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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 253–257
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A variant of Poincaré's inequality

Augusto C. Ponce [a,b,1]

[a] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, BC 187, 4, pl. Jussieu, 75252 Paris cedex 05, France*
[b] *Rutgers University, Dept. of Math., Hill Center, Busch Campus, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA*

Received and accepted 20 June 2003

Presented by Haïm Brézis

Abstract

We show that if $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded Lipschitz domain and $(\rho_n) \subset L^1(\mathbb{R}^N)$ is a sequence of nonnegative radial functions weakly converging to δ_0 then there exist $C > 0$ and $n_0 \geqslant 1$ such that

$$\int_\Omega \left| f - \fint_\Omega f \right|^p \leqslant C \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(\Omega)\ \forall n \geqslant n_0. \tag{1}$$

The above estimate was suggested by some recent work of Bourgain, Brezis and Mironescu (in: Optimal Control and Partial Differential Equations, IOS Press, 2001, pp. 439–455). As $n \to \infty$ in (1) we recover Poincaré's inequality. We also extend a compactness result of Bourgain, Brezis and Mironescu. ***To cite this article: A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Une variante de l'inégalité de Poincaré. Soit $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, un domaine lipschitzien borné. Étant donnée une suite de fonctions radiales positives $(\rho_n) \subset L^1(\mathbb{R}^N)$ qui converge vers la masse de Dirac δ_0 on montre qu'il existe $C > 0$ et $n_0 \geqslant 1$ tels que

$$\int_\Omega \left| f - \fint_\Omega f \right|^p \leqslant C \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(\Omega)\ \forall n \geqslant n_0. \tag{2}$$

Cette estimation a été motivée par un travail récent de Bourgain, Brezis et Mironescu (dans : Optimal Control and Partial Differential Equations, IOS Press, 2001, pp. 439–455). En prenant la limite dans (2) lorsque $n \to \infty$, on retrouve l'inégalité de Poincaré. On généralise aussi un théorème de compacité de Bourgain, Brezis et Mironescu. ***Pour citer cet article : A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

E-mail address: ponce@ann.jussieu.fr, augponce@math.rutgers.edu (A.C. Ponce).
[1] Supported by CAPES, Brazil, under grant no. BEX1187/99-6.

1631-073X/$ – see front matter
doi:10.1016/S1631-073X(03)00313-3

Version française abrégée

Soient $\Omega \subset \mathbb{R}^N,\ N \geqslant 2$, un ouvert connexe borné au bord lipschitzien et $1 \leqslant p < \infty$. Sous ces conditions, on a l'inégalité de Poincaré suivante :

$$\int_\Omega \left| f - \fint_\Omega f \right|^p \leqslant A_0 \int_\Omega |Df|^p \quad \forall f \in W^{1,p}(\Omega), \tag{3}$$

avec une constante A_0 qui dépend de p et de Ω.

D'autre part, soit $(\rho_n) \subset L^1(\mathbb{R}^N)$ une suite de fonctions *radiales* qui vérifient (8). Alors, on peut montrer que pour une certaine constante $K_{p,N}$ on a (voir [2,5])

$$\lim_{n\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y = K_{p,N} \int_\Omega |Df|^p \quad \forall f \in W^{1,p}(\Omega). \tag{4}$$

Motivés par (3) et (4), on établi le théorème suivant (voir [3,4] pour des cas particuliers) :

Théorème 0.1. *Étant donné $\delta > 0$, il existe $n_0 \geqslant 1$ tel que*

$$\int_\Omega \left| f - \fint_\Omega f \right|^p \leqslant \left(\frac{A_0}{K_{p,N}} + \delta \right) \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(\Omega)\ \forall n \geqslant n_0. \tag{5}$$

L'inégalité *(5)* est en fait une conséquence du résultat de compacité suivant :

Théorème 0.2. *Soit $(f_n) \subset L^p(\Omega)$ une suite bornée. On suppose qu'il existe $B > 0$ tel que*

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant B, \quad \forall n \geqslant 1. \tag{6}$$

Alors (f_n) est relativement compacte dans $L^p(\Omega)$.

Soit (f_{n_j}) une sous-suite qui converge vers f dans $L^p(\Omega)$. Alors : (a) $f \in W^{1,p}(\Omega)$ *si* $1 < p < \infty$; (b) $f \in BV(\Omega)$ *si* $p = 1$. *En plus,* $\int_\Omega |Df|^p \leqslant B/K_{p,N}$.

Ce théorème a été démontré par Bourgain, Brezis et Mironescu [2] en supposant que les fonctions ρ_n sont radiales décroissantes.

En ce qui concerne le cas $N = 1$ et $\Omega =]0, 1[$, alors (3) et (4) sont toujours vraies. En revanche, il faut imposer une condition supplémentaire sur les fonctions ρ_n pour que les Théorèmes 0.1 et 0.2 restent valables (voir [7] et aussi [2, Contre-exemple 1]).

Les démonstrations détaillées sont présentées dans [7].

1. Introduction and main results

Assume $\Omega \subset \mathbb{R}^N,\ N \geqslant 2$, is a bounded domain with Lipschitz boundary and let $1 \leqslant p < \infty$. It is a well-known fact that there exists a constant $A_0 = A_0(p, \Omega) > 0$ such that the following form of Poincaré's inequality holds:

$$\int_\Omega \left| f - \fint_\Omega f \right|^p \leqslant A_0 \int_\Omega |Df|^p \quad \forall f \in W^{1,p}(\Omega). \tag{7}$$

On the other hand, let $(\rho_n) \subset L^1(\mathbb{R}^N)$ be a sequence of *radial* functions satisfying

$$\rho_n \geqslant 0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_n = 1 \quad \forall n \geqslant 1, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N \setminus B_\delta} \rho_n = 0 \quad \forall \delta > 0. \tag{8}$$

In this case, we have the following pointwise limit (see [2], see also [5] for a simpler proof)

$$\lim_{n\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y = K_{p,N}\int_\Omega |Df|^p \quad \forall f \in W^{1,p}(\Omega), \tag{9}$$

where $K_{p,N} = ⨍_{S^{N-1}} |e_1 \cdot \sigma|^p \,\mathrm{d}\sigma$.

Motivated by this, we show the following estimate related to (7):

Theorem 1.1. *Given $\delta > 0$, there exists $n_0 \geqslant 1$ sufficiently large such that*

$$\int_\Omega \left| f - ⨍_\Omega f \right|^p \leqslant \left(\frac{A_0}{K_{p,N}} + \delta \right) \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(\Omega)\ \forall n \geqslant n_0. \tag{10}$$

The choice of $n_0 \geqslant 1$ depends not only on $\delta > 0$, but also on p, Ω and on the sequence $(\rho_n)_{n\geqslant 1}$. Special cases of this inequality have been used in the study of the Ginzburg–Landau model (see [3,4]; see also Corollaries 2.1–2.4 below).

We first point out that (10) is stronger than (7), in the sense that the right-hand side of (10) can be always estimated by $\int_\Omega |Df|^p$. In fact, given $f \in W^{1,p}(\Omega)$, we first extend f to $\mathbb{R}^N$ so that $f \in W^{1,p}(\mathbb{R}^N)$. It is then easy to see that (see, e.g., [2, Theorem 1])

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\mathbb{R}^N} |Df|^p \leqslant C\int_\Omega |Df|^p. \tag{11}$$

If $N = 1$ and $\Omega = (0,1)$, then both (7) and (9) still hold. However, it is possible to construct examples of sequences $(\rho_n) \subset L^1(\mathbb{R})$ for which (10) fails (see [2, Counterexample 1]). In this case, we need to impose an additional condition on (ρ_n) (see [7] and also Remark 1 below).

Theorem 1.1 can be deduced from the following compactness result:

Theorem 1.2. *If $(f_n) \subset L^p(\Omega)$ is a bounded sequence such that*

$$\int_\Omega\int_\Omega \frac{|f_n(x)-f_n(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant B \quad \forall n \geqslant 1, \tag{12}$$

then (f_n) is relatively compact in $L^p(\Omega)$. Assume that $f_{n_j} \to f$ in $L^p(\Omega)$. Then

(a) $f \in W^{1,p}(\Omega)$ *if* $1 < p < \infty$;
(b) $f \in BV(\Omega)$ *if* $p = 1$.

In both cases, we have $\int_\Omega |Df|^p \leqslant B/K_{p,N}$, where B is given by (12).

This result was already known under the additional assumption that ρ_n is radially nondecreasing for every $n \geqslant 1$ (see [2, Theorem 4]).

Detailed proofs of Theorems 1.1 and 1.2 will appear in [7].

2. Some examples

We now state some inequalities arising from Theorem 1.1. We denote by $Q=(0,1)^N$ the N-dimensional unit cube and by $\sigma_n = |S^{N-1}|$ the $(N-1)$-Hausdorff measure of S^{N-1}.

For every $N \geqslant 2$ we then have the following corollaries:

Corollary 2.1 (Bourgain, Brezis and Mironescu [3]). *For every* $0 < s_0 < s < 1$,

$$\int_Q \left| f - \int_Q f \right|^p \leqslant C_{s_0}(1-s)p \iint_{Q\,Q} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(Q). \tag{13}$$

Proof. We simply apply Theorem 1.1 with $\rho_n(h) = \frac{1}{\sigma_N}\frac{1-s_n}{|h|^{N-(1-s_n)p}}$ $\forall h \in B_1$, where (s_n) is any sequence such that $s_n \uparrow 1$.

This inequality takes into account the correction factor $(1-s)^{1/p}$ we should put in front of the Gagliardo seminorm $|f|_{W^{s,p}}$ as $s \uparrow 1$. In [3], the authors study related estimates arising from the Sobolev imbedding $L^q \hookrightarrow W^{s,p}$ for the critical exponent $\frac{1}{q} = \frac{1}{p} - \frac{s}{N}$; see also [6] for a more elementary approach.

Corollary 2.2 (Bourgain, Brezis and Mironescu [4]). *For every* $0 < \varepsilon < \varepsilon_0$,

$$\int_Q \left| f - \int_Q f \right|^p \leqslant C_{\varepsilon_0}\frac{1}{|\log \varepsilon|} \iint_{Q\,Q} \frac{|f(x)-f(y)|^p}{|x-y|^p}\frac{\mathrm{d}x\,\mathrm{d}y}{(|x-y|+\varepsilon)^N} \quad \forall f \in L^p(Q). \tag{14}$$

Proof. This follows from Theorem 1.1 with $\rho_n(h) = \frac{1}{\sigma_N|\log\varepsilon_n|}\frac{1}{(|h|+\varepsilon_n)^N}$ $\forall h \in B_1$ where $\varepsilon_n \downarrow 0$.

We observe that in the two previous cases the functions ρ_n are radially decreasing. A stronger form of this last inequality is the following

Corollary 2.3. *For every* $0 < \varepsilon < \varepsilon_0 \ll 1$,

$$\int_Q \left| f - \int_Q f \right|^p \leqslant C_{\varepsilon_0}\frac{1}{|\log \varepsilon|} \iint_{\substack{Q\,Q\\ |x-y|>\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(Q). \tag{15}$$

Proof. For any sequence $\varepsilon_n \downarrow 0$ we take

$$\rho_n(h) = 0 \quad \text{if } |h| \leqslant \varepsilon_n \quad \text{and} \quad \rho_n(h) = \frac{1}{\sigma_N|\log\varepsilon_n|}\frac{1}{|h|^N} \quad \text{if } \varepsilon_n < |h| \leqslant 1. \tag{16}$$

We have been informed by H. Brezis that Bourgain and Brezis [1] have proved that

$$\int_Q \left| f - \int_Q f \right|^p \leqslant C_{\varepsilon_0}\frac{1}{|\log \varepsilon|} \iint_{Q\,Q} \frac{|f(x)-f(y)|^p}{(|x-y|+\varepsilon)^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(Q), \tag{17}$$

for every $0 < \varepsilon < \varepsilon_0$, using a Paley–Littlewood decomposition of f. Note that this estimate can be deduced instead from the corollary above.

Here is another example

Corollary 2.4. *For every* $0 < \varepsilon < \varepsilon_0$,

$$\int_Q \left| f - \int_Q f \right|^p \leqslant C_{\varepsilon_0} \frac{N+p}{\varepsilon^{N+p}} \iint_{\substack{Q\,Q \\ |x-y|<\varepsilon}} \left| f(x) - f(y) \right|^p \mathrm{d}x\,\mathrm{d}y \quad \forall f \in L^p(Q). \tag{18}$$

Proof. We use Theorem 1.1 applied to

$$\rho_n(h) = \frac{1}{\sigma_N} \frac{N+p}{\varepsilon_n^{N+p}} |h|^p \quad \text{if } |h| < \varepsilon_n \quad \text{and} \quad \rho_n(h) = 0 \quad \text{if } |h| \geqslant \varepsilon_n. \tag{19}$$

Concerning the behavior of the constants in these inequalities, let A_0 denote the best constant in (7). Then in Corollary 2.1 the constant C_{s_0} can be chosen so that C_{s_0} tends to $\frac{A_0}{K_{p,N}\sigma_N}$ as $s_0 \uparrow 1$. Similarly, in Corollaries 2.2–2.4 we have C_{ε_0} converging to the same limit as $\varepsilon_0 \downarrow 0$.

Remark 1. In [7] we show that Corollaries 2.1–2.4 still hold when $N = 1$ and $\Omega = (0, 1)$.

Applying Theorem 1.1 to $p = 1$ and $f = \chi_E$, where $E \subset Q$ is any measurable set, we get (see also [3] for related results):

Corollary 2.5. *Let* $N \geqslant 2$. *Given a sequence of radial functions* $(\rho_n) \subset L^1(\mathbb{R}^N)$ *satisfying* (8), *then for any* $C > A_0/K_{1,N}$ *there exists* $n_0 \geqslant 1$ *such that*

$$|E||Q \setminus E| \leqslant C \int_E \int_{Q\setminus E} \frac{\rho_n(|x-y|)}{|x-y|} \mathrm{d}x\,\mathrm{d}y \quad \forall E \subset Q \textit{ measurable } \forall n \geqslant n_0. \tag{20}$$

Acknowledgements

The author is grateful to H. Brezis for bringing this problem to his attention. He would also like to thank H. Brezis for extremely interesting discussions. Part of this work was done during a visit to ICTP and SISSA in Trieste, Italy; the author acknowledges the invitation and hospitality of both institutions.

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