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Partial Differential Equations

Ginzburg–Landau minimizers with prescribed degrees: dependence on domain

Leonid Berlyand [a], Petru Mironescu [b]

[a] *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*
[b] *Département de mathématiques, Université Paris-Sud, 91405 Orsay cedex, France*

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Abstract

We study minimizers of the Ginzburg–Landau functional in an annular type domain with holes. We assume degrees 1 and -1 on the boundary of the annulus, degree 0 on the boundaries of the holes. Two types of qualitatively different behavior of minimizers occur, depending on the value of the H^1-capacity of the domain. We also describe the asymptotic behavior of minimizers as the coherency length tends to ∞. ***To cite this article: L. Berlyand, P. Mironescu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Minimiseurs de l'énergie de Ginzburg–Landau avec degrés prescrits : dépendance du domaine. On étudie des minimiseurs de l'énergie de Ginzburg–Landau dans un domaine annulaire à trous. Les conditions aux limites sont des degrés prescrits : degrés 1 et -1 sur le bord du domaine annulaire, degré 0 sur les bords des trous. En fonction de la H^1-capacité du domaine, les minimiseurs ont deux types de comportement, qualitativement différents. On décrit aussi le comportement des minimiseurs quand le paramètre de Ginzburg–Landau tend vers ∞. ***Pour citer cet article : L. Berlyand, P. Mironescu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

On étudie le problème variationnel suivant :

$$m_\kappa = \operatorname{Inf}\left\{\frac{1}{2}\int_A |\nabla u|^2 + \frac{\kappa^2}{4}\int_A \left(1-|u|^2\right)^2;\ u \in \mathcal{K}\right\}. \tag{1}$$

E-mail addresses: berlyand@math.psu.edu (L. Berlyand), petru.mironescu@math.u-psud.fr (P. Mironescu).

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Ici, A est un domaine perforé 2D, c'est-à-dire

$$A=\Omega\setminus\left(\bigcup_{j=0}^{k}\omega_j\right),\quad \overline{\omega_j}\subset\Omega,\ j=0,\ldots,k,\ \overline{\omega_j}\cap\overline{\omega_k}=\emptyset,\ j\neq l, \tag{2}$$

et les domaines $\Omega,\ \omega_j,\ j=0,\ldots,k,$ sont simplement connexes, bornés et réguliers. La classe des fonctions test est

$$\mathcal{K}=\bigl\{u\in H^1\bigl(A;\mathbb{R}^2\bigr);\ |u|=1 \text{ p.p. sur } \partial A,\ \deg(u,\partial\Omega)=1,\\ \deg(u,\partial\omega_0)=-1,\ \deg(u,\partial\omega_j)=0,\ j=1,\ldots,k\bigr\}. \tag{3}$$

Plus précisément, on s'intéresse aux deux questons suivantes :

1° Y-a-t-il un minimiseur de (1), pour tout κ ?
2° Si oui, quel est le comportement des minimiseurs de (1), pour $\kappa\to\infty$?

On montre que les réponses dépendent de la valeur d'un problème de minimisation pour applications harmoniques

$$I_0=\mathrm{Min}\left\{\frac{1}{2}\int_A|\nabla u|^2;\ u\in\mathcal{K}\cap H^1\bigl(A;S^1\bigr)\right\}. \tag{4}$$

On prouve les résultats suivants :

Théorème 0.1. *Si $I_0\leqslant 2\pi$, alors* :

(a) *m_κ est atteint pour tout κ* ;
(b) *si u_κ est un minimiseur de* (1)*, alors, à une sous-suite près, $u_\kappa\to u_0$ dans $C^{1,\alpha}(\bar{A})$, $\forall 0<\alpha<1$, où u_0 est un minimiseur de I_0* ;
(c) *il existe un κ_0 tel que, pour $\kappa>\kappa_0$, les minimiseurs de* (1) *soient uniques à une rotation près, c'est-à-dire, si $\kappa>\kappa_0$ et $u_\kappa,\ v_\kappa$ sont deux minimiseurs de* (1)*, alors $u_\kappa=\beta v_\kappa$, pour un $\beta\in S^1$.*

Théorème 0.2. *Si $I_0>2\pi$, alors* :

(a) *il existe un $\kappa_1\in[0,\infty]$ tel que* : *m_κ soit toujours atteint pour $\kappa<\kappa_1$ et jamais atteint pour $\kappa>\kappa_1$* ;
(b) *si m_κ est toujours atteint (c'est-à-dire si $\kappa_1=\infty$), alors, à une sous-suite près, les minimiseurs u_κ de* (1) *convergent, faiblement dans H^1 et fortement dans $C^{1,\alpha}_{\mathrm{loc}}(\bar{A}\setminus(\partial\Omega\cup\partial\omega_0))$, $\forall 0<\alpha<1$, vers une constante $\beta\in S^1$. De plus, il existe κ_0 tel que, pour $\kappa>\kappa_0$, u_κ ait au moins deux zéros de degré non-nul, l'un tendant vers $\partial\Omega$, l'autre vers $\partial\omega_0$ quand $\kappa\to\infty$.*

À ce stade, nous ne pouvons pas décider, pour un domaine A donné, si $\kappa_1=0$, κ_1 est positif, ou $\kappa_1=\infty$. En particulier, nous ne savons pas si les minimiseurs existent pour tout κ. Néanmoins, pour tout κ, on peut considérer des quasi-minimiseurs u_κ^n, vérifiant les conditions (5)–(7) ci-dessous. (À κ fixé, les u_κ^n forment une suite minimisante.) On obtient alors la généralisation suivante du Théorème 0.2(b) :

Théorème 0.3. *On suppose $I_0>2\pi$. Soit (u_κ^n) une famille d'applications vérifiant les conditions* (5)–(7)*. Alors, à une sous-suite près, les u_κ^n convergent, faiblement dans H^1 et fortement dans $C^{1,\alpha}_{\mathrm{loc}}(\bar{A}\setminus(\partial\Omega\cup\partial\omega_0))$, $\forall 0<\alpha<1$, vers une constante $\beta\in S^1$. De plus, il existe κ_0 tel que, pour $\kappa>\kappa_0$, u_κ ait au moins deux zéros de degré non-nul, l'un tendant vers $\partial\Omega$, l'autre vers $\partial\omega_0$ quand $\kappa\to\infty$.*

La valeur de la constante I_0 est liée à la H^1-capacité de A. Par exemple, si $A=\Omega\setminus\omega_0$, alors $I_0=2\pi^2/\mathrm{cap}(A)$. Si A a plusieurs trous, alors on peut exprimer I_0 en fonction d'une capacité généralisée.

Les preuves détaillées des Théorèmes 0.1–0.3 apparaîtront dans [3].

1. Introduction

Our study is motivated by the following problem. In [1,2], a mathematical model of an *ideal* superconductor reinforced by a large number of thin insulating rods was introduced. For a cylindrical superconductor with a coaxial cylindrical hole (often used in experimental settings), this model led to a minimization problem for *harmonic maps* in a 2D annular domain with many small holes. The distinguished mathematical feature of this problem is that the physical insulating conditions lead to prescribing degree (winding number) boundary conditions. Even though this problem is nonlinear, it has an underlying *linear* problem for the multi-valued phase of the harmonic maps.

2. The problem

This study led to a natural question: what if the superconductor in the composite described below is not ideal (e.g., of type II)? Mathematically, this means that, in the above minimization problem, the Dirichlet integral for harmonic maps should be replaced by the Ginzburg–Landau functional and we arrive to the following problem

$$m_\kappa=\mathrm{Inf}\left\{\frac{1}{2}\int_A|\nabla u|^2+\frac{\kappa^2}{4}\int_A\left(1-|u|^2\right)^2;\ u\in\mathcal{K}\right\}. \tag{1}$$

Here, A is a 2D perforated domain, i.e.,

$$A=\Omega\setminus\left(\bigcup_{j=0}^{k}\omega_j\right),\quad \overline{\omega_j}\subset\Omega,\ j=0,\dots,k,\ \overline{\omega_j}\cap\overline{\omega_k}=\emptyset,\ j\neq l, \tag{2}$$

with $\Omega,\ \omega_j,\ j=0,\dots,k,$ simply connected bounded smooth domains. The class $\mathcal{K}$ of testing maps is

$$\begin{aligned}\mathcal{K}=\bigl\{u\in H^1\left(A;\mathbb{R}^2\right);\ |u|=1 \text{ a.e. on } \partial A, \deg(u,\partial\Omega)=1,&\\ \deg(u,\partial\omega_0)=-1,\ \deg(u,\partial\omega_j)=0,\ j=1,\dots,k\bigr\}.&\end{aligned} \tag{3}$$

We thus consider a domain with finitely many holes ω_j of fixed size. The homogenization limit for this problem will be addressed in a subsequent paper.

When $g\in H^{1/2}(\Gamma;S^1)$, with Γ a simply closed planar curve, g has a well-defined degree (see [9]). Thus, the definition of $\mathcal{K}$ is meaningful. Here, the degrees are computed with respect to the direct orientation of ∂A.

An important feature of (1) is that $\mathcal{K}$ is not closed under weak H^1-convergence (see example below), so that it is not clear whether there exists a minimizer in $\mathcal{K}$. By contrast, in the case of a fixed Dirichlet boundary data, extensively studied in [7], the existence of minimizers is straightforward. The existence of minimizers for the above mentionned harmonic map problem is also straightforward, see [7]. We shall illustrate this phenomenon in the special case where $\Omega=\{z;\ |z|<2\}$ and $\omega_0=\{z;\ |z|<1\}$ (this example can be adapted to the general case). Set $\varphi_n(z)=z/|z|$, if $|z|>1-1/n,\ \varphi_n(z)=nz/(n-1)$, if $|z|\leqslant 1-1/n$. Consider the maps

$$u_n(z)=\varphi_n\left(2\frac{\bar z-(1+1/n^2)}{1-(1+1/n^2)\bar z}\,\frac{z-(2-1/n^2)}{4-(2-1/n^2)z}\right).$$

Then, $u_n\in\mathcal{K}$, for sufficiently large n. Moreover, it is easy to check that $u_n\rightharpoonup 1$ weakly in H^1. Clearly, this weak limit is not in $\mathcal{K}$.

The objective of this paper is to answer the following questions for problem (1):

1° Is m_κ attained (for any κ)?
2° If so, what is the behavior of minimizers u_κ of (1) as $\kappa \to \infty$?

A first guess for question 2° is the following: consider the minimization problem

$$I_0 = \operatorname{Min}\left\{\frac{1}{2}\int_A |\nabla u|^2;\ u \in \mathcal{K} \cap H^1\big(A; S^1\big)\right\}. \tag{4}$$

Then one may expect $u_\kappa \rightharpoonup u_0$ at least weakly in H^1, where u_0 is one of the minimizers of (4). It turns out that the answer depends on I_0.

Theorem 2.1. *Assume* $I_0 < 2\pi$ (*subcritical case*) *or* $I_0 = 2\pi$ (*critical case*). *Then*:

(a) m_κ *is attained for each* κ;
(b) *if* u_κ *is a minimizer of* (1), *then, up to a subsequence,* $u_\kappa \to u_0$ *in* $C^{1,\alpha}(\bar{A})$, $\forall 0 < \alpha < 1$, *where* u_0 *is a minimizer of* I_0;
(c) *there exists some* κ_0 *such that, for* $\kappa > \kappa_0$, *the minimizers of* (1) *are unique up to a a phase shift, i.e., if* $\kappa > \kappa_0$ *and* u_κ, v_κ *are minimizers of* (1), *then* $u_\kappa = \beta v_\kappa$ *for some* $\beta \in S^1$.

In [3], we estimate κ_0 in terms of the geometry of A.

Theorem 2.2. *Assume* $I_0 > 2\pi$ (*supercritical case*). *Then*:

(a) *there is some* $\kappa_1 \in [0, \infty]$ *such that* m_κ *is always attained for* $\kappa < \kappa_1$ *and never attained for* $\kappa > \kappa_1$;
(b) *assume* m_κ *always attained* (*i.e., assume* $\kappa_1 = \infty$). *Then, up to a subsequence, minimizers* u_κ *of* (1) *converge, as* $\kappa \to \infty$, *weakly in* H^1 *and strongly in* $C^{1,\alpha}_{\mathrm{loc}}(\bar{A} \setminus (\partial\Omega \cup \partial\omega_0))$, $\forall 0 < \alpha < 1$, *to some constant* $\beta \in S^1$. *Moreover, these minimizers have, for large values of* κ, *at least two zeroes of nonzero degree, one tending to* $\partial\Omega$, *the other one to* $\partial\omega_0$, *as* $\kappa \to \infty$.

At this point, we are unable to establish, for a given domain A, whether κ_1 is 0, finite positive or ∞. In particular, we do not know whether minimizers exist for each value of κ. However, for each κ we may clearly construct quasi-minimizers, i.e., maps u_κ^n such that

$$E_\kappa\big(u_\kappa^n\big) = \frac{1}{2}\int_A \big|\nabla u_\kappa^n\big|^2 + \frac{\kappa^2}{4}\int_A \big(1 - |u|^2\big)^2 \leqslant m_\kappa + \frac{1}{n+\kappa}, \tag{5}$$

$$u_\kappa^n \in \mathcal{K}, \tag{6}$$

$$E_\kappa\big(u_\kappa^n\big) \leqslant E_\kappa(u) \quad \text{if } u = u_\kappa^n \text{ on } \partial A. \tag{7}$$

Clearly, for fixed κ, (u_κ^n) is a minimizing sequence. We have the following generalization of Theorem 2.2(b):

Theorem 2.3. *Assume* $I_0 > 2\pi$ *and let* u_κ^n *satisfy* (5)–(7). *Then, up to a subsequence,* u_κ^n *converge, as* $\kappa \to \infty$, *weakly in* H^1 *and strongly in* $C^{1,\alpha}_{\mathrm{loc}}(\bar{A} \setminus (\partial\Omega \cup \partial\omega_0))$, $\forall 0 < \alpha < 1$, *to some constant* $\beta \in S^1$. *Moreover, these minimizers have, for large values of* κ, *at least two zeroes of nonzero degree, one tending to* $\partial\Omega$, *the other one to* $\partial\omega_0$, *as* $\kappa \to \infty$.

Note that the qualitative change of behavior of minimizers illustrated by Theorems 2.1–2.3 does not appear if we consider instead a fixed Dirichlet boundary condition. It follows from the above results that this behavior crucially depends on I_0. We now explain the geometrical meaning of I_0. We begin with the case of a circular annulus with no holes, i.e., $A = \{z;\ r < |z| < R\}$. Then it is easy to see that $I_0 = \pi \ln(R/r)$, so that A is: subcritical if $R/r < e^2$, critical if $R/r = \mathrm{e}^2$ and supercritical if $R/r > \mathrm{e}^2$. In this specific case, we have the following additional result:

Proposition 2.4. *Assume $A = \{z;\ r < |z| < R\}$, with $R/r \leqslant \mathrm{e}^2$. Then there is some κ_0 such that, for $\kappa > \kappa_0$, the minimizers of* (1) *are radially symmetric, i.e., of the form $u_\kappa(\rho\, \mathrm{e}^{\imath\theta}) = \beta f_\kappa(\rho)\, \mathrm{e}^{\imath\theta}$ for some $\beta \in S^1$. Moreover, up to a subsequence, $u_\kappa \to \beta\, \mathrm{e}^{\imath\theta}$.*

On the other hand, an immediate consequence of Theorem 2.2 is the following result, previously established in [4]:

Corollary 2.5. *Assume $A = \{z;\ r < |z| < R\}$, with $R/r > \mathrm{e}^2$. Then there is some κ_0 such that, for $\kappa > \kappa_0$, quasi-minimizers are not radially symmetric.*

The conclusions of Proposition 2.4, when R/r is sufficiently close to 1 and κ is arbitrary, were obtained in [11]. However, the techniques used in [4] and [11] require circular symmetry on domains and, in what concerns Proposition 2.4, do not provide the sharp threshold $R/r = \mathrm{e}^2$. Corollary 2.5 should be compared to a result in [12] that asserts that radially symmetric solutions of the Ginzburg–Landau equation belonging to $\mathcal{K}$ are local minimizers of the Ginzburg–Landau energy.

For S^2-valued harmonic maps with Dirichlet boundary conditions, the existence of a critical value of R/r determining a qualitative change in the behavior of minimizers was established in [5]. Similar split in behavior was described in physical context, see, e.g., [10].

Consider now a domain with a single hole, i.e., $A = \Omega \setminus \omega_0$. In this case, the role of R/r is played by the capacity of ω_0 in Ω. Recall (see [14]) that the H^1-capacity is defined as follows:

$$\operatorname{cap}(A) = \operatorname{Min}\left\{\int_A |\nabla v|^2;\ v \in H^1(A;\mathbb{R}),\ v = 0 \text{ on } \partial\Omega,\ v = 1 \text{ on } \partial\omega_0\right\}.$$

It is easy to see that

$$I_0 = \frac{2\pi^2}{\operatorname{cap}(A)}. \tag{8}$$

Thus,

$$\begin{aligned}&\text{subcritical corresponds to } \operatorname{cap}(A) > \pi \text{ (“thin domains”)},\\ &\text{critical to } \operatorname{cap}(A) = \pi,\\ &\text{supercritical to } \operatorname{cap}(A) < \pi \text{ (“thick domains”)}.\end{aligned} \tag{9}$$

Finally, we turn to general perforated domains $A = \Omega \setminus (\bigcup_{j=0}^{k} \omega_j)$. In this case, we introduce a generalized capacity, defined as follows:

$$\operatorname{cap}(A) = \operatorname{Min}\left\{\int_A |\nabla v|^2;\ v \in H^1(A;\mathbb{R}),\ v = 0 \text{ on } \partial\Omega,\ v = 1 \text{ on } \partial\omega_0,\ v = C_j \text{ on } \partial\omega_j,\ j \geqslant 1\right\}.$$

(Here, the constants C_j are arbitrary.) As above, the three cases are defined by (9).

3. Outline of the proofs

Step 1. Energy bounds. By considering appropriate testing maps, it is easy to see that

$$m_\kappa \leqslant 2\pi, \quad m_\kappa < I_0. \tag{10}$$

Step 2. Existence of minimizers. Theorem 2.1(a) is an immediate consequence of (10) and of the following

Lemma 3.1. *Assume that*

$$m_\kappa < 2\pi. \tag{11}$$

Then m_κ is attained.

Conditions of type (11) play a crucial role in PDEs with possible lack of compactness for minimizing sequences; see, e.g., [8].

Step 3. Asymptotic behavior of (quasi-)minimizers. Using the bound (10) and the methods developed in [15] we obtain that, up to subsequences, (quasi-)minimizers converge, in $C^l_{\mathrm{loc}}(A)$, to some $u \in H^1(A)$. Using once again (10) we prove that, when $I_0 < 2\pi$, then u is a minimizer of (4), while, when $I_0 > 2\pi$, u is a constant. When $I_0 = 2\pi$, u is a minimizer of (4), but the proof is much more involved in this case. Theorems 2.2(b) and 2.3 follow from Step 3. When $I_0 \leqslant 2\pi$, we also have $u_k \to u$ strongly in $H^1(A)$. Using this fact, Theorem 2.1(b), (c) follow by adapting the methods in [6,13].

The detailed proofs of Theorems 2.1–2.3 will appear in [3].

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