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Complex Analysis

A differential criterium for regularity of quaternionic functions

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Abstract

Let $\Omega \subseteq \mathbb{C}^2$. We prove that there exist differential operators T and N , with complex coefficients, such that a function $f : \overline{\Omega} \rightarrow \mathbb{H}$ of class C^1 is regular if and only if $(N - jT)f = 0$ on $\partial\Omega$ (j a basic quaternion) and f is harmonic on Ω . At the same time we generalize a result of Kytmanov and Aizenberg. We show that a complex harmonic function h on Ω ($\partial\Omega$ connected) is holomorphic if and only if $\bar{\partial}_n h = \overline{aL(h)}$ on $\partial\Omega$, where $\bar{\partial}_n$ is the normal part of $\bar{\partial}$, L is a tangential Cauchy–Riemann operator and $a \in \mathbb{C}$. **To cite this article:** A. Perotti, C. R. Acad. Sci. Paris, Ser. I 337 (2003).

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Résumé

Une condition différentielle de régularité pour les fonctions quaternioniennes. Soit $\Omega \subseteq \mathbb{C}^2$. Nous montrons l'existence de deux opérateurs différentiels T et N , à coefficients complexes, telle que une fonction $f : \Omega \rightarrow \mathbb{H}$ de classe C^1 est régulière si et seulement si $(N - jT)f = 0$ sur $\partial\Omega$ (j un quaternion de base de \mathbb{H}) et f est harmonique. Nous obtenons aussi une généralisation d'un résultat de Kytmanov et Aizenberg. Nous montrons qu'une fonction harmonique complexe h sur Ω ($\partial\Omega$ connexe) est holomorphe si et seulement si $\bar{\partial}_n h = \overline{aL(h)}$ sur $\partial\Omega$, où $\bar{\partial}_n$ est la composante normale de $\bar{\partial}$, L est un opérateur différentiel tangential de Cauchy–Riemann et $a \in \mathbb{C}$. **Pour citer cet article :** A. Perotti, C. R. Acad. Sci. Paris, Ser. I 337 (2003).

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1. Introduction

Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a bounded domain in \mathbb{C}^2 with boundary of class C^m , $m \geq 1$. We assume $\rho \in C^m$ on \mathbb{C}^2 and $d\rho \neq 0$ on $\partial\Omega$. Let ν denote the outer unit normal to $\partial\Omega$ and $\tau = i\nu$. For every $F \in C^1(\overline{\Omega})$, we set $\bar{\partial}_n F = \frac{1}{2}(\frac{\partial F}{\partial \nu} + i\frac{\partial F}{\partial \tau})$ (see [1], §§ 3.3 and 14.2). In a neighbourhood of $\partial\Omega$ we have the decomposition of $\bar{\partial}F$ in the tangential and the normal parts: $\bar{\partial}F = \bar{\partial}_b F + \bar{\partial}_n F \bar{\partial}\rho/|\bar{\partial}\rho|$.

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The normal part of $\bar{\partial}F$ on $\partial\Omega$ can also be expressed by means of the Hodge $*$ -operator and the Lebesgue surface measure $d\sigma$, as $\bar{\partial}_n F d\sigma = *\bar{\partial}F|_{\partial\Omega}$. We denote by L the tangential Cauchy–Riemann operator

$$L = \frac{1}{|\bar{\partial}\rho|} \left(\frac{\partial\rho}{\partial\bar{z}_2} \frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1} \frac{\partial}{\partial\bar{z}_2} \right).$$

Let \mathbb{H} be the algebra of quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k denote the basic quaternions. We identify the space \mathbb{C}^2 with the set \mathbb{H} by means of the mapping that associates to $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ the quaternion $q = z_1 + z_2j$. The commutation rule is then $aj = j\bar{a}$ for every $a \in \mathbb{C}$. We refer to [9] for the basic facts of quaternionic analysis. We will denote by D the left Cauchy–Riemann–Fueter operator

$$D = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}.$$

A quaternionic C^1 function f is (*left*-)regular on a domain Ω if $Df = 0$ on Ω . We will use also another class of regular functions, which are in the kernel of the following differential operator defined by the structural vector $\psi = \{1, i, j, -k\}$:

$$D' = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3} = 2 \left(\frac{\partial}{\partial\bar{z}_1} + j \frac{\partial}{\partial\bar{z}_2} \right).$$

A quaternionic C^1 function $f = f^1 + f^2j$, is called (*left*-) ψ -regular on a domain Ω if $D'f = 0$ on Ω . This condition is equivalent to the following system of complex differential equations for the complex components (f^1, f^2) of f :

$$\frac{\partial f^1}{\partial\bar{z}_1} = \frac{\partial \bar{f}^2}{\partial z_2}, \quad \frac{\partial f^1}{\partial\bar{z}_2} = -\frac{\partial \bar{f}^2}{\partial z_1} \tag{1}$$

or to the equation $*\bar{\partial}f^1 = -\frac{1}{2}\partial(\bar{f}^2 d\bar{z}_1 \wedge d\bar{z}_2)$. Note that the $(1, 2)$ -form $*\bar{\partial}f^1$ is ∂ -closed on Ω when f^1 is harmonic. Then, if Ω is a pseudoconvex domain in \mathbb{C}^2 , every complex harmonic function f^1 on Ω is a complex component of a ψ -regular function $f = f^1 + f^2j$ (cf. [5] and [7] for this result and its converse).

Remark 1. Any holomorphic mapping (f^1, f^2) on Ω defines a ψ -regular function $f = f^1 + f^2j$. Moreover, the complex components of a ψ -regular function are either both holomorphic or both not-holomorphic.

We refer, for instance, to [8,4] or [6] for the properties of structural vectors and ψ -regular functions. Regular and ψ -regular functions are real analytic on Ω , and they are harmonic with respect to the Laplace operator in \mathbb{R}^4 .

Remark 2. Let γ be the transformation of \mathbb{C}^2 defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if, and only if, $f \circ \gamma$ is ψ -regular on $\gamma^{-1}(\Omega)$.

2. A differential criterium for regularity

Theorem 2.1. *Let Ω be a bounded domain in \mathbb{H} , with smooth boundary of class C^1 . Let $f = f^1 + f^2j : \bar{\Omega} \rightarrow \mathbb{H}$ be a function of class C^1 . Then, f is (*left*-) ψ -regular on Ω if, and only if, f is an harmonic function on Ω and*

$$(\bar{\partial}_n - jL)f = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Proof. If f is a ψ -regular function on Ω , then from Eqs. (1) we get that it satisfies the equation $(\bar{\partial}_n - jL)f = 0$ on $\partial\Omega$, since $\bar{\partial}_n f^1 = -L(\bar{f}^2)$, $\bar{\partial}_n f^2 = L(\bar{f}^1)$ on $\partial\Omega$.

Conversely, assume that $\bar{\partial}_n f = jL(f)$ on $\partial\Omega$. For every $z \in \Omega$, it follows from the Bochner–Martinelli integral representation for complex harmonic functions on Ω (see for example [1], §1.1), that $f(z) = f^1(z) + f^2(z)j$ is equal to

$$\int_{\partial\Omega} U(\zeta, z) f^1(\zeta) + 2 \int_{\partial\Omega} g(\zeta, z) \bar{\partial}_n f^1(\zeta) \, d\sigma + \int_{\partial\Omega} U(\zeta, z) f^2(\zeta) j + 2 \int_{\partial\Omega} g(\zeta, z) \bar{\partial}_n f^2(\zeta) j \, d\sigma,$$

where $U(\zeta, z)$ is the two-dimensional Bochner–Martinelli form and $g(\zeta, z) = \frac{1}{4\pi^2} |\zeta - z|^{-2}$ is the fundamental solution of the complex Laplacian on \mathbb{C}^2 . Then we obtain

$$f(z) = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + 2 \int_{\partial\Omega} g(\zeta, z) \bar{\partial}_n f(\zeta) \, d\sigma = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + 2 \int_{\partial\Omega} g(\zeta, z) jL(f(\zeta)) \, d\sigma.$$

In [10] (see also [4]) it was shown that the two-dimensional Bochner–Martinelli form can be considered as a first complex component of the Cauchy–Fueter kernel $G'(p - q)$ associated to ψ -regular functions. Let $q = z_1 + z_2 j$, $p = \zeta_1 + \zeta_2 j$, $\sigma(q) = dx[0] - i dx[1] + j dx[2] + k dx[3]$, where $dx[k]$ denotes the product of dx_0, dx_1, dx_2, dx_3 with dx_k deleted. Then $G'(p - q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j$, where $\omega(\zeta, z)$ is the complex (1, 2)-form $\omega(\zeta, z) = -\frac{1}{4\pi^2} |\zeta - z|^{-4} ((\bar{\zeta}_1 - \bar{z}_1) d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2) d\zeta_2) \wedge d\bar{\zeta}$. From Proposition 6.3 in [4], we get that

$$f(z) = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + \int_{\partial\Omega} \omega(\zeta, z) j f(\zeta) = \int_{\partial\Omega} G'(p - q)\sigma(p) f(p).$$

The Cauchy–Fueter integral formula gives the ψ -regularity of f .

Let N and T be the differential operators, defined in a neighbourhood of $\partial\Omega$,

$$N = \frac{\partial\rho}{\partial z_1} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial\rho}{\partial \bar{z}_2} \frac{\partial}{\partial z_2}, \quad T = \frac{\partial\rho}{\partial z_2} \frac{\partial}{\partial \bar{z}_1} - \frac{\partial\rho}{\partial \bar{z}_1} \frac{\partial}{\partial z_2}.$$

T is a tangential (not Cauchy–Riemann) operator to $\partial\Omega$, while N is non-tangential, such that $N(\rho) = |\bar{\partial}\rho|^2$, $\text{Re}(N) = |\bar{\partial}\rho| \text{Re}(\bar{\partial}_n)$. The Remark 2 shows that Theorem 2.1 gives also a boundary condition for regularity of a harmonic function on Ω .

Corollary 2.2. *Let Ω be a C^1 -bounded domain in \mathbb{H} . Let $f = f^1 + f^2 j : \bar{\Omega} \rightarrow \mathbb{H}$ be a function of class C^1 . Then, f is (left-)regular on Ω if, and only if, f is harmonic on Ω and*

$$(N - jT)f = 0 \quad \text{on } \partial\Omega.$$

3. A differential criterium for holomorphicity in \mathbb{C}^2

Theorem 3.1. *Let Ω be a bounded domain in \mathbb{C}^2 , with connected boundary of class C^1 . Let $a \in \mathbb{C}$. If $h \in C^1(\bar{\Omega})$ is complex harmonic on Ω and satisfies the differential condition $\bar{\partial}_n h = aL(h)$ on $\partial\Omega$, then h is holomorphic on Ω .*

Proof. We use the same notation of the proof of Theorem 2.1. Let F^+ and F^- be the ψ -regular functions defined respectively on Ω and on $\mathbb{C}^2 \setminus \bar{\Omega}$ by the Cauchy–Fueter integral of $f = ah + hj$:

$$F^\pm(z) = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + \int_{\partial\Omega} \omega(\zeta, z) j f(\zeta).$$

From the equalities $U(\zeta, z) = -2*\partial_\zeta g(\zeta, z)$, $\omega(\zeta, z) = -\partial_\zeta(g(\zeta, z) d\bar{\zeta})$, we get that

$$\overline{F^-(z)} = -2 \int_{\partial\Omega} (\overline{ah(\zeta)} - \overline{jh(\zeta)}) * \bar{\partial}_\zeta g(\zeta, z) + \int_{\partial\Omega} (jah(\zeta) + h(\zeta)) \bar{\partial}_\zeta (g(\zeta, z) d\zeta)$$

for every $z \notin \bar{\Omega}$. By application of the Stokes' Theorem, of the complex Green formula and of the equality $\bar{\partial}h \wedge d\zeta|_{\partial\Omega} = 2L(h) d\sigma$ on $\partial\Omega$ we see that the equation $\bar{\partial}_n h = \overline{aL(h)}$ on $\partial\Omega$ implies that

$$\int_{\partial\Omega} \overline{h(\zeta)} * \bar{\partial}_\zeta g(\zeta, z) = -\frac{a}{2} \int_{\partial\Omega} h(\zeta) \bar{\partial}_\zeta (g(\zeta, z) d\zeta).$$

Therefore, F^- is a complex-valued, ψ -regular function on $\mathbb{C}^2 \setminus \bar{\Omega}$. Eqs. (1) then imply that F^- is a holomorphic function. Since $\partial\Omega$ is connected, F^- can be holomorphically continued to the whole space. Let \tilde{F}^- be such extension. Then $F = F^+ - \tilde{F}^-$ is a ψ -regular function on Ω , continuous on $\bar{\Omega}$, such that $F|_{\partial\Omega} = f|_{\partial\Omega}$. This means that the pair (ah, h) is equal on Ω to the two complex components (f^1, f^2) of a ψ -regular function. From Eqs. (1) we get that h satisfies

$$a \frac{\partial h}{\partial \bar{z}_1} = \frac{\partial h}{\partial \bar{z}_2}, \quad \bar{a} \frac{\partial h}{\partial \bar{z}_2} = -\frac{\partial h}{\partial \bar{z}_1}$$

from which it follows $\bar{\partial}h = 0$. \square

Remark 3. The result stated in Theorem 3.1 generalizes the following result of Kytmanov and Aizenberg [3], which holds in \mathbb{C}^n , $n > 1$: if h is a complex harmonic function on Ω , of class C^1 on $\bar{\Omega}$, such that $\bar{\partial}_n h = 0$ on $\partial\Omega$, then h is holomorphic. See also [2] for a similar generalization.

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