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Topology

Twisted unknots

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Abstract

Let K be a knot in the 3-sphere S^3, and D a disk in S^3 meeting K transversely in the interior. For non-triviality we assume that $|D \cap K| \geqslant 2$ over all isotopies of K in $S^3 - \partial D$. Let $K_{D,n} (\subset S^3)$ be the knot obtained from K by n twisting along the disk D. If the original knot is unknotted in S^3, we call $K_{D,n}$ a *twisted unknot*. We describe for which pairs (K, D) and integers n, the twisted unknot $K_{D,n}$ is a torus knot, a satellite knot or a hyperbolic knot. ***To cite this article: M. Aït Nouh et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Nœuds twistés Soient K un nœud dans la 3-sphère S^3, et D un disque dans S^3 rencontrant K transversalement dans son intérieur. Pour des raisons de non-trivialité, on peut supposer que $|D \cap K| \geqslant 2$ pour toutes les isotopies de K dans $S^3 - \partial D$. Soit $K_{D,n}$ le nœud de S^3 obtenu en effectuant n twists sur K le long du disque D. Si le nœud original K n'est pas noué dans S^3, on dit que $K_{D,n}$ est un *nœud twisté*. Nous décrivons les paires (K, D) et les entiers n, pour lesquels le nœud twisté $K_{D,n}$ est un nœud torique, satellite, ou hyperbolique. ***Pour citer cet article : M. Aït Nouh et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

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Soient K un nœud dans la 3-sphère S^3, et D un disque dans S^3 rencontrant K transversalement dans son intérieur. On suppose que $|D \cap K| \geqslant 2$ et minimal pour toutes les isotopies de K dans $S^3 - \partial D$. Nous appelons D *disque de twist* pour K. Soit $K_{D,n}$ le nœud de S^3 obtenu en effectuant n twists sur K le long du disque D. Si le nœud original K n'est pas noué dans S^3, on dit que (K, D) est une *paire de twist* et que $K_{D,n}$ est un *nœud twisté* (voir la Fig. 1).

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Par la décomposition des 3-variétés en tores de Jaco–Shalen Johanson [9,10] et le théorème d'uniformisation de Thurston [15,22] un nœud dans la 3-sphère est un nœud torique, satellite (i.e. son extèrieur contient un tore incompressible, et non parallèle au bord), ou hyperbolique (i.e. son complément admet une structure hyperbolique complète de volume fini).

Le but de cette Note est de donner une description des paires (K, D) et des entiers n, pour lesquels le nœud twisté $K_{D,n}$ est un nœud torique, satellite ou hyperbolique.

Pour toute paire de twist (K, D), l'extèrieur $S^3 - \operatorname{int} N(K \cup \partial D)$ est irréductible et bord-irréductible. Par conséquent, $S^3 - \operatorname{int} N(K \cup \partial D)$ est un espace fibré de Seifert, toroidal ou hyperbolique. Nous dirons qu'une paire de twist (K, D) est de type *de Seifert*, *toroidal* ou *hyperbolique* si $S^3 - \operatorname{int} N(K \cup \partial D)$ est fibrée de Seifert, toroidale ou hyperbolique, respectivement.

Théorème 0.1. *Soit (K, D) une paire de twist.*

(1) *Si (K, D) est une paire de twist de type hyperbolique, alors $K_{D,n}$ est un nœud hyperbolique, pour tout entier n vérifiant $|n| > 1$.*
(2) *Si (K, D) est une paire de twist de type de Seifert, alors $K_{D,n}$ est un nœud torique, pour tout entier relatif n.*
(3) *Si (K, D) est une paire de twist de type toroidal, alors $K_{D,n}$ est un nœud satellite pour tout entier non nul n, sauf si (K, D) est une paire décrite en Fig.* 2(1) (*resp.* (2))*, où $V - \operatorname{int} N(K)$ est un espace fibré de Seifert ou hyperbolique, et $n = -1$ (resp. $n = 1$).*
(4) *Supposons que (K, D) est une paire décrite en Fig.* 2(1) (*resp.* (2))*. Si $V - \operatorname{int} N(K)$ est un espace fibré de Seifert, alors $K_{D,-1}$ (resp. $K_{D,1}$) est un nœud torique. Si $V - \operatorname{int} N(K)$ est hyperbolique, alors $K_{D,-1}$ (resp. $K_{D,1}$) est un nœud hyperbolique.*

On peut noter qu'il existe des exemples correspondants au (1) du Théorème 0.1 avec $|n| = 1$, tels que les nœuds $K_{D,\pm 1}$ ne sont pas hyperboliques. Par exemple, dans la Fig. 1, (K, D) est une paire de type hyperbolique, mais $K_{D,1}$ est un nœud de trèfle. Dans [3], [23, p. 2293], se trouvent d'autres exemples de paire de twist (K, D) de type hyperbolique telle que $K_{D,1}$ ou $K_{D,-1}$ soit un nœud torique. Les détails de la preuve du Théorème 0.1 se trouvent dans [2].

1. Introduction

Let K be a knot in the 3-sphere S^3 and D a disk in S^3 meeting K transversely in the interior. We assume that $|D \cap K|$ is greater than one and minimal over all isotopies of K in $S^3 - \partial D$. We call such a disk D a *twisting disk* for K. Let $K_{D,n} (\subset S^3)$ be a knot obtained from K by n twisting along the disk D, in other words, $-\frac{1}{n}$-surgery on the trivial knot ∂D. In particular, if K is a trivial knot in S^3, then we call (K, D) a *twisting pair* and call $K_{D,n}$ a *twisted unknot*, see Fig. 1.

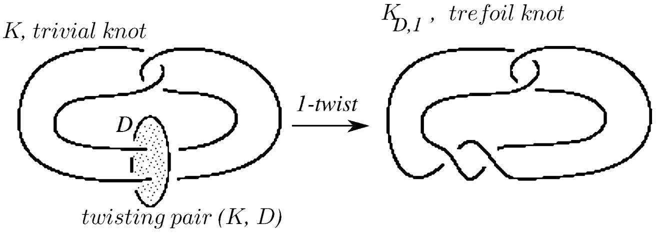

Fig. 1. Left: knot K; right: a twisted unknot.

Fig. 1. Gauche : le nœud K ; droite : on nœud twisté.

Let $\mathcal{K}$ be the set of all knots in S^3 and $\mathcal{K}_1$ the set of all twisted unknots. In [18, Theorem 4.1] Ohyama demonstrated that each knot in $\mathcal{K}_2 = \mathcal{K} - \mathcal{K}_1$ can be obtained from a trivial knot by twistings along exactly two properly chosen disks.

Following Thurston's uniformization theorem [15,22] and the torus theorem [9,10], every knot in the 3-sphere is a torus knot, a satellite knot (i.e., a knot whose exterior contains non-boundary-parallel, incompressible tori), or a hyperbolic knot (i.e., a knot whose complement admits a complete hyperbolic structure of finite volume). The purpose in the present paper is describing for which pairs (K, D) and integers n, a twisted unknot $K_{D,n}$ is a torus knot, a satellite knot or a hyperbolic knot.

For any twisting pair (K, D), the exterior $S^3 - \operatorname{int} N(K \cup \partial D)$ is irreducible and boundary-irreducible. It follows from Thurston's uniformization theorem [15,22] and the torus theorem [9,10] that $S^3 - \operatorname{int} N(K \cup \partial D)$ is Seifert fibered, toroidal or hyperbolic. We say that a twisting pair (K, D) is *Seifert fibered, toroidal* or *hyperbolic* if $S^3 - \operatorname{int} N(K \cup \partial D)$ is Seifert fibered, toroidal or hyperbolic, respectively.

Then our result can be stated as follows.

Theorem 1.1. *Let (K, D) be a twisting pair.*

(1) *If (K, D) is a hyperbolic pair, then $K_{D,n}$ is a hyperbolic knot for any integer n with $|n| > 1$.*
(2) *If (K, D) is a Seifert fibered pair, then $K_{D,n}$ is a torus knot for any integer n.*
(3) *If (K, D) is a toroidal pair, then $K_{D,n}$ is a satellite knot for any non-zero integer n unless (K, D) is a pair shown in Fig.* 2(1) (*resp.*(2)), *where $V - \operatorname{int} N(K)$ is Seifert fibered or hyperbolic, and $n = -1$* (*resp. $n = 1$*).
(4) *Suppose that (K, D) is a pair shown in Fig.* 2(1) (*resp.* (2)). *If $V - \operatorname{int} N(K)$ is Seifert fibered, then $K_{D,-1}$* (*resp. $K_{D,1}$*) *is a torus knot. If $V - \operatorname{int} N(K)$ is hyperbolic, then $K_{D,-1}$* (*resp. $K_{D,1}$*) *is a hyperbolic knot.*

Note that in Theorem 1.1 (1) with $|n| = 1$, the knot $K_{D,\pm 1}$ may be non-hyperbolic: see Examples 1 and 2.

Example 1 (*Producing torus knots from hyperbolic pairs*). In Fig. 1, (K, D) is a hyperbolic pair, but $K_{D,1}$ is a trefoil knot. In [3], [23, p. 2293], we find other examples of hyperbolic pairs (K, D) such that $K_{D,1}$ or $K_{D,-1}$ is a torus knot.

Example 2 (*Producing satellite knots from hyperbolic pairs*). In Fig. 3(1) $K_{D,1}$ is a connected sum of two torus knots [16]; we find other examples of composite twisted unknots in [4,21].

In Fig. 3(2), (K, D) is a hyperbolic pair [13], but $K_{D,1}$ is a $(23, 2)$-cable of a $(4, 3)$-torus knot [3,23].

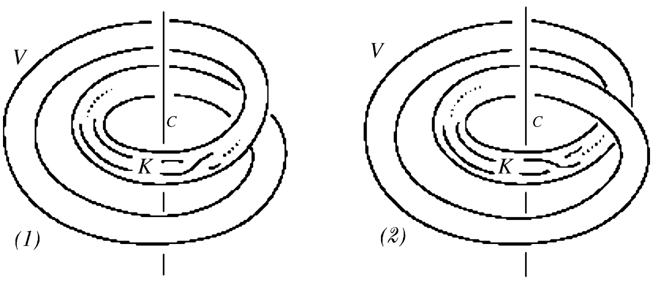

Fig. 2. A toroidal pair.

Fig. 2. Une paire de twist de type toroidal.

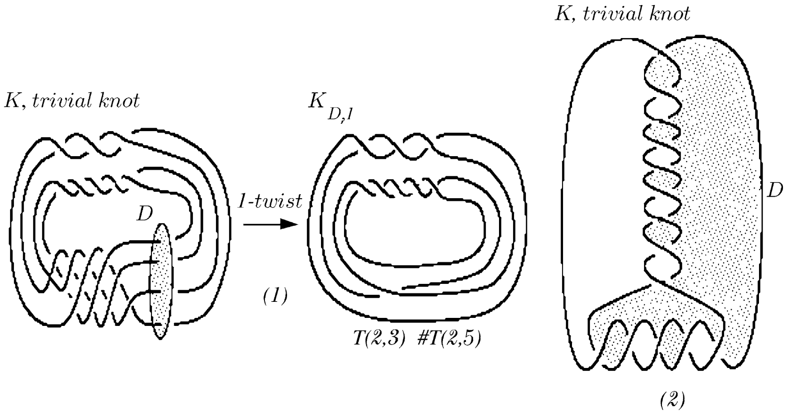

Fig. 3. Left: Connected sum of two torus knots; right: a (23, 2)-cable of a (4, 3)-torus knot.
Fig. 3. Gauche : la somme de deux nœuds toriques ; droite : un cable.

2. Proof of Theorem 1.1

(1) Assume that (K, D) is a hyperbolic twisting pair. It follows from [17, Theorem 3.8], [14] that $K_{D,n}$ cannot be a torus knot for any non-zero integer n.

Let us assume that $K_{D,n}$ is a satellite knot. Set $c = \partial D$, $V = S^3 - \operatorname{int} N(K)$. Then c is contained in the interior of V. The manifold obtained from V by Dehn surgery on a knot c with slope γ is denoted by $V(c;\gamma)$. Since $V \subset S^3$, we can parametrize slopes γ of c by $r \in \mathbb{Q} \cup \{\frac{1}{0}\}$, using the preferred meridian-longitude pair of $c \subset S^3$. We thus write $V(c;r)$ for $V(c;\gamma)$.

Recall that $E(K_{D,n}) = S^3 - \operatorname{int} N(K_{D,n})$ can be regarded as $V(c; -\frac{1}{n})$. Since $K_{D,n}$ is a satellite knot, $V(c; -\frac{1}{n})$ contains an essential torus T. Let D_V be a meridian disk of $V = V(c; \frac{1}{0})$. Assume that $|D_V \cap c|$ and $|T \cap c^*|$ is minimal, where c^* denotes the core of the filled solid torus. Further we may assume that the punctured surfaces $D_V - \operatorname{int} N(c) \subset V - \operatorname{int} N(c)$ and $T - \operatorname{int} N(c^*) \subset V - \operatorname{int} N(c)$ intersect transversely. Then as usual we obtain graphs G_{D_V} and G_T on D_V and T respectively. Analyzing these graphs, Gordon and Luecke have shown in [8, Corollary A.2] that the surgery is integral, i.e., $|n| = 1$, or $E(K_{D,n}) = V(c; -\frac{1}{n})$ is a union of two Seifert fiber spaces. In the latter case $K_{D,n}$ is a graph knot; since (K, D) is a hyperbolic pair, it follows from [1, Proposition 4.1] that $|n| = 1$.

Remark. In [8, Corollary A.2], Gordon and Luecke have shown the above result in more general setting in the sense that they do not assume the triviality of c in S^3 and consider not only $-\frac{1}{n}$-surgery but also general surgeries. In [2] we also gave a slightly different proof using graph theoretical arguments developed in [5–8].

(2) Assume that (K, D) is a Seifert fibered pair. Then it turns out that $S^3 - \operatorname{int} N(\partial D)$ is a $(1, p)$-fibered solid torus in which K is a regular fiber. Thus $K_{D,n}$ is a $(1 + np, p)$-torus knot in S^3.

(3) Let T be an essential torus in $S^3 - \operatorname{int} N(K \cup \partial D)$. Then there are two possibilities:

(i) T does not separate $\partial N(K)$ and $\partial N(\partial D)$,
(ii) T separates $\partial N(K)$ and $\partial N(\partial D)$.

Case (i). Let V be a solid torus bounded by T [19, p. 107]. As before $c = \partial D$. Since T is essential in $S^3 - \operatorname{int} N(K \cup c)$, K and c are contained in V and V is knotted in S^3. Furthermore, since K (resp. c) is unknotted in S^3, there is a 3-ball B_K (resp. B_c) in V which contains K (resp. c) in its interior; but there is no 3-ball in V which contains $K \cup c$.

Since the algebraic intersection number of $K_{D,n}$ and a meridian disk D_V of V coincides with that of K and D_V, which is zero, $K_{D,n}$ is not a core of V. Since V is knotted in S^3, the lemma below shows that $K_{D,n}$ is a satellite knot with a companion knot ℓ which is a core of V.

Lemma 2.1. *$K_{D,n}$ is not contained in a 3-ball in V for any non-zero integer n.*

Proof. Let M be a 3-manifold $V - \operatorname{int} N(K)$. Then $\partial V \subset \partial M$ is compressible in M, because the 3-ball B_K contains K, and $M - \operatorname{int} N(c) = V - \operatorname{int} N(K \cup c)$ is irreducible and boundary-irreducible. Assume for a contradiction that $K_{D,n}$ is contained in a 3-ball in V for some non-zero integer n. Then $M(c; -\frac{1}{n}) \cong V - K_{D,n}$ is reducible. Then from [20, Theorem 6.1], we see that c is cabled and the surgery slope $-\frac{1}{n}$ is the slope of the cabling annulus. Since c is unknotted in S^3, c is a $(1, p)$-cable of an unknotted circle for $|p| \geqslant 2$. Then the slope of the cabling annulus should be p. This then implies that $|p| = |n| = 1$, a contradiction. Thus $K_{D,n}$ is not contained in a 3-ball in V for any non-zero integer n. □

Case (ii). The torus T cuts S^3 into two 3-manifolds V and W. Without loss of generality, we may assume that $K \subset V$, $c \subset W$. Now we show that V is an unknotted solid torus in S^3. The solid torus theorem [19, p. 107] shows that V or W is a solid torus. Suppose first that V is a solid torus. Since T is essential in $S^3 - \operatorname{int} N(K \cup c)$, K is not contained in a 3-ball in V and not a core of V. Furthermore, since K is unknotted in S^3, V is unknotted in S^3. If W is a solid torus, then since c is also unknotted in S^3, the above argument shows that W is unknotted in S^3, and hence V is an unknotted solid torus. Let ℓ be a core of V. Since T is essential in $S^3 - \operatorname{int} N(K \cup c)$, ℓ intersects the twisting disk D more than once: (ℓ, D) is also a twisting pair.

If $\ell_{D,n}$ is knotted in S^3, then $K_{D,n}$ is a satellite knot with a companion knot $\ell_{D,n}$. Assume that $\ell_{D,n}$ is unknotted in S^3 for some non-zero integer n. Then from [12, Corollary 3.1], [11, Theorem 4.2], we have the situation as in Fig. 2(1) and $n = -1$ or Fig. 2(2) and $n = 1$.

Thus, in particular, we have:

Lemma 2.2. *For any toroidal pair (K, D), $K_{D,n}$ is a satellite knot if $|n| > 1$.*

Now we suppose that (K, D) is a pair shown in Fig. 2(1) (resp. (2)). Then since $\ell_{D,-1}$ (resp. $\ell_{D,1}$) is also unknotted in S^3 and the linking number of ℓ and ∂D is two, we see that $K_{D,-1}$ (resp. $K_{D,1}$) can be regarded as the result of -4-twist (resp. 4-twist) along the meridian disk D_V of V: $K_{D,-1} = K_{D_V,-4}$ (resp. $K_{D,1} = K_{D_V,4}$).

To finish the proof of Theorem 1.1(3), we assume that $V - \operatorname{int} N(K)$ is neither Seifert fibered nor hyperbolic, i.e., it is toroidal. Then $K_{D,-1} = K_{D_V,-4}$ (resp. $K_{D,1} = K_{D_V,4}$) is a satellite knot by Lemma 2.2.

(4) Suppose that (K, D) is a pair shown in Fig. 2(1) (resp. (2)). If $V - \operatorname{int} N(K)$ is Seifert fibered, $K_{D,-1} = K_{D_V,-4}$ (resp. $K_{D,1} = K_{D_V,4}$) is a torus knot by (2) above. If $V - \operatorname{int} N(K)$ is hyperbolic, then by (1), $K_{D,-1} = K_{D_V,-4}$ (resp. $K_{D,1} = K_{D_V,4}$) is a hyperbolic knot. □

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