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BMO is the intersection of two translates of dyadic BMO

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Abstract

Let $\mathbb{T}$ be the unit circle on $\mathbb{R}^2$. Denote by $\mathrm{BMO}(\mathbb{T})$ the classical BMO space and denote by $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$ the usual dyadic BMO space on $\mathbb{T}$. Then, for suitably chosen $\delta \in \mathbb{R}$, we have

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \backsimeq \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})} + \|\varphi(\cdot - 2\delta\pi)\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}, \quad \forall \varphi \in \mathrm{BMO}(\mathbb{T}).$$

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Résumé

BMO est l'intersection de deux translatés de BMO dyadique. Soit $\mathbb{T}$ le cercle unité dans $\mathbb{R}^2$. On note $\mathrm{BMO}(\mathbb{T})$ l'espace BMO classique et l'on note $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$ l'espace BMO dyadique usuel sur $\mathbb{T}$. Pour certaines valeurs de $\delta \in \mathbb{R}$, nous montrons que l'espace $\mathrm{BMO}(\mathbb{T})$ coïncide avec l'intersection de $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$ et du translaté par δ de $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$, en d'autres termes que l'on a

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \backsimeq \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})} + \|\varphi(\cdot - 2\delta\pi)\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}, \quad \forall \varphi \in \mathrm{BMO}(\mathbb{T}).$$

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1. Introduction

Let $\mathbb{T}$ be the unit circle on $\mathbb{R}^2$, identified with $(0, 2\pi]$. Recall that

$$\mathrm{BMO}(\mathbb{T}) = \left\{\varphi \in L^1(\mathbb{T}):\ \|\varphi\|_{\mathrm{BMO}(\mathbb{T})} = \sup\left\{\frac{1}{|I|}\int\limits_I |\varphi - \varphi_I|\,\mathrm{d}\theta\right\} < \infty\right\},$$

where the supremum runs over all intervals I on $\mathbb{T}$ and $\varphi_I = \frac{1}{|I|}\int_I \varphi(s)\,\mathrm{d}s$. Let $\mathcal{D} = \{\mathcal{D}_n\}_{n\geqslant 0}$ be the family of the usual dyadic σ-algebra on $\mathbb{T}$, i.e.,

$$\mathcal{D}_n = \sigma\left\{\left(D_n^k\right)_{0\leqslant k<2^n}\right\}, \quad D_n^k = \left(2\pi k 2^{-n}, 2\pi(k+1)2^{-n}\right]; \quad n \geqslant 0.$$

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Recall that the usual dyadic BMO space is defined by

$$\mathrm{BMO}_{\mathcal{D}}(\mathbb{T}) = \left\{\varphi \in L^1(\mathbb{T}):\ \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})} = \sup_{n,k}\left\{\frac{2^n}{2\pi}\int_{D_n^k} |\varphi - \varphi_{D_n^k}|\,\mathrm{d}\theta\right\} < \infty\right\}.$$

BMO($\mathbb{T}$) and the dyadic BMO space $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$ have many similarities, but nevertheless certain differences. The dyadic BMO space is usually much easier to study. Some works have been done to study the relationship between the two kinds of BMO spaces (see [1,4]). In this paper, we show that, for any positive δ suitably chosen (more precisely satisfying $d(\delta) > 0$, with $d(\delta)$ as defined below) φ is in BMO($\mathbb{T}$) if and only if $\varphi(\cdot)$ and $\varphi(\cdot - 2\pi\delta)$ are in $\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})$. Clearly the analogous result holds on $\mathbb{R}$ with the same proof (see the final remark below).

2. The main result

Let A be the collection of all dyadic rationals. For $0 < \delta < 1$, define its relative distance to A, denoted by $d(\delta)$ in this paper, as follows

$$d(\delta) := \inf\left\{2^n\left|\delta - k2^{-n}\right| \mid n \geqslant 0,\ k \in \mathbb{Z}\right\}.$$

Let $\mathbb{T}$ be the unit circle on $\mathbb{R}^2$. For δ with $d(\delta) > 0$, we consider the filtration $\mathcal{D}^\delta = \{\mathcal{D}_n^\delta\}_{n\geqslant 0}$ on $\mathbb{T}$ obtained from the usual dyadic filtration after translation by $2\pi\delta$. More precisely:

$$\mathcal{D}_n^\delta = \sigma\left\{\left(D_n^{\delta,k}\right)_{0\leqslant k<2^n}\right\}, \quad D_n^{\delta,k} = \left(2\delta\pi + 2\pi k2^{-n}, 2\delta\pi + 2\pi(k+1)2^{-n}\right], \quad \forall n \geqslant 0.$$

Hence, if we define $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}^\delta}(\mathbb{T})}$ in the usual way, we have $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}^\delta}(\mathbb{T})} = \|\varphi(\cdot - 2\delta\pi)\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}$.

In this paper, we will say $\mathcal{D}$ (resp. $\mathcal{D}^\delta$) "fits" an interval $I \subset \mathbb{T}$ with fit-constant c if there exist $n \geqslant 0,\ 0 \leqslant k_I < 2^n$ such that $I \subset D_n^{k_I}$ (resp. $I \subset D_n^{\delta,k_I}$) and $|D_n^{k_I}| \leqslant c|I|$ (resp. $|D_n^{\delta,k_I}| \leqslant c|I|$). Our key observation is the following simple fact.

Proposition 2.1. *For any interval $I \subset \mathbb{T}$, either $\mathcal{D}$ or $\mathcal{D}_\delta$ fits I with fit-constant $2/d(\delta)$.*

Proof. If $|I| \geqslant 2\pi d(\delta)$, let $n = k_I = 0$, then $I \subset D_0^0 = (0, 2\pi]$.

If $|I| < 2\pi d(\delta)$, let $n \geqslant 0$ be the integer such that $d(\delta)2\pi 2^{-n-1} \leqslant |I| < d(\delta)2\pi 2^{-n}$. Set

$$A_n = \left\{k2\pi 2^{-n};\ 0 \leqslant k < 2^n\right\}, \qquad A_n^\delta = \left\{2\delta\pi + k2\pi 2^{-n};\ 0 \leqslant k < 2^n\right\}.$$

Note that for any two points $a, b \in A_n \cup A_n^\delta$, we have $|a-b| \geqslant d(\delta)2\pi 2^{-n} > |I|$. Thus there is at most one element of $A_n \cup A_n^\delta$ belonging to I. Then $I \cap A_n = \phi$ or $I \cap A_n^\delta = \phi$. Therefore, I must be contained in some $D_n^{k_I}$ or D_n^{δ,k_I} and $|D_n^{k_I}| = |D_n^{\delta,k_I}| = 2\pi/2^n \leqslant 2/d(\delta)|I|$. □

Remark 1. From the above proposition, a number of "classical" results become immediate consequences of their "probabilistic" counterparts. For instance, Doob's maximal inequality implies the Hardy–Littlewood maximal inequality immediately.

Theorem 2.2. *For $\varphi \in L^1(\mathbb{T}),\ 0 < \delta < 1$, with $d(\delta) > 0$, we have*

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \leqslant \frac{4}{d(\delta)}\max\left\{\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}, \left\|\varphi(\cdot - 2\delta\pi)\right\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}\right\}.$$

Proof. By the above proposition, for every interval $I \subset \mathbb{T}$, there exist N, k_I such that $I \subseteq D_N^{k_I}$ or $I \subseteq D_N^{\delta,k_I}$ and $\frac{2\pi}{2^N} \leqslant \frac{2}{d(\delta)}|I|$. If D_N^{δ,k_I} contains I, then

$$\frac{1}{|I|}\int_I \left|\varphi(\theta)-\varphi_I\right| \mathrm{d}\theta \leqslant \frac{1}{|I|}\int_I \left|\varphi(\theta)-\varphi_{D_N^{\delta,k_I}}\right| \mathrm{d}\theta + \left|\varphi_{D_N^{\delta,k_I}}-\varphi_I\right| \leqslant \frac{2}{|I|}\int_I \left|\varphi(\theta)-\varphi_{D_N^{\delta,k_I}}\right| \mathrm{d}\theta$$
$$\leqslant \frac{4}{d(\delta)|D_N^{\delta,k_I}|}\int_{D_N^{\delta,k_I}} \left|\varphi(\theta)-\varphi_{D_N^{\delta,k_I}}\right| \mathrm{d}\theta \leqslant \frac{4}{d(\delta)}\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}^\delta}(\mathbb{T})} = \frac{4}{d(\delta)}\left\|\varphi(\cdot-2\delta\pi)\right\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}.$$

If $D_N^{k_I}$ contains I, then similarly

$$\frac{1}{|I|}\int_I \left|\varphi(\theta)-\varphi_I\right| \mathrm{d}\theta \leqslant \frac{4}{d(\delta)}\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}.$$

Thus, taking the supremum over all intervals $I \subset \mathbb{T}$, we get

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \leqslant \frac{4}{d(\delta)}\max\left\{\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}, \left\|\varphi(\cdot-2\delta\pi)\right\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}\right\}. \qquad \square$$

Example 1. Let $\delta = 1/3$, then $d(\delta) = 1/3$, and then

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \leqslant 12\max\left\{\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}, \left\|\varphi\left(\cdot-\frac{2\pi}{3}\right)\right\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T})}\right\}.$$

Remark 2. Let $\varphi^{\#}(t) = \sup_{I\ni t}\frac{1}{|I|}\int_I|\varphi-\varphi_I|\,\mathrm{d}\theta$ and $\varphi^{\#}_{\mathcal{D}}(t) = \sup_{D_n^k\ni t}\frac{1}{|D_n^k|}\int_{D_n^k}|\varphi-\varphi_{D_n^k}|\,\mathrm{d}\theta$. It is easy to see that $\{\delta, d(\delta)>0\}$ is exactly the set of all δ's such that $\varphi^{\#} \leqslant c\max\{\varphi^{\#}_{\mathcal{D}}, \varphi^{\#}_{\mathcal{D}}(\cdot-2\pi\delta)\}$ for some $c>0$. The same statement trivially remains valid in the Banach space valued case and is particularly useful in the operator valued case: see [3] for some results in that direction.

Remark 3. One can check that the set $\{\delta, d(\delta)>0\}$ is dense in $(0,1)$ while its measure is zero.

Corollary 2.3. $\mathrm{BMO}(\mathbb{T}) = \mathrm{BMO}_{\mathcal{D}}(\mathbb{T}) \cap \mathrm{BMO}_{\mathcal{D}^\delta}(\mathbb{T})$ *with equivalent norms.*

Denote by $H^1_{\mathcal{D}}$ (resp. $H^1_{\mathcal{D}^\delta}$) the dyadic Hardy space with respect to $\mathcal{D}$ (resp. $\mathcal{D}^\delta$). By duality, we have

Corollary 2.4. $H^1 = H^1_{\mathcal{D}} + H^1_{\mathcal{D}^\delta}$ *with equivalent norms.*

Remark 4. There is another way to see Corollary 2.4. Denote by $H^{1,at}$ the classical atomic Hardy space. Denote by $H^{1,at}_{\mathcal{D}}$ (resp. $H^{1,at}_{\mathcal{D}^\delta}$) the dyadic atomic Hardy space with respect to $\mathcal{D}$ (resp. $\mathcal{D}^\delta$). From Proposition 2.1, we see that any atom is a dyadic atom (up to a fixed factor) with respect to either $\mathcal{D}$ or $\mathcal{D}^\delta$. Thus $H^{1,at} = H^{1,at}_{\mathcal{D}} + H^{1,at}_{\mathcal{D}^\delta}$ with equivalent norms. Since $H^{1,at} = H^1$ and $H^{1,at}_{\mathcal{D}} = H^1_{\mathcal{D}}$, we obtain Corollary 2.4.

Remark 5. See [4] for a recent result (of the same flavor) comparing Hilbert transforms and martingale transforms proved by averaging shifted and dilated dyadic filtrations.

Remark 6. John Garnett kindly informed us that he already knew that $\mathrm{BMO}(\mathbb{T})$ coincides with the intersection of three (suitably chosen) translates of dyadic $\mathrm{BMO}(\mathbb{T})$ (the idea for this can be traced back to p. 417 of [2]), but our main result seems new.

We now turn to the case of dimension $m>1$. By a straightforward product argument, one can deduce from the above proposition that $\mathrm{BMO}(\mathbb{T}^m)$ coincides with the intersection of a family of 2^m translates of the dyadic version of $\mathrm{BMO}(\mathbb{T}^m)$. However, we wish to show below that the number of translates can be reduced to $m+1$.

In the following, we always suppose $\{\delta_i\}_{i=0}^m$ is a sequence in $(0,1)$ such that

$$d\bigl(\{\delta_i\}_{i=0}^m\bigr) := \min_{i\neq j} d(\delta_i-\delta_j) > 0.$$

Let $\mathcal{D}^{\delta_i}$ be the translation by $2\pi\delta_i$ of the family of the usual (one dimensional) dyadic σ-algebra. Set $\mathcal{F}^i = (\mathcal{D}^{\delta_i})^m$, $0\leqslant i\leqslant m$. Then we get $m+1$ families of increasing dyadic σ-algebras on $\mathbb{T}^m$.

Proposition 2.5. *Let $\mathcal{F}^i$, $0\leqslant i\leqslant m$, be as above and let $c = 2/d(\{\delta_i\}_{i=0}^m)$. Then, for any cube $J\subset\mathbb{T}^m$, there exists some $\mathcal{F}^i$ which fits J with fit-constant c^m.*

Proof. Write $J\subset\mathbb{T}^m$ as $J = J_1\times J_2\times\cdots\times J_m$, where J_i are intervals in $\mathbb{T}$, $1\leqslant i\leqslant m$. Let $\{\delta_i\}_{i=0}^m$ be such that $d(\{\delta_i\}_{i=0}^m) > 0$. By Proposition 2.1, for every J_i, there is at most one k_i, $0\leqslant k_i\leqslant m$, such that $\mathcal{D}^{\delta_{k_i}}$ does not fit J_i with constant c. Then there is at least one $\mathcal{D}^{\delta_k}$ which fits all J_i with constant c. Thus (with an obvious extension of our terminology) we may say that $\mathcal{F}^k$ fits J with fit-constant c^m. □

From Proposition 2.5 we have

Theorem 2.6 (In the case of $\mathbb{T}^m$). *Let $\{\delta_i\}_{i=0}^m$ be a sequence in $(0,1)$ such that $d(\{\delta_i\}_{i=0}^m) > 0$. Let ${}^i\delta = (\delta_i,\delta_i,\ldots,\delta_i)$. Then, for $\varphi\in L^1(\mathbb{T}^m)$, we have*

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T}^m)} \leqslant 2\bigl(2/d\bigl(\{\delta_i\}_{i=0}^m\bigr)\bigr)^m \max_{0\leqslant i\leqslant m}\bigl\{\bigl\|\varphi\bigl(\cdot - {}^i\delta 2\pi\bigr)\bigr\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{T}^m)}\bigr\}.$$

Remark 7. To extend our results to $\mathbb{R}^m$, denote by $\mathcal{D}(\mathbb{R})$ the family of the usual dyadic σ-algebra on $\mathbb{R}$. For $0<\delta<1$ with $d(\delta)>0$, choose an increasing family of dyadic σ-algebra $\mathcal{D}^\delta(\mathbb{R}) = (\mathcal{D}_n^\delta)_{n\in\mathbb{Z}}(\mathbb{R})$ such that, for n even,

$$\mathcal{D}_n^\delta(\mathbb{R}) = \sigma\bigl(\bigl\{D_n^{\delta,k}\bigr\}_{k\in\mathbb{Z}}\bigr),$$
$$D_n^{\delta,k}(\mathbb{R}) = \begin{cases} \bigl(\frac{k}{2^n}+\delta, \frac{k+1}{2^n}+\delta\bigr], & n\geqslant 0,\\ \bigl(\frac{k}{2^n}+\delta+\sum_{j=n+2}^0\frac{1}{2^j}, \frac{k+1}{2^n}+\delta+\sum_{j=n+2}^0\frac{1}{2^j}\bigr], & n<0.\end{cases}$$

Note that all $\mathcal{D}_n^\delta(\mathbb{R})$'s are given after fixing $\mathcal{D}_n^\delta(\mathbb{R})$'s for all even n's. Let $\{\delta_i\}_{i=0}^m$ be a sequence in $(0,1)$ such that $d(\{\delta_i\}_{i=0}^m) > 0$. Let ${}^i\mathcal{D}^\delta(\mathbb{R}^m) = \{{}^i\mathcal{D}_n^\delta(\mathbb{R}^m)\}_{n\in\mathbb{Z}}$, where ${}^i\mathcal{D}_n^\delta(\mathbb{R}^m)$ is the m times product of the σ-algebra $\mathcal{D}_n^{\delta_i}(\mathbb{R})$. Then, by the same idea as above, we can get

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{R}^m)} \simeq \max_{0\leqslant i\leqslant m}\|\varphi\|_{\mathrm{BMO}_{{}^i\mathcal{D}^\delta}(\mathbb{R}^m)}\quad \forall\varphi\in L^1\bigl(\mathbb{R}^m\bigr).$$

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