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A complete orthonormal system of divergence

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Presented by Jean-Pierre Kahane

Dedicated to Professor Daniel Waterman on the occasion of his 75th anniversary.

Abstract

A complete orthonormal system of functions $\{\Theta_n\}_{n=1}^{\infty}$, $\Theta_n \in L^{\infty}_{[0,1]}$ defined on the closed interval $[0,1]$ is constructed such that $\sum_{n=1}^{\infty} a_n \Theta_n$ diverges almost everywhere for any $\{a_n\}_{n=1}^{\infty} \notin l^2$. For the constructed system the following result is true: Any nontrivial series by the system $\{\Theta_n\}_{n=1}^{\infty}$ which converges in measure to zero diverges almost everywhere. ***To cite this article: K. Kazarian, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Un système orthonormal complet de divergence. On construit un système complet orthonormal $\{\Theta_n\}_{n=1}^{\infty}$, $\Theta_n \in L^{\infty}_{[0,1]}$ tel que $\sum_{n=1}^{\infty} a_n \Theta_n$ diverge presque partout pour n'importe quel $\{a_n\}_{n=1}^{\infty} \notin l^2$. Pour le système construit le résultat suivant est vrai : Toute série suivant le système $\{\Theta_n\}_{n=1}^{\infty}$ non triviale et qui converge en mesure vers zéro diverge presque partout. ***Pour citer cet article : K. Kazarian, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction and the main theorem

If $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal system (ONS) of functions defined on the closed interval $[0,1]$, then it is well known that, the series $\sum_{n=1}^{\infty} c_n\phi_n$ converges in $L^2_{[0,1]}$ if and only if $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$. In many questions it is of interest to study the behaviour of an orthogonal series $\sum_{n=1}^{\infty} c_n\phi_n$ when the above condition on the coefficients is not satisfied (cf. $\{c_n\}_{n=1}^{\infty} \notin l^2$). For a particular class of ONS, namely if $\{\phi_n\}_{n=1}^{\infty}$ is an ONS of independent functions such that

$$\sup_n \|\phi_n\|_{L^{\infty}_{[0,1]}} < +\infty, \qquad \int_0^1 \phi_n(t)\,\mathrm{d}t = 0 \quad \forall n \in \mathbf{N},$$

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Kolmogorov [7] has proved that $\sum_{n=1}^{\infty} c_n\phi_n$ converges in measure on $[0,1]$ if and only if $\{c_n\}_{n=1}^{\infty} \in l^2$. Note that an ONS with such a property can not be complete in $L_E^2,\ E \subset [0,1]$, if the measure of the set E is very close to one (see [5]).

Definition 1.1. Let $\{\phi_n\}_{n=1}^{\infty}$ be an ONS of functions defined on the closed interval $[0,1]$. We will say that $\{\phi_n\}_{n=1}^{\infty}$ is a divergence system if the series $\sum_{n=1}^{\infty} a_n\phi_n$ diverges almost everywhere (a.e.) for any $\{a_n\}_{n=1}^{\infty} \notin l^2$.

Definition 1.2. Let $\{\phi_n\}_{n=1}^{\infty}$ be an ONS of functions defined on the closed interval $[0,1]$. We will say that $\{\phi_n\}_{n=1}^{\infty}$ is a divergence system in the weak sense if for any $\{a_n\}_{n=1}^{\infty} \notin l^2$ the series $\sum_{n=1}^{\infty} a_n\phi_n$ diverges on a set of positive measure.

Recall that an ONS of functions defined on the closed interval $[0,1]$ $\{\phi_n\}_{n=1}^{\infty}$ is called a system of convergence if for any $\{c_n\}_{n=1}^{\infty} \in l^2$ the series $\sum_{n=1}^{\infty} c_n\phi_n$ converges almost everywhere. The existence of a complete ONS (CONS) which is a divergence system in the weak sense was proved by Kashin [3,4]. Problems about the existence of complete ONS which are divergence system or divergence system in the weak sense have been posed by Ulyanov ([10], p. 17]). Precisely, he asked if there exists a complete ONS of convergence which is a divergence system (divergence system in the weak sense). The most interesting part in these problems is the question about the existence of a complete ONS which is a divergence system (divergence system in the weak sense). That is why it is not strange that in the Kashin's original article and also in the book written jointly with Saakian it is given only the construction of a CONS which is a divergence system in the weak sense and there are given some indications how to modify the construction to make it also a system of convergence. We construct a complete orthonormal system $\{\Theta_n\}_{n=1}^{\infty}$ defined on the closed interval $[0,1]$ and prove that the following result holds.

Main Theorem. *There exists a complete ONS* $\{\Theta_n\}_{n=1}^{\infty},\ \Theta_n \in L_{[0,1]}^{\infty}$ *of functions defined on* $[0,1]$ *which is a divergence system.*

We prefer not to give any indication by which one can modify our construction to obtain a complete solution of Ulyanov's problem. A forthcoming paper will be dedicated to that question.

2. Some corollaries

One can formulate several corollaries of the above theorem in the theory of representation of functions by series. We proceed to formulate the principal question in this theory. Let $\{f_n\}_{n=1}^{\infty}$ be a system of functions from a linear topological space $\mathbf{F}_0$ such that for any $f \in \mathbf{F}_0$ there exists a sequence of scalars $\{a_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} a_n f_n = f$ in the topology of $\mathbf{F}_0$ or in some other sense (convergence almost everywhere, convergence in measure, etc.) and suppose $\mathbf{F}_1 \supset \mathbf{F}_0$ is another linear topological space with weaker topology. Then the main question is: can any $g \in \mathbf{F}_1$ be represented by some series $\sum_{n=1}^{\infty} b_n f_n$ which converges in the topology of $\mathbf{F}_1$ or in another sense (convergence almost everywhere, convergence in measure, convergence on sets of positive measure, etc.). The history of representation of functions by series goes back to Bernoulli, Euler, Fourier and many others whose works give rise to contemporary harmonic analysis. The development of the Lebesgue theory of measure gave a new impulse to the theory of representation of functions allowing Luzin to study the questions of summability of trigonometric series to any given measurable function almost everywhere and to pose several open problems which afterwards have been studied by himself, Privalov, Meñshov, Talalyan, Ulyanov and many others. In the late fifties Talayan [9] proved the first result about the general complete ONS (see [6] for more recent results).

Theorem 2.1 (A.A. Talalyan). *Let* $\{f_n\}_{n=1}^{\infty}$ *be a complete ONS of functions defined on* $[0,1]$. *Then for any measurable function* f *defined on* $[0,1]$ *there exists a sequence of scalars* $\{a_n\}_{n=1}^{\infty}$ *such that* $\sum_{n=1}^{\infty} a_n f_n = f$, *where the series converges in measure.*

Observe that when $\{f_n\}_{n=1}^{\infty}$ is a complete ONS then we automatically have that any function $g \in L^2_{[0,1]}$ is represented by a series with respect to the system that converges to g in the $L^2_{[0,1]}$-norm and, consequently, in measure. We will say that a series $\sum_{n=1}^{\infty} b_n f_n$ is nontrivial if $b_k \neq 0$ for some $k \in \mathbf{N}$. The proof of Talalyan's theorem remain valid if one discards any finite number of elements from the complete ONS $\{f_n\}_{n=1}^{\infty}$. Thus it follows that for any complete ONS there exist nontrivial series $\sum_{n=1}^{\infty} b_n f_n$ which converge to zero in measure. The last comment permits us to formulate the following corollary of the Main Theorem.

Corollary 2.2. *Any nontrivial series by the system $\{\Theta_n\}_{n=1}^{\infty}$ which converges in measure to zero diverges almost everywhere.*

We will not formulate here several other corollaries asserting that the system $\{\Theta_n\}_{n=1}^{\infty}$ is not a representation system for the classes $L^r_{[0,1]}$, $0 \leqslant r < 1$ if we want to represent the functions from those classes by a series with respect to the system $\{\Theta_n\}_{n=1}^{\infty}$ which converges (or is summable for some methods) pointwise on sets of positive measure, even if those sets depend on the function.

The following corollary is related with the modification of functions.

Corollary 2.3. *For any $Q \subset [0,1]$, $|Q| > 0$ and any $\phi \in L^p_{[0,1]}$, $1 \leqslant p < 2$ such that $\phi|_Q \notin L^2_Q$ we have that the Fourier series with respect to the system $\{\Theta_n\}_{n=1}^{\infty}$ of any function $g \in L^1_{[0,1]}$, $g(x) = \phi(x)$ if $x \in Q$ diverges almost everywhere on* $[0,1]$.

The above result shows that the constructed system can serve also as an counterexample if one considers the possibility of generalization for the complete ONS of the following (see [1])

Theorem 2.4 (D.E. Meñshov). *Let $f \in L^1_{[0,1]}$ and $P \subset [0,1]$ is any perfect nowhere dense set. Then there exists $g \in L^1_{[0,1]}$, $g(x) = f(x)$ if $x \in P$ such that the Fourier series of the function g with respect to the trigonometric system $\{e^{2\pi i n x}\}_{n=-\infty}^{+\infty}$ converges almost everywhere.*

3. Main tools of the construction

In the construction of the system $\{\Theta_n\}_{n=1}^{\infty}$ we have used the properties of the collections of the Meñshov functions. Let $[a,b] \subset \mathbf{R}$ and $k \in \mathbf{N}$, where $\mathbf{N}$ is the set of the natural numbers. We denote by $\mathcal{E}^k_{[a,b]}$ the inner-product space of the step functions

$$\mathcal{E}^k_{[a,b]} = \left\{ f\colon\ f(x) = a_i \text{ if } x \in \left(a + (i-1)\frac{b-a}{2^k}, a + i\frac{b-a}{2^k} \right),\ 1 \leqslant i \leqslant 2^k \right\},$$

where the inner product is defined in the same way as in $L^2_{[a,b]}$. The Meñshov function $M_k \in \mathcal{E}^{k+1}_{[-1,1]}$, for any natural $k \geqslant 3$, is an odd 2-periodic function on the real line $\mathbf{R}$ defined by the following equations:

$$M_k(x) = \begin{cases} \frac{1}{8i} \cdot 2^{k/2}, & \text{if } x \in \left(\dfrac{i-1}{2^k}, \dfrac{i}{2^k} \right),\ 1 \leqslant |i| \leqslant 2^k - 1, \\ 0, & \text{if } x \in \left(-\dfrac{1}{2^k}, 0 \right) \cup \left(1 - \dfrac{1}{2^k}, 1 \right). \end{cases}$$

Denote $M_{k,i}(x) = M_k(x - i \cdot 2^{-k})$ $\forall i \in \mathbf{N}$ and let

$$S^*_k(\mathbf{a}, \omega)(x) = \sup_{0 \leqslant j \leqslant 2^k - 1} \left| \sum_{i=0}^{j} a_i \left(M_{k,i}(x) + \omega \right) \right|,$$

where $\mathbf{a} = \{a_i\}_{i=0}^{2^k-1}$ is a set of 2^k real numbers and $\omega \in \mathbf{R}$. Then the following lemmas are true.

Lemma 3.1. *For any $k \geqslant 2$ and any $\mathbf{a} = \{a_i\}_{i=0}^{2^k-1}$*

$$\int_{-1}^{1} S_k^*(\mathbf{a},\omega)(x)\,\mathrm{d}x \geqslant \frac{1}{6}\big(k+|\omega|\big)2^{-k/2}\left|\sum_{i=0}^{2^k-1} a_i\right|.$$

Lemma 3.2. *For any $k \in \mathbf{N}$ there exist an orthonormal system $\{f_k^i\}_{i=0}^{2^k-1}$ in $L^2_{[-2,2]}$ such that*

$$f_k^i(x) = \begin{cases} M_{k,i}(x), & \text{if } x \in [-1,1], \\ 0, & \text{if } x \in [-2,-1), \end{cases} \qquad \int_{-2}^{2} f_k^i(x)\,\mathrm{d}x = 0 \quad \big(0 \leqslant i \leqslant 2^k-1\big),$$

and $f_k^i|_{[1,2]} \in \mathcal{E}_{[1,2]}^{k+1}$.

For the proof of the main theorem we use also the following

Lemma 3.3. *Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal system of independent functions on $[0,1]$ such that*

$$\int_{[0,1]} \phi_i(t)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_{[0,1]} \big|\phi_i(t)\big|\,\mathrm{d}t > \alpha > 0 \quad \text{for all } i \in \mathbf{N}.$$

Then there exist $C_l > 0$, $l = 1,2$ such that $|\{x \in [0,1]\colon\ |\sum_{i=1}^{k} a_i\phi_i(x)| \geqslant C_1(\sum_{i=1}^{k} a_i^2)^{1/2}\}| \geqslant C_2$, for any collection of numbers $\{a_i\}_{i=1}^{k} \subset \mathbf{R}$.

We prove Lemma 3.3 using a lemma from [2] (see p. 8) and the following lemma which should be attributed to Pisier [8] (see also [2], p. 22) where the proof is given for the L^p-norm, $p > 1$.

Lemma 3.4. *Let $\{F_i\}_{i=1}^{\infty}$ be a system of symmetric independent functions defined on a measurable set $G \subset \mathbf{R}^n$, $|G|_n = 1$ and $\int_G |F_i(\mathbf{x})|\,\mathrm{d}\mathbf{x} \geqslant \alpha \geqslant 0$ $\forall i \in \mathbf{N}$. Then $\int_G |\sum_{i=1}^{k} a_i F_i(\mathbf{x})|\,\mathrm{d}\mathbf{x} \geqslant \alpha \int_G |\sum_{i=1}^{k} a_i r_i(t)|\,\mathrm{d}t$, for any collection of numbers $\{a_i\}_{i=1}^{k} \subset \mathbf{R}$.*

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