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Number Theory/Algebra

Cesàro asymptotics for the orders of $\mathrm{SL}_k(\mathbb{Z}_n)$ and $\mathrm{GL}_k(\mathbb{Z}_n)$ as $n \to \infty$

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Abstract

Given an integer $k > 0$, our main result states that the sequence of orders of the groups $\mathrm{SL}_k(\mathbb{Z}_n)$ (respectively, of the groups $\mathrm{GL}_k(\mathbb{Z}_n)$) is Cesàro equivalent as $n \to \infty$ to the sequence $C_1(k)n^{k^2-1}$ (respectively, $C_2(k)n^{k^2}$), where the coefficients $C_1(k)$ and $C_2(k)$ depend only on k; we give explicit formulas for $C_1(k)$ and $C_2(k)$. This result generalizes the theorem (which was first published by I. Schoenberg) that says that the Euler function $\varphi(n)$ is Cesàro equivalent to $n\frac{6}{\pi^2}$. We present some experimental facts related to the main result. ***To cite this article: A.G. Gorinov, S.V. Shadchin, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Formules asymptotiques au sens de Cesàro pour les ordres de $\mathrm{SL}_k(\mathbb{Z}_n)$ **et** $\mathrm{GL}_k(\mathbb{Z}_n)$ **quand** $n \to \infty$**.** Fixons un entier $k > 0$. Notre resultat principal dit que la suite des ordres des groupes $\mathrm{SL}_k(\mathbb{Z}_n)$ (respectivement, des groupes $\mathrm{GL}_k(\mathbb{Z}_n)$) est equivalente au sens de Cesàro quand $n \to \infty$ à la suite $C_1(k)n^{k^2-1}$ (respectivement, $C_2(k)n^{k^2}$), où les coefficients $C_1(k)$ et $C_2(k)$ ne dependent que de k ; on donne des formules explicites pour $C_1(k)$ et $C_2(k)$. Ce resultat généralise le théorème (publié pour la première fois par I. Schoenberg) disant que la fonction d'Euler $\varphi(n)$ est equivalente au sens de Cesàro à $n\frac{6}{\pi^2}$. On présente quelques faits experimentaux liés au resultat principal. ***Pour citer cet article : A.G. Gorinov, S.V. Shadchin, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

0. Introduction

The article is organized as follows: in Section 1 we introduce some notation and formulate our main result. Then, in Section 2, we prove this result. Finally, in Section 3 we discuss some interesting related facts.

1. The main theorem

Two sequences of real numbers $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be *Cesàro equivavent*, if $\lim_{n\to\infty} \frac{x_1+\cdots+x_n}{y_1+\cdots+y_n} = 1$.

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For any finite set X we shall denote by $\#(X)$ the cardinality of X. We shall use the symbol $\prod_p$ to denote the product over all prime numbers.

Our main result is the following theorem:

Theorem 1.1. *For any fixed integer $k > 0$ the sequence $(\#(\mathrm{SL}_k(\mathbb{Z}_n)))_{n\in\mathbb{N}}$ (resp., the sequence $(\#(\mathrm{GL}_k(\mathbb{Z}_n)))_{n\in\mathbb{N}}$) is Cesàro equivalent as $n \to \infty$ to $C_1(k)n^{k^2-1}$ (resp., $C_2(k)n^{k^2}$), where $C_1(1) = 1$, $C_2(1) = \prod_p(1 - \frac{1}{p^2})$, and for any $k > 1$ we have*

$$C_1(k) = \prod_p \left(1 - \frac{1}{p}\left(1 - \prod_{i=2}^{k}\left(1 - \frac{1}{p^i}\right)\right)\right), \qquad C_2(k) = \prod_p \left(1 - \frac{1}{p}\left(1 - \prod_{i=1}^{k}\left(1 - \frac{1}{p^i}\right)\right)\right).$$

Remark. In particular, $\#(\mathrm{GL}_1(\mathbb{Z}_n))$ and $\#(\mathrm{SL}_2(\mathbb{Z}_n))$ are Cesàro equivalent to $\frac{n}{\zeta(2)}$ and $\frac{n^3}{\zeta(3)}$ respectively. We do not know if the asymptotics given by Theorem 1.1 can be expressed in terms of values of the Riemann zeta-function (or any other remarkable function) at algebraic points in any of the other cases.

To the best of our knowledge, the fact that the Euler function $\varphi(n) = \#(\mathrm{GL}_1(\mathbb{Z}_n))$ is Cesàro equivalent to $n\frac{6}{\pi^2}$ was first published in [1] by Schoenberg, who attributes the result to Schur. This result was probably already known to Gauss. An explicit formula for the cumulative distribution function of the sequence $(\varphi(n)/n)_{n\in\mathbb{N}}$ is given in [2] by Venkov.

2. Proof of Theorem 1.1

Let us first recall the explicit formulas for $\#(\mathrm{SL}_k(\mathbb{Z}_n))$ and $\#(\mathrm{GL}_k(\mathbb{Z}_n))$. For any positive integer k denote by $\tilde{\varphi}_k$[1] the map $\mathbb{N} \to \mathbb{R}$ given by the formula $\tilde{\varphi}_k(p_1^{l_1} \cdots p_m^{l_m}) = (1 - 1/p_1^k) \cdots (1 - 1/p_m^k)$ (here $p_1, \ldots, p_m$ are pairwise distinct primes).

Lemma 2.1. *We have $\#(\mathrm{GL}_1(\mathbb{Z}_n)) = n\tilde{\varphi}_1(n)$, and for any integer $k > 1$ we have $\#(\mathrm{SL}_k(\mathbb{Z}_n)) = n^{k^2-1}\tilde{\varphi}_2(n) \cdots \tilde{\varphi}_k(n)$, $\#(\mathrm{GL}_k(\mathbb{Z}_n)) = n^{k^2}\tilde{\varphi}_1(n) \cdots \tilde{\varphi}_k(n)$.*

The proof is an exercise in linear algebra. □

Now let us calculate the limits of the averages of the sequences $(\tilde{\varphi}_1(n) \cdots \tilde{\varphi}_k(n))_{n\in\mathbb{N}}$ and $(\tilde{\varphi}_2(n) \cdots \tilde{\varphi}_k(n))_{n\in\mathbb{N}}$. More generally, let ℓ be a finite (nonempty) ordered collection of positive integers: $\ell = (i_1, \ldots, i_l)$. For any $n \in \mathbb{N}$ set $\tilde{\varphi}_\ell(n) = \tilde{\varphi}_{i_1}(n) \cdots \tilde{\varphi}_{i_l}(n)$. For any sequence $x = (x_n)_{n\in\mathbb{N}}$ denote by $\langle x\rangle$ the Cesàro limit of x, i.e., the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} x_m$.

Theorem 2.2. *For any $\ell = (i_1, \ldots, i_l)$ the limit $\langle\tilde{\varphi}_\ell\rangle$ exists and is equal to $\prod_p f_\ell(\frac{1}{p})$, where $f_\ell(t) = 1 - t(1 - \prod_{j=1}^{l}(1 - t^{i_j}))$.*

Sketch of a proof of Theorem 2.2. We shall first give an informal proof of the theorem; we shall then show what changes should be made to make our informal proof rigorous.

The idea of the proof of Theorem 2.2 is to give a probabilistic interpretation to some complicated expressions (such as $\frac{1}{n}\sum_{m=1}^{n} \tilde{\varphi}_\ell(m)$). This idea goes back to Euler.

[1] This notation can be explained as follows: the function $\tilde{\varphi}_k$ generalizes the function $n \mapsto \varphi(n)/n = \tilde{\varphi}_1(n)$.

Let us note that for any positive integer q the "probability" that a "random" positive integer is a not a multiple of q is $1 - 1/q$. If q_1 and q_2 are coprime integers, the events "r is not divisible by q_1" and "r is not divisible by q_2" (r being a "random" positive integer) are independent, which implies that for any positive integers $m,\ k$ the expression $\tilde{\varphi}_k(m)$ is the "probability" that a "randomly chosen" positive integer is not divisible by k-th powers of the prime divisors of m.

Analogously, for any fixed positive integer m the expression $\tilde{\varphi}_\ell(m)$ can be seen as the "probability" to find an element $(x_1, \dots, x_l) \in \mathbb{N}^l$ that satisfies the following conditions: x_1 is not divisible by the i_1-th powers of the prime factors of m, x_2 is not divisible by the i_2-th powers of the prime factors of m etc.

Using the total probability formula, we obtain that $\frac{1}{n}\sum_{m=1}^{n} \tilde{\varphi}_\ell(m)$ is the "probability" that a "random" element of the set $\{(x_0, x_1, \dots, x_l) \mid x_0, \dots, x_l \in \mathbb{N},\ x_0 \leqslant n\}$ satisfies the following condition: any $x_j,\ j = 1, \dots, l$ is not divisible by the i_j-th powers of the prime divisors of x_0. So the limit $\langle \tilde{\varphi}_\ell \rangle$ is the "probability" of the limit event, which can be described as the intersection for all prime p of the following events: "either (x_0 is not divisible by p), or (none of $x_j,\ j = 1, \dots, l$, is divisible by p^{i_j})". These events are independent, and the "probability" of each of them is $f_\ell(\frac{1}{p}) = 1 - \frac{1}{p}\bigl(1 - \prod_{j=1}^{l}(1 - \frac{1}{p^{i_j}})\bigr)$. This gives the desired expression for $\langle \tilde{\varphi}_\ell \rangle$.

This idea is formalized as follows. Let l be a positive integer, and let A and B be subsets of $\mathbb{N}^l$ such that there exists $\lim_{k\to\infty} \frac{\#(A\cap B\cap C_k)}{\#(B\cap C_k)}$, where $C_k = \{(x_1, \dots, x_l) \in \mathbb{N}^l \mid x_1 \leqslant k, \dots, x_l \leqslant k\}$. This limit will be called the *density* of A in B and will be denoted by $p_B(A)$. For any $B \subset \mathbb{N}^l$ the correspondence $B \supset A \mapsto p_B(A)$ defines a measure on B.[2]

Using the same argument as above (and replacing "probabilities" with "densities" and "events" with "sets"), we can represent $\frac{1}{n}\sum_{m=1}^{n} \tilde{\varphi}_\ell(m)$ as the density of a certain subset of the set $\{(x_0, x_1, \dots, x_l) \mid x_0, \dots, x_l \in \mathbb{N},\ x_0 \leqslant n\}$. This interpretation does not allow us to pass immediately to the limit as $n \to \infty$, but it enables us to write the following combinatorial formula for $\frac{1}{n}\sum_{m=1}^{n} \tilde{\varphi}_\ell(m)$. Define the sequence $(a_k)_{k\in\mathbb{N}}$ by the formula $\sum_{k=1}^{\infty} a_k t^k = 1 - \prod_j (1 - t^{i_j})$. We have $\frac{1}{n}\sum_{m=1}^{n} \tilde{\varphi}_\ell(m) = 1 + \sum_{r=2}^{\infty} \frac{1}{r}(-1)^{pr(r)} a(r) b_{r,n}$, where for any $r = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ we define $pr(r) = s$, $a(r) = a_{\alpha_1} \cdots a_{\alpha_s}$, $b_{r,n} = [\frac{n}{p_1\cdots p_s}]\frac{1}{n}$ (in particular, $a(r) = 0$, if $\max\{\alpha_1, \dots, \alpha_s\} > i_1 + \cdots + i_l$). Now let us note that this expression has the form $\sum_{k=1}^{\infty} b'_{k,n} c_r$, where c_k is the k-th term of the absolutely convergent series obtained by multiplying out the product $\prod_p \bigl(1 - \frac{1}{p}\bigl(1 - \prod_{j=1}^{l}(1 - \frac{1}{p^{i_j}})\bigr)\bigr)$, and every $b'_{k,n}$ has the form $\frac{p_1\cdots p_s}{n}[\frac{n}{p_1\cdots p_s}]$. We have $0 \leqslant b'_{k,n} \leqslant 1$ for any k, n, and the limit $\lim_{n\to\infty} b'_{k,n}$ is equal to 1 for any k. This implies Theorem 2.2. $\square$

Theorem 1.1 can be obtained from Theorem 2.2, from Lemma 2.1 and from the following lemma.

Lemma 2.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and suppose that $\langle x \rangle$ exists. Then, for any nonnegative integer k, we have* $\lim_{n\to\infty} \frac{x_1 + 2^k x_2 + \cdots + n^k x_n}{1 + 2^k + \cdots + n^k} = \langle x \rangle$.

Proof of Lemma 2.3. The proof is by induction on k. If $k = 0$, there is nothing to prove. Suppose Lemma 2.3 holds for some k. For any sequence $y = (y_n)_{n\in\mathbb{N}}$ set $S_n^k[y] = y_1 + 2^k y_2 + \cdots + n^k y_n$. We have $S_n^k[x] = n^{k+1}(\frac{\langle x \rangle}{k+1} + \varepsilon_n)$, where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Note that for any sequence $y = (y_n)_{n\in\mathbb{N}}$ we have

$$S_n^{k+1}[y] = n S_n^k[y] - \sum_{m=1}^{n-1} S_m^k[y]. \qquad (*)$$

Thus, we can write $S_n^{k+1}[x] = \frac{\langle x \rangle}{k+1}(n^{k+2} - \sum_{m=1}^{n-1} m^{k+1}) + \varepsilon_n n^{k+2} - S_{n-1}^{k+1}[\varepsilon]$. We have $\lim_{n\to\infty} \frac{S_{n-1}^{k+1}[\varepsilon]}{n^{k+2}} = 0$, and hence $\lim_{n\to\infty} \frac{S_n^{k+1}[x]}{n^{k+2}} = \frac{\langle x \rangle}{k+1}(1 - \frac{1}{k+2}) = \frac{\langle x \rangle}{k+2}$, which implies the statement of Lemma 2.3. $\square$

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[2] Unfortunately, this measure is not σ-additive, which is why we prefer to speak rather of densities than of probabilities.

3. Convergence rates and the distribution of the values of $\tilde{\varphi}_\ell$

Let ℓ be a finite (nonempty) ordered collection of positive integers: $\ell = (i_1, \dots, i_l)$. In this section we briefly discuss the convergence rate of the sequences $(\frac{1}{n^{s+1}} \sum_{k=1}^{n} k^s \tilde{\varphi}_\ell(k))_{n \in \mathbb{N}}$ for different fixed $s \in \mathbb{N}$ and the distribution of the values of the function $\tilde{\varphi}_\ell$.

Set $\Phi_\ell = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{\varphi}_\ell(k) = \prod_p f_\ell(\frac{1}{p})$, $\xi_{\ell,s}(n) = \frac{1}{n^s}\bigl(\sum_{k=1}^{n} k^s \tilde{\varphi}_\ell(n) - \frac{n^{s+1}}{s+1}\Phi_\ell\bigr)$. It follows immediately from these definitions that $\sum_{k=1}^{n} k^s \tilde{\varphi}_\ell(k) = \frac{n^{s+1}}{s+1}\Phi_\ell + n^s \xi_{\ell,s}(n)$.

Theorem 3.1. *If $\langle \xi_{\ell,0}\rangle$ exists, then for all integers $s > 0$ the limit $\langle \xi_{\ell,s}\rangle$ exists and is equal to $\frac{1}{2}\Phi_\ell$.*

Proof of Theorem 3.1. Set $\eta_{\ell,s}(n) = \frac{1}{n^s}\sum_{k=1}^{n} k^s(\tilde{\varphi}_\ell(k) - \Phi_\ell)$. Note that $\xi_{\ell,0} = \eta_{\ell,0}$, hence $\langle \eta_{\ell,0}\rangle$ exists. Using formula $(*)$ we get $\eta_{\ell,s+1}(n) = \eta_{\ell,s}(n) - \frac{1}{n^{s+1}}\sum_{k=1}^{n} k^s \eta_{\ell,s}(k)$. Hence we obtain using Lemma 2.3 that $\langle \eta_{\ell,s+1}\rangle = \frac{s}{s+1}\langle \eta_{\ell,s}\rangle$ for any integer $s \geqslant 0$. Thus, $\langle \eta_{\ell,s}\rangle = 0$ for any integer $s > 0$.

For any integer $s \geqslant 1$ we have $\sum_{k=1}^{n} k^s = \frac{n^{s+1}}{s+1} + \frac{1}{2}n^s + \mathrm{O}(n^{s-1})$. Hence we get the following relation: $\xi_{\ell,s}(n) = \eta_{\ell,s}(n) + \frac{1}{2}\Phi_\ell + \mathrm{O}(\frac{1}{n})$, which implies that $\langle \xi_{\ell,s}\rangle = \frac{1}{2}\Phi_\ell$. The theorem is proven. □

Let us now consider the distribution of the values of the function $\tilde{\varphi}_\ell$. Using the argument from [1, §5], one can prove that for any $t \in [0,1]$ the limit $\lim_{n\to\infty} \frac{1}{n}\#\{k \in \mathbb{N} \mid k \leqslant n,\ \tilde{\varphi}_\ell(k) \leqslant t\}$ exists, and that the function F_ℓ defined by the formula $F_\ell(t) = \lim_{n\to\infty} \frac{1}{n}\#\{k \in \mathbb{N} \mid k \leqslant n,\ \tilde{\varphi}_\ell(k) \leqslant t\}$ is continuous (I. Schoenberg considers only the case $\ell = (1)$, but his argument can be easily extended to the case of an arbitrary ℓ). The function F_ℓ is the analogue of the cumulative distribution function in probability theory. Given a nonnegative integer s, the s-th moment of F_ℓ is defined as follows: $\mu_{\ell,s} = \int_0^1 t^s \,\mathrm{d}F_\ell(t)$. It is easy to prove (see [1, Satz I]) that $\mu_{\ell,s} = \langle (\tilde{\varphi}_\ell)^s\rangle$. Due to Theorem 2.2, we have $\mu_{\ell,s} = \Phi_{\ell^s}$ where ℓ^s is the following collection of positive integers: $\ell^s = (i_1, i_1, \dots, i_1$ (s times), $i_2, i_2, \dots, i_2$ (s times), $\dots)$.

The Fourier series for $F_\ell(t)$ is equal to $\sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n t}$, where $u_0 = 1 - \Phi_\ell = \frac{1}{2} - \sum_{k\neq 0} \frac{\langle \mathrm{e}^{-2\pi \mathrm{i} k\tilde{\varphi}_\ell}\rangle}{2\pi \mathrm{i} k}$ (the sum of the series in the latter formula is to be taken in Cesàro sence), and the Fourier coefficients u_n for $n \neq 0$ can be calculated using either the formula $u_n = -\sum_{m=1}^{\infty} \frac{(-2\pi \mathrm{i} n)^{m-1}}{m!}\Phi_{\ell^m}$, or the formula $u_n = \frac{1}{2\pi \mathrm{i} n}(\langle \mathrm{e}^{-2\pi i n \tilde{\varphi}_\ell}\rangle - 1)$. Since F_ℓ is continuous, its Fourier series converges in Cesàro sence to F_ℓ uniformly on every compact subset of the open interval $(0,1)$.

Note added in proof. Recently we proved that for any $\ell = (i_1, \dots, i_l)$ such that all $i_j > 1$, the limit $\langle \xi_{\ell,0}\rangle$ exists and $\langle \xi_{\ell,0}\rangle = \frac{1}{2}\Phi_\ell - \frac{1}{2\zeta(i_1)\cdots\zeta(i_l)}$. After the article has been accepted for publication, we learn from P. Moree an alternative proof of Theorem 1.1 based on a lemma in [3, p. 108] (the proof of that lemma given in [1] is due to Erdős).

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