



### Optimal Control

# Insensitizing controls for a large-scale ocean circulation model

Enrique Fernández-Cara<sup>a</sup>, Galina C. Garcia<sup>b,c</sup>, Axel Osses<sup>c,d</sup>

<sup>a</sup> Dpto. E.D.A.N., Universidad de Sevilla, Apto. 1160, 41080 Sevilla, Spain

<sup>b</sup> Dpto. Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

<sup>c</sup> Centro de Modelamiento Matemático, UMR 2071 CNRS-Universidad de Chile, Santiago, Chile

<sup>d</sup> Dpto. Ingeniería Matemática, Universidad de Chile, Casilla 170/3 Correo 3, Santiago, Chile

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#### Abstract

We consider here a linear quasi-geostrophic ocean model. We look for controls insensitizing (resp.  $\varepsilon$ -insensitizing) an observation function of the state. The existence of such controls is equivalent to a null controllability property (resp. an approximate controllability property) for a cascade Stokes-like system. Under reasonable assumptions on the spatial domains where the observation and the control are performed, we are able to prove these properties. **To cite this article:** E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).

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#### Résumé

**Contrôle insensibilisant pour un modèle de circulation océanique pour les grandes échelles.** Nous considérons un modèle océanique quasi-géostrophique linéarisé. On fixe une fonctionnelle d'observation de l'état et on cherche des contrôles insensibilisants (resp.  $\varepsilon$ -insensibilisants). L'existence de ces contrôles est équivalente à la contrôlabilité exacte à zéro (resp. la contrôlabilité approchée) d'un système en cascade de type Stokes. Sous des hypothèses géométriques raisonnables sur les domaines où l'on exerce l'observation et le contrôle, on est capable de montrer ces propriétés. **Pour citer cet article :** E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).

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#### Version française abrégée

Soit  $\Omega$  un ouvert connexe borné non vide de  $\mathbb{R}^2$ , avec frontière  $\Gamma$  de classe  $C^2$  et normale unitaire extérieure  $\nu = \nu(x)$ . Soient  $\omega$  et  $\mathcal{O}$  deux sous-ensembles non vides de  $\Omega$ ,  $T > 0$ ,  $Q = \Omega \times (0, T)$  et  $\Sigma = \Gamma \times (0, T)$ . On considère un modèle océanique quasi-géostrophique linéarisé [9,10], décrit par les équations

$$\begin{cases} u_t - A\Delta u + \gamma u + (f_0 + \beta x_2)k \wedge u + \frac{1}{\rho_0} \nabla p = \mathcal{T} + h1_\omega & \text{dans } Q, \\ \operatorname{div} u = 0 & \text{dans } Q, \\ u = 0 & \text{sur } \Sigma, \\ u|_{t=0} = u_0 + \tau \hat{u}_0 & \text{dans } \Omega, \end{cases} \quad (1)$$

E-mail addresses: [cara@numer.us.es](mailto:cara@numer.us.es) (E. Fernández-Cara), [galina@ing-mat.udec.cl](mailto:galina@ing-mat.udec.cl) (G.C. Garcia), [axosses@dim.uchile.cl](mailto:axosses@dim.uchile.cl) (A. Osses).

où  $u = u(x, t)$  et  $p = p(x, t)$  sont la vitesse et la pression du fluide. Dans ce modèle,  $A$  est un coefficient de *viscosité turbulente*,  $\gamma$  est le coefficient de *friction* au fond,  $(f_0 + \beta x_2)k \wedge u$  est le terme de Coriolis avec  $k \wedge u = (-u_2, u_1)$  et  $\rho_0$  est la densité du fluide. À droite,  $1_\omega$  représente la *fonction caractéristique* de l'ensemble  $\omega$  et  $\mathcal{T}$  est une source donnée. Le terme  $\tau \hat{u}_0$ , avec  $\tau \in \mathbb{R}$ , est une petite perturbation de la donnée initiale  $u_0$  et  $h = h(x, t)$  est le contrôle à déterminer. Le terme de Coriolis introduit un couplage d'ordre zéro dans les équations et entraîne un comportement différent du système selon la direction de l'espace. Par simplicité, on supposera que  $A = \gamma = f_0 = \beta = \rho_0 = 1$ .

On introduit l'espace  $H = \{v \in L^2(\Omega)^2: \operatorname{div} v = 0 \text{ dans } \Omega, v \cdot \nu = 0 \text{ sur } \Gamma\}$  et on suppose que  $u_0, \tau \hat{u}_0 \in H$ , avec  $\|\hat{u}_0\|_{0,\Omega} = 1$ . On suppose aussi que  $\mathcal{T} \in L^2(Q)$  et  $h \in L^2(\omega \times (0, T))$ . La solution satisfait  $u \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H) \cap H^1(0, T; V')$  et  $p \in \mathcal{D}'(0, T; L_0^2(\Omega))$ , où  $V = \{v \in H_0^1(\Omega)^2: \operatorname{div} v = 0 \text{ dans } \Omega\}$ .

Dans cette Note, on s'intéresse par l'existence de contrôles insensibilisants et  $\varepsilon$ -insensibilisants pour le système (1). Signalons que, dans les cas similaires des équations de la chaleur linéaire et semilinéaire, cette question a déjà été étudiée et résolue dans [1,2,12].

Soit  $\mathcal{O}$  un sous-ensemble ouvert non vide de  $\Omega$  et considérons la fonctionnelle  $\Phi$  définie par (4). On imposera l'hypothèse géométrique (7). Les résultats principaux de cette Note sont :

**Théorème 0.1.** *On suppose que  $T > 0$  et (7) est vérifiée. Alors, pour chaque  $\varepsilon > 0$ , il existe un contrôle  $h \in L^2(\omega \times (0, T))$  qui est  $\varepsilon$ -insensible pour  $\Phi$ .*

**Théorème 0.2.** *Dans les conditions du Théorème 0.1, si  $u_0 = 0$ , il existe une constante  $M(\Omega, \omega, \mathcal{O}, T) > 0$  telle que, si*

$$\int_0^T \int_{\Omega} \exp\left(\frac{M}{t^4}\right) \mathcal{T}^2 \, dx \, dt < \infty,$$

*alors il existe un contrôle  $h \in L^2(\omega \times (0, T))$  qui est insensible pour  $\Phi$ .*

Montrons les idées générales des arguments qui permettent démontrer ces résultats.

Tout d'abord, on peut caractériser les propriétés d'insensibilité (5) et (6) en termes de la contrôlabilité exacte à zéro et la contrôlabilité approchée d'un système en cascade. Plus précisément, on a

**Proposition 0.3.** *Soit  $(\bar{u}, \bar{p}, q, \pi)$  la solution du système couplé (8). Alors le problème de trouver un contrôle insensibilisant (resp.  $\varepsilon$ -insensibilisant) est équivalent au problème de trouver  $h$  tel que*

$$q|_{t=0} = 0 \quad (\text{resp. } \|q|_{t=0}\|_{0,\Omega} \leq \varepsilon). \quad (2)$$

En général, pour des problèmes linéaires, il est bien connu que la contrôlabilité approchée est équivalente à un résultat d'unicité du système adjoint associé et que la contrôlabilité exacte équivaut à une estimation d'observabilité appropriée.

**Proposition 0.4.** *On suppose (7). Soit  $(z, r, \phi, \theta)$  la solution du système (10). Si  $z = 0$  dans  $\omega \cap \mathcal{O} \times (0, T)$ , alors  $\phi \equiv 0$  et  $\nabla \theta \equiv 0$  dans  $Q$ .*

Le résultat d'unicité précédent entraîne le Théorème 0.1. D'autre part, le Théorème 0.2 est une conséquence de l'inégalité d'observabilité suivante :

**Proposition 0.5.** *Il existe des constantes  $M, C > 0$  telles que*

$$\int_0^T \int_{\Omega} \exp\left(-\frac{M}{t^4}\right) z^2 \, dx \, dt \leq C \int_0^T \int_{\omega} z^2 \, dx \, dt,$$

pour toute solution de (10) avec  $\phi_0 \in V$ .

La Proposition 0.5 repose sur une inégalité globale de Carleman pour le système en cascade (10) :

**Proposition 0.6.** *On suppose (7). On considère les fonctions (12), où  $\eta_0$  est une fonction positive convenablement choisie dans  $C^4(\overline{\Omega})$ . Il existe  $\hat{s} = \hat{s}(\Omega, \omega, \mathcal{O}, T)$ , avec  $\hat{s} > 0$ , tel que pour tout  $s > \hat{s}$  et  $\hat{\gamma} < 1$  il existe  $\hat{\lambda} > 1$ ,  $\hat{\beta} > 1$  et  $C > 0$  tel que pour  $\lambda > \hat{\lambda}$ ,  $\beta > \hat{\beta}$  et  $\gamma < \hat{\gamma}$ , on a l'inégalité (13) pour toute solution de (10) avec  $\phi_0 \in V$ .*

### 1. Introduction and main result

Let  $\Omega$  be a nonempty open bounded connected subset of  $\mathbb{R}^2$ , with boundary  $\Gamma$  of class  $\mathcal{C}^2$  and outwards unit normal vector  $\nu = \nu(x)$ . Let  $\omega$  be a nonempty open subset of  $\Omega$ ,  $T > 0$ ,  $Q = \Omega \times (0, T)$  and  $\Sigma = \Gamma \times (0, T)$ . We consider a linear quasi-geostrophic ocean model [9,10], described by the following equations

$$\begin{cases} u_t - A\Delta u + \gamma u + (f_0 + \beta x_2)k \wedge u + \frac{1}{\rho_0} \nabla p = \mathcal{T} + h1_{\omega} & \text{in } Q, \\ \operatorname{div} u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u|_{t=0} = u_0 + \tau \hat{u}_0 & \text{in } \Omega. \end{cases} \tag{3}$$

Here,  $u = u(x, t)$  and  $p = p(x, t)$  are the velocity field and the pressure of the fluid. In this model,  $A$  is the horizontal eddy viscosity coefficient,  $\gamma$  is the bottom friction coefficient,  $\rho_0$  is the fluid density and  $(f_0 + \beta x_2)k \wedge u$  is the Coriolis term, with  $k \wedge u = (-u_2, u_1)$ . In the right-hand side,  $1_{\omega}$  denotes the characteristic function of  $\omega$  and  $\mathcal{T}$  is a given source. The term  $\tau \hat{u}_0$ , where  $\tau \in \mathbb{R}$ , represents a small unknown perturbation of the initial condition  $u_0$  and  $h = h(x, t)$  is a control term to be determined.

Notice that the Coriolis force is modelled by a zero order coupling term in the equations. It introduces a different behavior of the system depending on the direction in space. In order to simplify the presentation of the results, we will assume that  $A = \gamma = f_0 = \beta = \rho_0 = 1$ .

Let us recall the classical spaces  $H = \{v \in L^2(\Omega)^2: \operatorname{div} v = 0 \text{ in } \Omega, v \cdot \nu = 0 \text{ on } \Gamma\}$  and  $V = \{v \in H_0^1(\Omega)^2: \operatorname{div} v = 0 \text{ in } \Omega\}$ .

Assume that  $u_0, \tau \hat{u}_0 \in H$ , with  $\|\hat{u}_0\|_{0,\Omega} = 1$ ,  $\mathcal{T} \in L^2(Q)$  and  $h \in L^2(\omega \times (0, T))$ . Then problem (3) possesses exactly one solution  $(u, p)$ , with  $u \in L^2(0, T; V) \cap H^1(0, T; V) \cap C([0, T]; H)$  and  $p \in \mathcal{D}'(0, T; L_0^2(\Omega))$ . This can be easily proved by adapting the arguments of [11].

Let  $\mathcal{O}$  be a nonempty subset of  $\Omega$  and let us introduce the functional  $\Phi$ , with

$$\Phi(u) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |u(x, t)|^2 \, dx \, dt. \tag{4}$$

The notion of *insensitizing control* was introduced by J.-L. Lions in [8]. In the context of (3), it can be formulated as follows:

**Definition 1.1.** We say that the control  $h$  is insensitizing  $\Phi$  if

$$\frac{d}{d\tau} \Phi(u) \Big|_{\tau=0} = 0, \quad \forall \hat{u}_0 \in H, \quad \|\hat{u}_0\|_{0,\Omega} = 1. \tag{5}$$

On the other hand, we say that  $h$  is  $\varepsilon$ -insensitizing  $\Phi$  if

$$\left| \frac{d}{d\tau} \Phi(u) \Big|_{\tau=0} \right| \leq \varepsilon, \quad \forall \hat{u}_0 \in H, \quad \|\hat{u}_0\|_{0,\Omega} = 1. \tag{6}$$

Of course, in (5) and (6),  $u$  is, together with  $p$ , the unique solution to (3).

The existence of insensitizing and  $\varepsilon$ -insensitizing controls for linear and semilinear heat equations has been studied in [1,2,12]. In this Note, we will be concerned with this question in the case of the previous Stokes-like system.

As in the previous references, we will impose the following geometrical hypothesis:

$$\omega \cap \mathcal{O} \neq \emptyset. \tag{7}$$

The main results of this Note are the following:

**Theorem 1.2.** *Let  $T > 0$  and assume that (7) is satisfied. Then, for each  $\varepsilon > 0$ , there exists a control  $h \in L^2(\omega \times (0, T))$  which is  $\varepsilon$ -insensitizing  $\Phi$ .*

**Theorem 1.3.** *Under the assumptions of Theorem 1.2, if we also have  $u_0 = 0$ , there exists a constant  $M(\Omega, \omega, \mathcal{O}, T) > 0$  such that, whenever*

$$\iint_{0 \leq t \leq T} \exp\left(\frac{M}{t^4}\right) T^2 \, dx \, dt < \infty,$$

*there exists a control  $h \in L^2(\omega \times (0, T))$  which is insensitizing  $\Phi$ .*

## 2. Sketch of proof of the main results

Let us first give a characterization of the insensitivity (resp.  $\varepsilon$ -insensitivity) properties in terms of the null controllability (resp. approximate controllability) of an appropriate cascade system. This is obtained by differentiating  $\Phi$  with respect to the parameter  $\tau$  and following a standard argument, see [1,12]:

**Proposition 2.1.** *Let  $(\bar{u}, \bar{p}, q, \pi)$  be the solution of the following system:*

$$\begin{cases} \bar{u}_t - \Delta \bar{u} + \bar{u} + (1 + x_2) \mathbf{k} \wedge \bar{u} + \nabla \bar{p} = \mathcal{T} + h 1_\omega & \text{in } Q, \\ -q_t - \Delta q + q - (1 + x_2) \mathbf{k} \wedge q + \nabla \pi = \bar{u} 1_{\mathcal{O}} & \text{in } Q, \\ \operatorname{div} \bar{u} = 0, \quad \operatorname{div} q = 0 & \text{in } Q, \\ \bar{u} = q = 0 & \text{on } \Sigma, \\ \bar{u}|_{t=0} = u_0, \quad q|_{t=T} = 0 & \text{in } \Omega. \end{cases} \tag{8}$$

*Then the control  $h$  is insensitizing  $\Phi$  (resp.  $\varepsilon$ -insensitizing  $\Phi$ ) if and only if*

$$q|_{t=0} = 0 \quad (\text{resp. } \|q|_{t=0}\|_{0,\Omega} \leq \varepsilon). \tag{9}$$

In general, for linear problems, it is well known that approximate controllability is equivalent to a *unique continuation property* of the associate adjoint system and that exact controllability reduces to suitable observability estimates. We will now present the main steps in the proofs of Theorems 1.2 and 1.3, which are inspired by these general principles. Full versions of the proofs will appear in a forthcoming paper.

**Proposition 2.2.** Assume (7). Let  $(z, r, \phi, \theta)$  be the solution to

$$\begin{cases} -z_t - \Delta z + z - (1 + x_2)k \wedge z + \nabla r = \phi 1_{\mathcal{O}} & \text{in } Q, \\ \phi_t - \Delta \phi + \phi + (1 + x_2)k \wedge \phi + \nabla \theta = 0 & \text{in } Q, \\ \operatorname{div} z = 0, \quad \operatorname{div} \phi = 0 & \text{in } Q, \\ z = 0, \quad \phi = 0 & \text{on } \Sigma, \\ z|_{t=T} = 0, \quad \phi|_{t=0} = \phi_0 & \text{in } \Omega, \end{cases} \tag{10}$$

where  $\phi_0 \in V$ . Then, if  $z = 0$  in  $\omega \times (0, T)$ , we necessarily have  $z \equiv \phi \equiv 0$  and  $\nabla r \equiv \nabla \theta \equiv 0$  in  $Q$ .

**Sketch of the proof.** Let us put  $\tilde{\omega} = \omega \cap \mathcal{O}$ . We will prove a more general result saying that

$$\begin{cases} -z_t - \Delta z + z - (1 + x_2)k \wedge z + \nabla r = \phi 1_{\mathcal{O}} & \text{in } Q, \\ \phi_t - \Delta \phi + \phi + (1 + x_2)k \wedge \phi + \nabla \theta = 0 & \text{in } Q, \\ \operatorname{div} z = 0, \quad \operatorname{div} \phi = 0 & \text{in } Q, \\ \phi_1 = 0 & \text{on } \Sigma_1, \quad \phi|_{t=0} \in V, \end{cases} \tag{11}$$

where  $\Sigma_1 = (\Gamma \cap C_1) \times (0, T)$ ,  $C_1 = \{(x_1, x_2) \in \Omega : \exists x_1^0 \text{ s.t. } (x_1^0, x_2) \in \tilde{\omega}\}$ . Then, if  $z = 0$  in  $\omega \times (0, T)$ , we necessarily have  $\phi \equiv 0$  in  $Q$ . We first notice that  $\operatorname{curl} \phi = 0$  in  $\tilde{\omega} \times (0, T)$ . Then, applying the curl operator to the second equation of (10), in view of the presence of the Coriolis term and the fact that  $\operatorname{div} \phi = 0$ , we deduce that  $\phi_2 = 0$  in  $\tilde{\omega} \times (0, T)$  and  $\phi_1$  is a constant in  $\tilde{\omega} \times (0, T)$ .

Let us introduce  $\psi = \partial \phi / \partial x_1$  and  $\pi = \partial \theta / \partial x_1$ . We have

$$\begin{cases} \psi_t - \Delta \psi + \psi + (1 + x_2)k \wedge \psi + \nabla \pi = 0 & \text{in } Q, \\ \operatorname{div} \psi = 0 & \text{in } Q, \\ \psi = 0 & \text{in } \tilde{\omega} \times (0, T), \\ (\psi, \pi) \in L^2(0, T; H) \times \mathcal{D}'(Q). \end{cases}$$

From the uniqueness property in [4], one has that  $\psi \equiv 0$  in  $Q$ . Since  $\partial \phi_i / \partial x_1 \equiv 0$ ,  $i = 1, 2$ , and  $\operatorname{div} \phi = 0$  in  $Q$ , we have  $\nabla \phi_2 = 0$  in  $Q$ , and from  $\phi_2 = 0$  in  $\tilde{\omega} \times (0, T)$ , we deduce that  $\phi_2 \equiv 0$  in  $Q$ . On the other hand, since  $\partial \phi_1 / \partial x_1 = 0$  in  $Q$  and  $\phi_1 = 0$  on  $\Sigma_1$ , we obtain  $\phi_1 = 0$  in  $C_1 \times (0, T)$ . The uniqueness properties in [3] give  $\phi_1 \equiv 0$  in  $Q$ , as desired.  $\square$

**Remark 1.** In [3], appropriate unique continuation properties have been deduced for the Stokes system, where the main assumption is that all components of the velocity except one vanish in an open nonempty subset of  $Q$ . At present, we do not know whether a similar result remains true for (10).

The unique continuation property contained in the statement of Proposition 2.2 is equivalent to Theorem 1.2. This proves our first main result. On the other hand, Theorem 1.3 follows from the following observability estimates:

**Proposition 2.3.** There exist constants  $M, C > 0$  depending on  $\Omega$ ,  $\omega$ ,  $\mathcal{O}$  and  $T$  such that the following inequality holds true

$$\int_0^T \int_{\Omega} \exp\left(-\frac{M}{t^4}\right) z^2 \, dx \, dt \leq C \int_0^T \int_{\omega} z^2 \, dx \, dt$$

for every solution of (10) with  $\phi_0 \in V$ .

The proof of Proposition 2.3 relies on certain global Carleman estimates for the cascade system (10). More precisely, let us introduce the weight functions

$$\begin{cases} \alpha(x, t) = (e^{2\lambda\|\eta_0\|_\infty} - e^{\lambda\eta_0})t^{-4}(T-t)^{-4}, & \hat{\alpha}(t) = \min_{\overline{\Omega}} \alpha(x, t), \\ \varphi(x, t) = (e^{\lambda\eta_0})t^{-4}(T-t)^{-4}, & \hat{\varphi}(t) = \max_{\overline{\Omega}} \varphi(x, t), \end{cases} \quad (12)$$

where  $\eta_0 \in C^4(\overline{\Omega})$  is a function satisfying:

$$\eta_0 > 0 \quad \text{in } \Omega, \quad \eta_0 = 0 \quad \text{on } \Gamma, \quad |\nabla\eta_0| > 0 \quad \text{in } \Omega \setminus (\overline{\omega \cap \mathcal{O}}).$$

The existence of such a function  $\eta_0$  is guaranteed in [6].

**Proposition 2.4.** *Assume (7). There exists  $\hat{s} = \hat{\sigma}(\Omega, \omega, \mathcal{O}, T)$ , with  $\hat{\sigma} > 0$ , such that for all  $s > \hat{s}$  and for all  $\hat{\gamma} < 1$  there exist  $\hat{\lambda} > 1$ ,  $\hat{\beta} > 1$ , and  $C > 0$  such that for any  $\lambda > \hat{\lambda}$ ,  $\beta > \hat{\beta}$ , and  $\gamma < \hat{\gamma}$ , one has:*

$$\int_0^T \int_{\Omega} e^{-2s\alpha} \left( \frac{1}{s\varphi} (|\Delta z|^2 + |z_t|^2) + s\lambda^2 \varphi |\nabla z|^2 + s^3 \lambda^4 \varphi^3 |z|^2 \right) dx dt \leq C \int_0^T \int_{\omega} e^{-(1+\gamma)s\hat{\alpha}} \hat{\varphi}^\beta |z|^2 dx dt \quad (13)$$

for every solution of (10) with  $\phi_0 \in V$ , where the constant  $C$  depends on  $\Omega, \omega, \mathcal{O}$  and  $T$ , but is independent of  $(z, r, \phi, \theta)$ .

**Sketch of the proof.** We first write the global Carleman inequalities for the Stokes equations given in [7,5] for both systems in (10). The task is then to estimate the integral of  $\phi$  in terms of  $z$ . To this end, the main idea is to follow the steps of the proof of Proposition 2.2 in the reversed order. Thus, we estimate  $\phi_1$  in terms of  $\nabla\phi_1$ , then  $\nabla\phi_1$  in terms of  $\phi_2$ , then  $\phi_2$  in terms of  $\text{curl } \phi$  and, finally,  $\text{curl } \phi$  in terms of  $z$ . At each step, the weight increases.  $\square$

**Remark 2.** It must be emphasized that the presence of the Coriolis term plays a crucial role in this argument.

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