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Parallel pure spinors and holonomy

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Abstract

We characterize the spin pseudo-Riemannian manifolds which admit parallel pure spinors by their holonomy groups. In particular, we study the Lorentzian case. ***To cite this article: A. Ikemakhen, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Spineurs purs parallèles et holonomie. Nous caractérisons les variétés pseudo-riemanniennes spinorielles qui admettent des spineurs purs parallèles par leurs groupes d'holonomie. En particulier, nous étudions le cas des variétés lorentziennes. ***Pour citer cet article : A. Ikemakhen, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit (M,g) une variété pseudo-riemannienne connexe spinorielle de dimension m et de signature (p,q). Notons S son fibré spinoriel associé et $\cdot$ la multiplication de Clifford associée. Soient $\mathcal{C}_{p,q}$ l'algèbre de Clifford de $\mathbb{R}^{p,q}=(\mathbb{R}^m,\langle\rangle_{p,q})$ et $\mathcal{C}^{\mathbb{C}}_{p,q}$ sa complexification. Soient $\Delta_{p,q}\cong\mathbb{C}^{[n/2]}$ le module des spineurs de S et $0\neq u\in\Delta_{p,q}$. Puisque on peut injecter canoniquement $\mathbb{C}^m$ dans $\mathcal{C}^{\mathbb{C}}_{p,q}$, on peut définir

$$V_u:=\{X\in\mathbb{C}^m;\ X\cdot u=0\}\quad\text{et}\quad T_u:=\{X\in\mathbb{R}^m;\ X\cdot u=0\}.$$

Si on note $\langle\rangle^{\mathbb{C}}_{p,q}$ l'extension linéaire complexe de $\langle\rangle_{p,q}$, alors V_u et T_u sont respectivement des sous-espaces totalement isotropes de $(\mathbb{C}^m,\langle\rangle^{\mathbb{C}}_{p,q})$ et $\mathbb{R}^{p,q}$.

Définition 0.1. $u\in\Delta_{p,q}$ est dit un spineur pur (resp. spineur pur réel), si $\dim_{\mathbb{C}}V_u=n:=[m/2]$ (resp. si $\dim_{\mathbb{R}}T_u=n$).

Définition 0.2. Un spineur parallèle $\varphi\in\Gamma(S)$ est dit pur, si il est pur en un point (donc en tout point) de M.

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Dans ([7], Chapitre IV, §9), Lawson et Michelsohn ont caractérisé l'existence d'un spineur pur parallèle (sous forme abrégée s.p.p.) sur une variété riemannienne spinorielle par des restrictions sur son groupe d'holonomie restreint. Plus précisement ils ont démontré le résultat suivant :

Théorème 0.3. *Une variété riemannienne spinorielle* (M, g)*, de dimension* m*, admet un s.p.p., si et seulement si, son groupe d'holonomie restreint est contenu dans le groupe* $\mathrm{SU}([\frac{m}{2}])$*, si* m *est pair ou contenu dans* $\{1\} \times \mathrm{SU}([\frac{m}{2}])$ *si* m *est impair.*

Le but de cette Note est de généraliser ce résultat pour les variétés pseudo-riemanniennes spinorielles. Dans [5], Kath a associé à l'existence d'un s.p.p. sur une variété pseudo-riemannienne spinorielle une structure dite « structure optique orthogonale parallèle » et elle a caractérisé cette existence par des restrictions sur la courbure tensorielle de la variété. En utilisant ce résultat et le théorème d'holonomie d'Ambrose–Singer (voir [1]) nous caractérisons ce type de variétés par leurs groupes d'holonomie et nous obtenons le résultat suivant :

Théorème 0.4. *Soit* (M, g) *une variété pseudo-riemannienne connexe, simplement connexe, spinorielle. Alors* (M, g) *admet un s.p.p. si et seulement si, il existe un* $l \in \mathbb{N}$ *tel que l'algèbre d'holonomie* $\mathcal{H}$ *est contenue dans l'algèbre*

$$\left(\mathrm{sl}(l, \mathbb{R}) \oplus \mathrm{su}\left(\left[\frac{p-l}{2}\right], \left[\frac{q-l}{2}\right]\right)\right) \ltimes \left(\mathbb{R}^{l(m-2l)} \ltimes \wedge^2 \mathbb{R}^l\right).$$

Si (M, g) est une variété riemannienne, $l = 0$ et nous déduisons du Théorème 0.4, le Théorème 0.3. Pour le cas lorentzien indécomposable (c.à.d., la représentation d'holonomie ne laisse invariant aucun sous-espace propre non dégénérée) nous obtenons le corollaire suivant :

Soit (M, g) est une variété lorentzienne indécomposable, connexe, simplement connexe, spinorielle de signature $(1, 1+n)$. Alors (M, g) admet un s.p.p. si et seulement si, son groupe d'holonomie est contenu dans le groupe $\mathrm{SU}([\frac{n}{2}]) \ltimes N_n$, où N_n est le groupe

$$N_n := \left\{ \begin{pmatrix} 1 & {}^tX & -\frac{1}{2}{}^tX \cdot X \\ 0 & I_n & -X \\ 0 & 0 & 1 \end{pmatrix};\ X \in \mathbb{R}^n \right\}.$$

1. Introduction

Parallel spinors occur in several contexts. They are special solutions of the Dirac and twistor equation of a pseudo-Riemannian spin manifold, they occur as a technical tool in the construction pseudo-Riemannian spin manifolds that admit Killing spinors and they have applications in supergravity and string theories. The characterization of pseudo-Riemannian spin manifolds that admit parallel spinors by their holonomy groups is not known, except in the irreducible case [2] and partially in the Lorentzian case [6]. In this note we study the case of pseudo-Riemannian spin manifolds that admit parallel pure spinors. In [7], Chapter IV, §9), Lawson and Michelsohn proved that:

Theorem 1.1 [7]. *A* m*-dimensional spin Riemannian manifold admits a non-trivial parallel pure spinor* (*in abbreviate form p.p.s.*) *if and only if its restricted holonomy group is a subgroup of* $\mathrm{SU}([\frac{m}{2}])$ *if* m *is even, or a subgroup of* $\{1\} \times \mathrm{SU}([\frac{m}{2}])$ *if* m *is odd.*

Our aim is to generalize this result for the spin pseudo-Riemannian manifolds. In [5], Kath associated to every parallel pure spinor on a spin pseudo-Riemannian manifold a parallel orthogonal optical structure (see Definition 2.3) and she characterized a spin pseudo-Riemannian manifolds that admit p.p.s. by their curvature tensor. In this Note, we will investigate pseudo-Riemannian spin manifolds that admit p.p.s. Using the result of Kath (see Theorem 2.4) and the Ambrose–Singer holonomy theorem [1], we characterize this kind of manifolds by their holonomy groups.

2. Parallel pure spinors and holonomy

2.1. Parallel totally isotropic distribution and holonomy

Let (M,g) be a pseudo-Riemannian manifold of signature (p,q), $(p \leqslant q)$ and dimension $m = p+q \geqslant 3$. In the whole paper, (M,g) is supposed to be connected and simply connected. We denote by H the holonomy group of (M,g) at a point $\theta \in M$. We suppose the holonomy representation H preserves a totally isotropic subspace E of $T_\theta M$ of dimension $l \leqslant \min(p,q)$. At the point θ, we can suppose the metric g has the following form:

$$\langle\,,\rangle = g_\theta = 2\sum_{i=1}^{l} \mathrm{d}x_i\,\mathrm{d}y_i + g',$$

where g' is the canonical scalar product of signature $(p-l, q-l)$ on $E^\perp/E$. If we denote by $\mathcal{H}$ the Lie algebra of H, then $\mathcal{H}$ is a subalgebra of $\mathrm{so}(\langle,\rangle)$. Let L_l be the connected subgroup of $\mathrm{SO}^0(\langle,\rangle)$ which is the full stabilizer of E. Then the Lie algebra $\mathcal{L}_l$ of L_l contains $\mathcal{H}$ and consists of the following matrices:

$$(U,A,X,B) := \begin{pmatrix} U & {}^tX & B \\ 0 & A & -G'^{-1}X \\ 0 & 0 & -{}^tU \end{pmatrix}, \tag{1}$$

where $U \in \mathcal{M}(l)$, the space of $l \times l$ real matrices, $A \in \mathrm{so}(p-l, q-l)$, $X \in \mathcal{M}(l, m-2l)$ the space of $l \times (m-2l)$ real matrices, and $B \in \mathrm{so}(l)$. G' is the matrix of g' written in a orthonormal basis of $E^\perp / E$.

We remark that $\mathcal{L}_l = (\mathcal{U}_l \oplus \mathcal{T}_l) \ltimes (\mathcal{N}_{(l,p-l,q-l)} \ltimes \mathcal{B}_l)$, where $\mathcal{B}_l = \{(0,0,0,B);\ B \in \mathrm{so}(l)\}$ is in abelian ideal of $\mathcal{L}_l$, $\mathcal{N}_{(l,p-l,q-l)} \ltimes \mathcal{B}_l = \{(0,0,X,B);\ X \in \mathcal{M}(l,m-2l),\ B \in \mathrm{so}(l)\}$ is an ideal of $\mathcal{L}_l$, $\mathcal{T}_l = \{(0,A,0,0);\ A \in \mathrm{so}(g')\}$ is a subalgebra of $\mathcal{L}_l$ isomorphic to $\mathrm{so}(p-l,q-l)$ and $\mathcal{U}_l = \{(U,0,0,0);\ U \in \mathcal{M}(l)\}$ is a subalgebra of $\mathcal{L}_l$.

Now, let $T \subset TM$ be a l-dimensional parallel totally isotropic distribution, then $T^\perp$ is also a parallel distribution. In particular the Levi-Civita connection D of g induces a connection D' on the bundle $T^\perp / T$ defined by:

$$D'[X] = [DX], \quad \text{for } X \in \Gamma\big(T^\perp\big).$$

Furthermore the metric g' induced by g on $T^\perp / T$ is D'-parallel: $D'g' = 0$. From this we deduce

Proposition 2.1. *The holonomy algebra $\mathcal{H}'$ of the connection D' at the point θ is precisely the projection of $\mathcal{H}$ in the algebra $\mathcal{T}_l$.*

2.2. Parallel pure spinors and holonomy

In this subsection, we characterize the existence of a parallel pure spinor on a pseudo-Riemannian by the holonomy. We suppose that (M,g) is a spin manifold and we denote by S its spinor bundle, the Clifford multiplication by $\cdot$. Let $\langle\rangle$ be a scalar product on $\mathbb{R}^m$ $(m = p+q)$ of signature (p,q). Let $\mathcal{C}_{p,q}$ be the Clifford algebra of $\mathbb{R}^{p,q} = (\mathbb{R}^m, \langle\rangle_{p,q})$ and $\mathcal{C}^{\mathbb{C}}_{p,q}$ its complexification. Let $\Delta_{p,q} \cong \mathbb{C}^{[n/2]}$ the spinor module of S and $0 \neq u \in \Delta_{p,q}$. Since there is a natural embedding of $\mathbb{C}^m$ in $\mathcal{C}^{\mathbb{C}}_{p,q}$, we can define

$$V_u := \big\{X \in \mathbb{C}^m;\ X \cdot u = 0\big\} \quad \text{and} \quad T_u := \big\{X \in \mathbb{R}^m;\ X \cdot u = 0\big\}.$$

If we denote by $\langle\rangle^{\mathbb{C}}_{p,q}$ the complex linear extension of $\langle\rangle_{p,q}$ on $\mathbb{R}^m$, then V_u is a totally isotropic subspace of $(\mathbb{C}^m, \langle\rangle^{\mathbb{C}}_{p,q})$. Also T_u is a totally isotropic subspace of $\mathbb{R}^{p,q}$.

Definition 2.2. $u \in \Delta_{p,q}$ is said to be a pure spinor (resp. real pure spinor), if $\dim_{\mathbb{C}} V_u = n := [\frac{m}{2}]$ (resp., if $\dim_{\mathbb{C}} T_u = n$). A parallel spinor $\varphi \in \Gamma(S)$ is said to be a pure (resp. real pure), if it is pure (resp. real pure) at one point (then at every point) of M.

Definition 2.3. If the dimension m of (M, g) is even, (T, J) is called a parallel orthogonal optical structure on (M, g), if T is a parallel totally isotropic distribution on M and J a parallel orthogonal almost complex structure on $T^{\perp}/T$ (i.e., orthogonal with respect to the metric g' induced by g and $D'J = 0$).

If m is odd, (T, F, J) is called a parallel orthogonal optical structure on (M, g), if

- T is a parallel totally isotropic distribution on M;
- F is a parallel section of the bundle of the nondegenerate subspaces of codimension one of $T^{\perp}/T$;
- and J a parallel orthogonal almost complex structure on F.

In [5], Kath associated to every parallel pure spinor φ on a spin pseudo-Riemannian manifold a parallel orthogonal optical structure (T, J), or (T, F, J), where $T = \{X \in \Gamma(TM),\ X \cdot \varphi = 0\}$ and she proved:

Theorem 2.4 [5]. *Let (M, g) be a connected, simply connected, spin pseudo-Riemannian manifold of signature (p, q) and dimension $m = p + q$. Then (M, g) admits a non-trivial parallel pure spinor if and only if there exists a parallel orthogonal optical structure (T, J) (if m is even) or (T, F, J) (if m is odd) on (M, g) which satisfies the following conditions*:

(i) *the trace of the restriction of $R(X, Y)$ to T equals* 0:

$$\operatorname{tr} R(X, Y)_{|T} = 0, \quad \text{for all } X, Y \in \Gamma(TM), \tag{2}$$

where R is the curvature tensor of the metric g;

(ii) *the curvature R' of the induced connection D' on $T^{\perp}/T$, if m is even and on F if m is odd, satisfies*

$$\operatorname{tr}\big(J \circ R'(X, Y)\big) = 0, \quad \text{for all } X, Y \in \Gamma(TM). \tag{3}$$

From Theorem 2.4, we will deduce the following theorem which gives a relation between the existence of a parallel pure spinor and restrictions on the holonomy group.

Theorem 2.5. *Based on the assumptions of Theorem 2.4 and with the notation introduced in Subsection 2.1. (M, g) admits a non-trivial parallel pure spinor if and only if there exists an integer $l \in \mathbb{N}$ and a decomposition $\mathbb{R}^{p,q} = \mathbb{R}^l \oplus \mathbb{R}^{p-l,q-l} \oplus \mathbb{R}^l$ such that its holonomy algebra $\mathcal{H}$ is contained in $\mathcal{L}_l$ and satisfies the following conditions*:

(i) *the projection of $\mathcal{H}$ in $\mathcal{U}_l$ is contained in* $\mathrm{sl}(l, \mathbb{R})$;

(ii) *the projection $\mathcal{H}'$ of $\mathcal{H}$ in $\mathcal{T}_l$ satisfies*:

If m is even,

$$\mathcal{H}' \subset \mathrm{su}\left(\frac{p-l}{2}, \frac{q-l}{2}\right). \tag{4}$$

If m is odd, there exists a nonisotropic vector $x_0 \in \mathbb{R}^{p-l,q-l}$, such that

$$\mathcal{H}' x_0 = 0 \quad \text{and} \quad \mathcal{H}' \subset \mathrm{su}\left(\left[\frac{p-l}{2}\right], \left[\frac{q-l}{2}\right]\right). \tag{5}$$

Said otherwise, a m-dimensional, connected, simply connected, spin pseudo-Riemannian manifold of signature (p, q) admits a parallel pure spinor if and only if there exists an integer $l \in \mathbb{N}$ such that its holonomy algebra is contained in the algebra $(\mathrm{sl}(l, \mathbb{R}) \oplus \mathrm{su}([\frac{p-l}{2}], [\frac{q-l}{2}])) \ltimes (\mathbb{R}^{l(m-2l)} \ltimes \wedge^2 \mathbb{R}^l)$.

Remark 1. Let W be a vector bundle over a manifold M equipped with a pseudo-Riemannian metric g' of signature (r,s) and a linear connection D' such that $D'g'=0$. Let $J:W\to W$ be a D'-parallel orthogonal complex structure on W. Then, necessarily the index of g' is even ($(r,s)=(2p',2q')$). At a point θ,

$$u(p',q')=\{A\in \mathrm{so}(2p',2q');\ J_\theta\circ A=A\circ J_\theta\},$$

then the holonomy algebra $\mathcal{H}'$ of D' at the point θ satisfies:

$$\mathcal{H}'\subset u(p',q'). \tag{6}$$

We also have $\mathrm{su}(p',q')=\{A\in u(p',q');\ \mathrm{tr}(J_\theta\circ A)=0\}$.

Conversely, if we have (6), there exists a D'-parallel J orthogonal complex structure on W.

Proof of Theorem 2.5. Ambrose–Singer holonomy theorem asserts that the holonomy algebra $\mathcal{H}$ at the point θ of a linear connection D is generated by $R_{\tau(\gamma)}(X,Y):=\tau(\gamma)^{-1}\circ R(\tau(\gamma)X,\tau(\gamma)Y)\circ\tau(\gamma)$, where R is the curvature tensor of D, γ runs through all piecewise C^1 paths starting from θ and X, Y run through $T_\theta M$ (see [1]). Now, if $T\subset TM$ is a parallel distribution, $R_{\tau(\gamma)}(X,Y)$ stabilizes T_θ and $R(\tau(\gamma)X,\tau(\gamma)Y)$ stabilizes T_y, where y is the end of the curve γ. Hence the traces of the restrictions of $R_{\tau(\gamma)}(X,Y)$ and $R(\tau(\gamma)X,\tau(\gamma)Y)$ respectively to T_θ and T_y are equal. Consequently the condition (i) of Theorem 2.4 is equivalent to the condition (i) of Theorem 2.5. Since J is D'-parallel, we have

$$J_\theta\circ R'_{\tau(\gamma)}(X,Y)=\tau(\gamma)^{-1}\circ J_y\circ R'\big(\tau(\gamma)X,\tau(\gamma)Y\big)\circ\tau(\gamma).$$

Consequently, with the same argument as above, condition (ii) of Theorem 2.4 is equivalent to the condition:

$$\mathrm{tr}(J_\theta\circ A)=0,\quad \text{for all } A\in\mathcal{H}'\subset \mathrm{so}(p-l,q-l). \tag{7}$$

Then, if m is even, from Remark 1 and Proposition 2.1, the second condition of Theorem 2.4 is equivalent to (4). If m is odd, from Theorem 2.4, there exists a parallel optical structure (T,F,J), i.e., there exists a D'-parallel nonisotropic vector field $\xi\in T^\perp/T$ such that $F=\xi^\perp$ and $J:F\to F$ is a D'-parallel orthogonal complex structure on F. Then, according to the principle of holonomy, condition (ii) of Theorem 2.4 is equivalent to (5). □

For the case of the parallel real pure spinors, we get from Theorem 2.5 the following:

Corollary 2.6. *Based on the assumptions of Theorem* 2.5. *If g is of signature (n,n) or $(n,n+1)$, then (M,g) admits a parallel real pure spinor if and only if*

$$\mathcal{H}\subset\left\{\begin{pmatrix}U & B\\ 0 & -{}^tU\end{pmatrix};\ U\in \mathrm{sl}(n,\mathbb{R}),\ B\in\mathrm{so}(n)\right\},\quad \textit{if } m \textit{ is even},$$

and

$$\mathcal{H}\subset\left\{\begin{pmatrix}U & {}^tX & B\\ 0 & 0 & -X\\ 0 & 0 & -{}^tU\end{pmatrix};\ U\in \mathrm{sl}(n,\mathbb{R}),\ B\in\mathrm{so}(n),\ X\in\mathbb{R}^n\right\},\quad \textit{if } m \textit{ is odd}.$$

Now, if we suppose that (M,g) is irreducible or Riemannian, the distribution $T=0$. Then $l=0$ and from Theorem 2.5 we deduce Theorem 1.1 and the following corollary:

Corollary 2.7. *Let (M,g) be a connected, simply connected, spin irreducible pseudo-Riemannian manifold of signature (p,q) and dimension $m=p+q$. If (M,g) admits a non-trivial parallel pure spinor, then necessarily the dimension of M and the integers (p,q) are even and*

$$H\subset \mathrm{SU}\left(\frac{p}{2},\frac{q}{2}\right),\ \textit{i.e., } (M,g) \textit{ is Kähler and Ricci flat.}$$

Conversely, if $H \subset \mathrm{SU}(\frac{p}{2}, \frac{q}{2})$, (M, g) *admits a non-trivial parallel pure spinor.*

From Theorem 2.5, if we denote by N_n the subgroup

$$N_n := \left\{ \begin{pmatrix} 1 & {}^tX & -\frac{1}{2}{}^tX.X \\ 0 & I_n & -X \\ 0 & 0 & 1 \end{pmatrix};\ X \in \mathbb{R}^n \right\} \subset L_1,$$

we also deduce:

Corollary 2.8. *Let* (M, g) *be a* m*-dimensional connected, simply connected, spin indecomposable Lorentzian manifold of signature* $(1, 1+n)$. *Then* (M, g) *admits a non-trivial parallel pure spinor if and only if its holonomy group* H *is contained in* $\mathrm{SU}([\frac{n}{2}]) \ltimes N_n \subset L_1$.

Proof. To show this corollary, according to Theorem 2.5, it is sufficient to show that H stabilizes necessarily one nontrivial isotropic vector. Let us suppose the opposite. Then, from Theorem 2.5, either $H \subset \{1\} \times \mathrm{SU}([\frac{m}{2}])$ if m is odd, or $H \subset \mathrm{SU}([\frac{m}{2}])$ if m is even. Since (M, g) is indecomposable, the first case cannot occur. Also, the second case cannot occur, since the metric g is not of even index. □

Remark 2. (a) In [3], Bernard Bergery and Ikemakhen describe the possible holonomy groups of simply connected indecomposable-reducible Lorentzian manifolds. From this result and Corollary 2.8, we can get easily the list of the possible holonomy groups of simply connected indecomposable Lorentzian spin manifolds, of dimension $\leqslant 5$ which admit parallel pure spinors.

(b) Another similar list can be gotten from another classification of Bergery and Ikemakhen (see [4]), for the case of a spin 4-dimensional pseudo-Riemannian manifolds of signature $(2, 2)$.

Remark 3. Based on the assumptions of Corollary 2.8, Leistner proved in [6] that (M, g) admits a parallel spinor if and only if $\mathcal{H}$ and $\mathcal{H}'$ satisfy:

(i) $\mathcal{H} \subset \mathrm{so}(n) \ltimes \mathbb{R}^n$,

(ii) $V_{\mathcal{H}'} := \{v \in \Delta_n \mid \lambda_*^{-1}(\mathcal{H}') \cdot v = 0\} \neq 0$, where λ_* is the differential of the twofold covering $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$.

Since $V_{\mathrm{su}([n/2])} \neq 0$, then the Corollary 2.8 is a particular case of this result.

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