant des variantes d'une formule de Orlik et Solomon relative aux invariants d'un groupe de réflexions complexes G, nous redémontrons de façon élémentaire deux résultats de Lehrer and Springer, en particulier le fait qu'un entier est régulier pour G si et seulement si il divise le même nombre de degrés et de codegrés. Notre preuve évite l'analyse cas par cas. ***Pour citer cet article : G.I. Lehrer, J. Michel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit G un groupe fini engendré par des (pseudo-)réflexions dans un espace vectoriel V sur $\mathbb{C}$ de dimension ℓ. L'anneau des fonctions polynomiales sur V est identifié à l'algèbre symétrique S du dual V^*. L'anneau des invariants S^G est une algèbre de polynomes à ℓ générateurs. Les degrés d'un ensemble de générateurs homogènes algébriquement indépendants de S^G sont uniquement déterminés et sont appelés les degrés de G ; nous les noterons $d_1, \ldots, d_\ell$.

Si F est l'idéal de S engendré par les éléments de S^G s'annulant en $0 \in V$, le quotient $S_G := S/F$ réalise la représentation régulière de G et est appelé l'anneau des coinvariants de G. Il hérite de la graduation de S ; si q est une indéterminée, nous définissons pour tout G-module M le « degré fantôme » comme $\sum_i q^i \langle (S_G)_i, M\rangle_G = \sum_j q^{m_j(M)}$.

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Les entiers $m_j(M)$ sont appelés les M-exposants de G. Pour $M=V$ on les note simplement m_j et ils sont appelés les exposants de G. Pour le contragrédient $M=V^*$ on les note m_j^* et ils sont appelés les coexposants de G. On a, si on range les deux listes par ordre croissant, $m_i=d_i-1$. On définit les codegrés de G par $d_i^*=m_i^*-1$. Enfin, plus généralement pour $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ nous notons $m_j(\sigma)$ les V^σ-exposants de G.

Nous démarrons par le résultat suivant de [3, Theorem 3.7] :

Proposition 0.1. *Soient x et y des indéterminées sur $\mathbb{C}$. Pour tout $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a*

$$|G|^{-1}\sum_{g\in G}\frac{\det_V(1-yg^\sigma)}{\det_V(1-xg)}=\frac{\prod_{j=1}^r(1-yx^{m_j(\sigma)})}{\prod_{i=1}^\ell(1-x^{d_i})}.$$

Nous en déduisons, en utilisant la même méthode que [4] qui concerne le cas $\sigma=\mathrm{Id}$, le

Théorème 0.2. *Soit d un entier positif et soit ζ une racine primitive d-ième de l'unité. Soit $A(d)=\{i\mid\zeta^{d_i}=1\}$ et soit $B_\sigma(d)=\{j\mid\zeta^{-\sigma}=\zeta^{m_j(\sigma)}\}$. Alors, si l'on pose $a(d)=\#A(d)$ et $b_\sigma(d)=\#B_\sigma(d)$, on a $a(d)\leqslant b_\sigma(d)$, et si, pour une application linéaire $h:V\to V$, nous notons $\det'(h)$ le produit des valeurs propres non nulles de h, nous avons l'égalité polynomiale suivante dans $\mathbb{C}[T]$:*

$$\sum_{g\in G}T^{d(g,\zeta)}\det{}'\bigl(1-\zeta^{-1}g\bigr)^{\sigma-1}$$
$$=\begin{cases}\prod_{j\in B_\sigma(d)}\bigl(T+m_j(\sigma)\bigr)\prod_{j\notin B_\sigma(d)}\bigl(1-\zeta^{-\sigma-m_j(\sigma)}\bigr)\prod_{j\notin A(d)}\dfrac{d_j}{1-\zeta^{-d_j}} & \text{si } a(d)=b_\sigma(d),\\ 0 & \text{sinon},\end{cases}$$

où, pour $g\in G$, on a noté $V(g,\zeta)$ l'espace propre de g pour la valeur propre ζ et posé $d(g,\zeta)=\dim V(g,\zeta)$.

Le cas particulier où $\sigma=\mathrm{Id}$ est le résultat de [4] :

Corollaire 0.3. $\sum_{g\in G}T^{d(g,\zeta)}=\prod_{\{j\mid\zeta^{d_j}=1\}}(T+d_j-1)\prod_{\{j\mid\zeta^{d_j}\neq1\}}d_j$.

Le cas particulier où σ est la conjugaison complexe donne, pour $B(d)=B_\sigma(d)=\{i\mid d \text{ divise } d_i^*\}$ et $b(d)=\#B(d)$:

Corollaire 0.4.

$$(-\zeta)^\ell\sum_{g\in G}\det\bigl(g^{-1}\bigr)(-T)^{d(g,\zeta)}=\begin{cases}\prod_{j\in B(d)}(T+d_j^*+1)\prod_{j\notin B(d)}\bigl(1-\zeta^{-d_j^*}\bigr)\prod_{j\notin A(d)}\dfrac{d_j}{1-\zeta^{-d_j}} & \text{si } a(d)=b(d),\\ 0 & \text{sinon.}\end{cases}$$

En étudiant les termes de degré $a(d)$ dans 0.3 (resp. 0.4) nous redémontrons de façon élémentaire les deux théorèmes suivants :

Théorème 0.5 [1]. *Soit $g\in G$ tel que $E:=V(g,\zeta)$ soit maximal parmi les ζ-espaces propres d'éléments de G. Soit $N=\{x\in G\mid xE=E\}$ et $C=\{x\in G\mid xe=e$ pour tout $e\in E\}$. Alors N/C est un groupe de réflexion complexe dans son action sur E, qui est fidèle.*

L'entier d est *régulier* pour G (au sens de [5]) si E contient un point du complémentaire des hyperplans de réflexion de W. Du théorème [6, 1.5] de Steinberg, on déduit que d est régulier si et seulement si $C=\{1\}$.

Théorème 0.6 [2]. *L'entier d est régulier pour G si et seulement si $a(d)=b(d)$.*

1. Introduction

Let G be a finite group generated by (pseudo)reflections in a complex vector space V of dimension $\ell > 0$. It is well known that if S denotes the coordinate ring of V (identified with the symmetric algebra on the dual V^*) the ring S^G of polynomial invariants of G is free; if $f_1, f_2, \ldots, f_\ell$ is a set of homogeneous free generators of S^G, then the degrees $d_i = \deg f_i$ $(i = 1, \ldots, \ell)$ are determined by G, and are called the *invariant degrees* of G.

If F is the ideal of S generated by the elements of S^G which vanish at $0 \in V$, then S/F realises the regular representation of G. It is called the ring of coinvariants of G and denoted S_G; it inherits a grading from S. Let q be an indeterminate; for any G-module M, we define the "fake degree" of M as

$$\sum_i \big\langle (S_G)_i, M\big\rangle_G q^i = \sum_j q^{m_j(M)}. \tag{1}$$

Here $\langle, \rangle_G$ denotes the usual intertwining number for complex representations of G. The integers $m_1(M) \leqslant m_2(M) \leqslant \cdots \leqslant m_r(M)$, where $r = \dim M$ since S_G realises the regular representation of G, are called the *M-exponents* of G. This terminology is sometimes reserved for irreducible M, but applies generally. There are two noteworthy special cases of this definition. When $M = V$, usually called the reflection representation, the M-exponents are called simply the *exponents* $m_1, \ldots, m_\ell$ of G, so that $m_i = m_i(V)$. When $M = V^*$, the contragredient of V, the corresponding exponents $m_i^* := m_i(V^*)$ $(i = 1, \ldots, \ell)$ are referred to as the *coexponents* of G. It is well known that if the degrees $d_1, \ldots, d_\ell$ are also written in non-decreasing order, then

$$d_i = m_i + 1 \quad \text{for } i = 1, \ldots, \ell. \tag{2}$$

In analogy with (2), we define the *codegrees* d_i^* of G by

$$d_i^* = m_i^* - 1 \quad \text{for } i = 1, \ldots, \ell. \tag{3}$$

For any linear transformation $h : V \to V$ and element $\zeta \in \mathbb{C}$, denote by $V(h, \zeta)$ the ζ-eigenspace of h. Consider the following theorems.

Theorem 1.1 [1]. *Let d be any positive integer, and let $\zeta \in \mathbb{C}$ be a primitive d-th root of unity. Suppose $g \in G$ is such that $E := V(g, \zeta)$ is maximal among ζ-eigenspaces of elements of G. Define the subgroups $N = \{x \in G \mid xE = E\}$ and $C = \{x \in G \mid xe = e$ for all $e \in E\}$. Then the group N/C acts faithfully as a unitary reflection group on E.*

In the notation of Theorem 1.1, the integer d is said (cf. [5]) to be *regular* for G if the maximal eigenspace E contains a vector which is regular for G, i.e., which lies in no reflecting hyperplane. Equivalently, E is not contained in any reflecting hyperplane. By Steinberg's theorem [6, 1.5], we have in the notation of Theorem 1.1 that d is regular for G if and only if $C = \{1\}$.

Theorem 1.2 [2]. *Let $A(d) = \{i \mid d$ divides $d_i\}$ and let $B(d) = \{i \mid d$ divides $d_i^*\}$. Write $a(d), b(d)$ respectively for the cardinalities of $A(d), B(d)$. Then $a(d) \leqslant b(d)$, and we have equality if and only if d is regular for G.*

Theorem 1.1 was proved in [1], using intersection multiplicities of hypersurface intersections, and Bezout's theorem. Theorem 1.2 was proved in [2], where some case by case analysis was needed, which depended on the Shephard–Todd classification of the irreducible reflection groups.

It is our purpose in this Note to prove some variations on known formulae in the invariant theory of G, and to show how they may be used to give case-free elementary proofs of Theorems 1.1 and 1.2.

2. Invariant theory

We begin by recalling some results of Solomon and Orlik–Solomon, which may be found in [3].

For any $\mathbb{C}G$-module M, consider the $\mathbb{C}G$-module $S \otimes M^*$, which is canonically isomorphic to $\mathrm{Hom}(M, S)$. To determine the subspace of G-invariant elements, recall that by Chevalley's theorem, $S \cong S^G \otimes S_G$ as G-module, whence $(S \otimes M^*)^G \cong S^G \otimes (S_G \otimes M^*)^G$. This is a free S^G-module of rank $r = \dim M$ (since $\langle S_G, M\rangle_G = r$), which is generated over S^G by elements $\{u_i = \sum_{j=1}^r h_{ij} \otimes y_j \mid i = 1, \dots, r\}$, where $\{y_j \mid j = 1, \dots, r\}$ is a $\mathbb{C}$-basis of M^* and the h_{ij} are homogeneous elements of S_G whose degrees are the M-exponents $m_j(M)$ of G. Define $Q_M \in S$ by $Q_M = \det(h_{ij})$. The following properties of Q_M may be found in [3, pp. 79–82], where Q_M is denoted by j_M.

Lemma 2.1. (i) *The polynomial Q_M is uniquely defined by M up to multiplication by a non-zero constant, and has degree $q(M) := \sum_j m_j(M)$.*

(ii) *We have $gQ_M = \det_M(g)Q_M$ for any element $g \in G$.*
(iii) *We have $Q_M = PQ_{\Lambda^r M}$ for some $P \in S^G$.*
(iv) *If M is a Galois conjugate of the reflection representation V, then $Q_M \in \mathbb{C}^\times Q_{\Lambda^r M}$.*

Proposition 2.2. *Let G be a unitary reflection group and let M be a $\mathbb{C}G$ module such that $Q_M \in \mathbb{C}^\times Q_{\Lambda^r M}$, where $r = \dim M$. Let x and y be indeterminates over $\mathbb{C}$. Then*

$$|G|^{-1} \sum_{g\in G} \frac{\det_M(1 - yg)}{\det_V(1 - xg)} = \frac{\prod_{j=1}^r (1 - yx^{m_j(M)})}{\prod_{i=1}^\ell (1 - x^{d_i})}.$$

Proof. This is essentially [3, Theorem 3.7], which is the statement for Galois conjugates of V. The proof of the general case is the same; it uses the obvious bigrading of $(S \otimes \Lambda^* M^*)^G$ and op. cit. Theorem 3.1. □

It is proved in [3, Theorem 2.13] that the Galois conjugates V^σ, where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, satisfy the conditions of (2.2), and hence that the formula holds for $M = V^\sigma$. We shall exploit this to prove the following generalisation of [3, Theorem 3.3].

Theorem 2.3. *Let G be a unitary reflection group in V, with basic degrees $\{d_1, \dots, d_\ell\}$. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denote by $m_i(\sigma)$ the V^σ-exponents of G. Let d be a fixed positive integer and let ζ be a primitive d-th root of unity in $\mathbb{C}$. Let $A(d) = \{i \mid \zeta^{d_i} = 1\}$ and let $B_\sigma(d) = \{j \mid \zeta^{-\sigma} = \zeta^{m_j(\sigma)}\}$. Write $a(d)$, $b_\sigma(d)$ respectively for the cardinalities of $A(d)$, $B_\sigma(d)$. Then $a(d) \leqslant b_\sigma(d)$, and if, for any linear transformation $h : V \to V$, we denote by* $\det'(h)$ *the product of the non-zero eigenvalues of h, we have the following polynomial identity in $\mathbb{C}[T]$.*

$$\sum_{g\in G} T^{d(g,\zeta)} \det{}'\bigl(1 - \zeta^{-1}g\bigr)^{\sigma-1}$$
$$= \begin{cases} \prod_{j\in B_\sigma(d)} \bigl(T + m_j(\sigma)\bigr) \prod_{j\notin B_\sigma(d)} \bigl(1 - \zeta^{-\sigma-m_j(\sigma)}\bigr) \prod_{j\notin A(d)} \dfrac{d_j}{1 - \zeta^{-d_j}} & \text{if } a(d) = b_\sigma(d), \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $d(g, \zeta) = \dim V(g, \zeta)$ for $g \in G$.

Proof. We start with the formula (2.2) for $M = V^\sigma$; writing $\lambda_1(g), \dots, \lambda_\ell(g)$ for the eigenvalues of $g \in G$ on V, we get

$$|G|^{-1} \sum_{g\in G} \prod_{i=1}^{\ell} \frac{(1 - y\lambda_i(g)^\sigma)}{(1 - x\lambda_i(g))} = \prod_{j=1}^{\ell} \frac{(1 - yx^{m_j(\sigma)})}{(1 - x^{d_j})}. \tag{5}$$

The term corresponding to $g \in G$ on the left side has a zero of order $d(g, \zeta)$ at $x = \zeta^{-1}$ in the denominator. We therefore introduce the change of variable $y = \zeta^{-\sigma}(1 - T(1 - x\zeta))$, so that $1 - y\zeta^\sigma = T(1 - x\zeta)$, which ensures that the same is true of the numerator. The left side then becomes a rational function in x and T with no pole at $x=\zeta^{-1}$. Now the factor $1-x^{d_j}$ of the denominator of the right side has a zero at $x=\zeta^{-1}$ exactly when $j\in A(d)$, and this zero is simple. Similarly, the factor $1-yx^{m_j(\sigma)}$ of the numerator has a zero at $x=\zeta^{-1}$ exactly when $j\in B_\sigma(d)$, and this zero is simple. Since the pole at $x=\zeta^{-1}$ is removable, $a(d)\leqslant b_\sigma(d)$, and the right side is zero unless we have equality, as claimed.

Evaluating the left side at $x=\zeta^{-1}$ we get

$$|G|^{-1}\sum_{g\in G}\prod_{\{i\mid\lambda_i(g)=\zeta\}}T\prod_{\{i\mid\lambda_i(g)\neq\zeta\}}\frac{1-\zeta^{-\sigma}\lambda_i(g)^\sigma}{1-\zeta^{-1}\lambda_i(g)}=|G|^{-1}\sum_{g\in G}T^{d(g,\zeta)}\det{}'\bigl(1-\zeta^{-1}g\bigr)^{\sigma-1}. \tag{6}$$

For the right side, we find, by differentiation with respect to x, that if $j\in B_\sigma(d)$ and $i\in A(d)$, then the limit as $x\to\zeta^{-1}$ of $(1-yx^{m_j(\sigma)})/(1-x^{d_i})$ is $(T+m_j(\sigma))/d_i$. Hence the value of the right side at $x=\zeta^{-1}$ is

$$\prod_{j\in B_\sigma(d)}\bigl(T+m_j(\sigma)\bigr)\prod_{j\in A(d)}d_j^{-1}\prod_{j\notin B_\sigma(d)}\bigl(1-\zeta^{-\sigma-m_j(\sigma)}\bigr)\prod_{j\notin A(d)}\bigl(1-\zeta^{-d_j}\bigr)^{-1}. \tag{7}$$

The theorem now follows immediately by comparing (6) and (7), taking into account that $|G|=d_1\dots d_\ell$. □

We next state four corollaries of Theorem 2.3, the first two of which are well known.

Corollary 2.4 [3, Theorem 3.3]. *We have* $\sum_{g\in G}T^{d(g,1)}\det{}'(1-g)^{\sigma-1}=\prod_{j=1}^{\ell}(T+m_j(\sigma))$.

This is simply the case $d=1$ (whence $\zeta=1$) of the theorem.

Corollary 2.5 [4]. *We have* $\sum_{g\in G}T^{d(g,\zeta)}=\prod_{\{j\mid\zeta^{d_j}=1\}}(T+d_j-1)\prod_{\{j\mid\zeta^{d_j}\neq1\}}d_j$.

This is the case $\sigma=\mathrm{Id}$ of the theorem. Note that here the integers $m_j(\sigma)=m_j=d_j-1$ are the usual exponents. We believe the next two results are new.

Corollary 2.6. *Maintaining the above notation, we have $a(d)\leqslant b(d)$, and*

$$(-\zeta)^\ell\sum_{g\in G}\det\bigl(g^{-1}\bigr)(-T)^{d(g,\zeta)}=\begin{cases}\prod_{j\in B(d)}(T+d_j^*+1)\prod_{j\notin B(d)}\bigl(1-\zeta^{-d_j^*}\bigr)\prod_{j\notin A(d)}\dfrac{d_j}{1-\zeta^{-d_j}} & \text{if } a(d)=b(d),\\ 0 & \text{if } a(d)<b(d),\end{cases}$$

where the integers $d_j^*=m_j^*-1$ are the codegrees of G.

Proof. Take σ to be the complex conjugation in (4). Then by definition, $m_j(\sigma)=m_j^*=d_j^*-1$ and σ acts as inversion on roots of unity, thus for a root of unity $\lambda\neq1$ we have $(1-\lambda)^{\sigma-1}=-\lambda^{-1}$; so for $g\in G$, $\det{}'(1-\zeta^{-1}g)^{\sigma-1}=(-\zeta)^\ell(-1)^{d(g,\zeta)}\det(g^{-1})$. The formula (2.6) now follows directly from (4). □

There are of course many other special cases which might be explored. Here is another.

Corollary 2.7. *With the above notation, for $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$\sum_{g\in G}T^{d(g,-1)}\det{}'(1+g)^{\sigma-1}=\begin{cases}0, & \text{unless } \#\{j\mid m_j(\sigma)\text{ is odd}\}=\#\{j\mid d_j\text{ is odd}\},\\ \prod_{\{j\mid m_j(\sigma)\text{ is odd}\}}\bigl(T+m_j(\sigma)\bigr)\prod_{\{j\mid d_j\text{ is odd}\}}d_j, & \text{otherwise.}\end{cases}$$

This is obtained from (4) by putting $d=2$ and hence $\zeta=-1$.

3. Eigenspaces

In this section we show how the results above may be applied to questions concerning maximal eigenspaces of elements of G. We begin with the

Proof of Theorem 1.1. It follows from [5, Theorem 3.4] (whose proof is elementary) that the maximal ζ-eigenspaces $V(g,\zeta)$ all have dimension $a(d)$ and are conjugate under the action of G. We compute the coefficient of $T^{a(d)}$ on both sides of (2.5), obtaining in the notation of 1.1 the equality $\frac{|G||C|}{|N|} = \prod_{j:d\nmid d_j} d_j$, the left side being the number $|C|$ of elements $g \in G$ such that $V(g,\zeta) = E$ multiplied by the number $\frac{|G|}{|N|}$ of distinct conjugates of E. Since $|G| = \prod_{j=1}^{\ell} d_j$, it follows that $|\frac{N}{C}| = \prod_{j:d|d_j} d_j$, which by [5, Proposition 3.5(ii)] (whose proof is also elementary) proves Theorem 1.1.

The next result includes Theorem 1.2.

Theorem 3.1. (i) *Let* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *In the notation of Theorem* 4*, if* d *is regular for* G*, then* $b_\sigma(d) = a(d)$.

(ii) *When* σ *is the complex conjugation, so that* $b_\sigma(d) = b(d)$*, then the converse of* (i) *holds. That is,* d *is regular for* G *if and only if* $a(d) = b(d)$.

Proof. We keep the notation of the proof of Theorem 1.1 above. Clearly the set $\{g \in G \mid V(g,\zeta) = E\}$ is a coset g_0C of the pointwise stabiliser $C = G_E$ of E in G. Hence if $C = 1$, which is the regular case, there is a unique element g with $V(g,\zeta) = E$. If we write g_h for the corresponding element when E is replaced by a conjugate hE ($h \in G$) of E, we have clearly, that $g_h = hgh^{-1}$, so that in particular, all the elements g_h have the same eigenvalues.

Let us compare the coefficients of $T^{a(d)}$ in both sides of (4) when d is regular. The left side is $|G/N|\det'(1-\zeta^{-1}g)$, which is clearly non-zero, while the right side is zero unless $a(d) = b_\sigma(d)$, which proves (i).

To prove (ii), assume that d is not regular for G. Then by the observations above, the coefficient of $T^{a(d)}$ in the left side of (2.6) has a factor $\sum_{h\in C}\det^{-1}(h)$ which is zero if C is non-trivial, since by Steinberg's theorem C is a reflection subgroup of G. Hence the right side of (2.6) is zero, so $b(d) \gneq a(d)$. □

Note that generally whenever d is regular for G, $\sigma \neq \mathrm{Id}$ and $g \in G$ is such that $V(g,\zeta)$ is maximal for some primtive d-th root of unity ζ, evaluating the coefficient of $T^{d(g,\zeta)}$ in both sides of (4) gives information on the eigenvalues of g. The particular case when σ is the complex conjugation yields

Corollary 3.2. *Let* ζ *be a primitive* d*-th root of unity, where* d *is regular for* G*, and let* $g \in G$ *be such that* $V(g,\zeta)$ *is maximal. Then* $\det(g) = \zeta^{\ell}\prod_{j\notin B(d)}(\zeta^{-d_j^*}-1)^{-1}\prod_{j\notin A(d)}(1-\zeta^{-d_j}) = \zeta^{-\sum_{j=1}^{\ell} m_j}$.

The first equality is immediate from (2.6). The second, which follows from [5, Theorem 4.2(v)] may also be written $\prod_{j\notin B(d)}(1-\zeta^{-d_j^*}) = \prod_{j\notin A(d)}(1-\zeta^{d_j})$, which permits the simplification of 2.6 above.

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