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Derivation of the k–ε model for locally homogeneous turbulence by homogenization techniques

Tomás Chacón Rebollo, Daniel Franco Coronil

Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, 41012 Sevilla, Spain

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Presented by Olivier Pironneau

Abstract

We derive the incompressible and compressible k–ε model for locally homogeneous turbulence. The model is rigorously derived on formal mathematical grounds using the MPP modelling technique. This lets us calculate by either analytical or numerical means the closure constants of the model. ***To cite this article: T. Chacón Rebollo, D. Franco Coronil, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Dérivation du modèle k–ε de turbulence localement homogène par des techniques d'homogénéisation. Nous obtenons le modèle k–ε de turbulence incompressible et compressible. Le modèle est dérivé rigoureusement sur des bases mathématiques formelles, en utilisant la technique MPP de modélisation. Ceci nous permet de calculer, aussi bien analytiquement, que bien numériquement, les constantes de fermeture du modèle. ***Pour citer cet article : T. Chacón Rebollo, D. Franco Coronil, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Dans ce travail nous obtenons le modèle de turbulence k–ε pour une turbulence localement homogène, par des techniques mathématiques formelles.

Nous utilisons comme technique de départ le modèle MPP de turbulence (cf. [5]). Ce modèle utilise le formalisme des techniques d'homogénéisation des matériaux composites. Il constitue une technique systématique pour trouver des équations moyennes des écoulements à deux échelles. La perturbation turbulente est déterminée comme solution des Equations d'Euler 3D.

La principale difficulté de la procédure MPP est de trouver des conditions initiales et aux limites pour les équations d'Euler qui déterminent la perturbation. Ceci a pu conduire à considérer le modèle MPP comme un nouveau modèle dans le cadre de la modélisation de la turbulence (cf. [3]).

E-mail address: franco@us.es (D. Franco Coronil).

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Dans ce travail nous proposons de nouvelles conditions initiales et aux limites pour la perturbation. Ceci permet de la déterminer à partir d'une seule perturbation canonique. En plus, nous identifions les perturbations qui en résultent comme des fonctions quasi-périodiques. Les propriétés d'invariance de l'opérateur de moyennage de ces fonctions nous permettent de donner des définitions rigoureuses de turbulence localement homogène et isotrope.

Comme consequence, nous déterminons complètement la structure des termes de fermeture. Nous identifions le modèle MPP qui en résulte comme étant le modèle k–ε pour la turbulence localement homogène et, lorsque la perturbation initiale est invariante par changement de repère, pour la turbulence localement isotrope. De plus, notre formalisme nous permet de calculer par des méthodes numériques les constantes de fermeture du modèle.

1. Introduction

In this work we derive the k–ε model for locally homogeneous turbulence for incompressible and compressible flows, using only formal mathematical techniques.

For that, as a basic technique, we use the MPP turbulence modelling (cf. [5]). This technique consists in formally applying the homogenization techniques to the equations of fluid flows. This provides a systematic way of averaging flows in two space scales, with rigorous mathematical definitions of mean flow and perturbation. Moreover, this technique allows us to determine the perturbation, which appears to be the solution of a system of partial differential equations (the 3D Euler equations). The closure terms of the model are calculated from the perturbation, as usual when applying homogenization techniques to the analysis of multi-scale materials (cf. [2,3]).

The main difficulty of the MPP procedure is to set the initial and boundary conditions for the Euler equations satisfied by the perturbation. In preceding MPP models the perturbation was considered to be periodic and several initial conditions were proposed. This only allowed one to partially determine the structure of the closure terms. Essentially for this reason, MPP models were considered as new models in the context of turbulence modelling, although some links with the k–ε model were derived (cf. [3]).

In this work we give new initial and boundary conditions for the perturbation. The basic idea is to let the mean flow deform the period cell of the perturbation, and use the invariants of the Euler equations to set the initial conditions. This allows us to determine all perturbations in terms of a unique canonical perturbation. In addition, we observe that the perturbations that we determine in this way belong to the larger family of almost-periodic functions. The invariance properties of the average of almost-periodic functions leads us to rigorously define the notions of locally homogeneous and isotropic turbulence.

As a consequence, we fully determine the structure of the closure terms: We identify the resulting MPP model as the k–ε model for locally homogeneous turbulence and, for frame-invariant initial perturbations, locally isotropic turbulence. Furthermore, by our formalism, we obtain the closure constants of the k–ε model.

2. Incompressible model

2.1. Statement of the problem

We consider the initial formal framework of the MPP turbulence model (cf. [5]): we consider flows with two well-separated space scales of ratio δ, and turbulent perturbation located in the inertial range. We assume that this flow is governed by the Navier–Stokes equations for incompressible viscous fluid flows with kinematic viscosity of order δ^2:

$$u^{\delta}_{,t} + \left(u^{\delta} \cdot \nabla\right) u^{\delta} + \nabla p^{\delta} - \mu \delta^2 \Delta u^{\delta} = 0, \quad \nabla \cdot u^{\delta} = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{1}$$

with initial conditions in two scales given by

$$u^\delta(x,0)=u^0(x)=u_0(x)+\delta^{1/3}w^0\left(\frac{x}{\delta},x\right)\qquad \text{in } \mathbb{R}^3. \tag{2}$$

Here, u^δ and p^δ respectively denote velocity and pressure of the flow. The constant μ is a positive number of order one with respect to δ. Also, u_0 is a smooth velocity field in $\mathbb{R}^3$ and $w^0(y,x)$ is a smooth velocity field in $\mathbb{R}^3\times\mathbb{R}^3$, periodic in the variable y, with periodic cell $Y=[-\pi,\pi]^3$ and with zero mean in y:

$$\langle w^0\rangle=\frac{1}{|Y|}\int_Y w^0(y)\,\mathrm{d}y=0. \tag{3}$$

This formalism allows to rigorously define the initial mean field (u^0) and the initial turbulent perturbation ($\delta^{1/3}w^0$).

The MPP technique provides a systematic way to derive a set of partial differential equations that describes the asymptotic behavior of the solution (u^δ, p^δ) of (1), (2) as δ decreases to zero.

2.2. *Averaged equations model*

Following the procedure of the MPP model, we consider the following asymptotic expansions:

$$\left.\begin{aligned}
&u^\delta(x,t)\sim u(x,t)+\delta^{1/3}w\left(\frac{a(x,t)}{\delta},\frac{t}{\delta^{2/3}};x,t\right)+\mathrm{O}\big(\delta^{2/3}\big),\\
&p^\delta(x,t)\sim p(x,t)+\delta^{1/3}p^{(0)}\left(\frac{a(x,t)}{\delta},\frac{t}{\delta^{2/3}};x,t\right)+\delta^{2/3}\pi\left(\frac{a(x,t)}{\delta},\frac{t}{\delta^{2/3}};x,t\right)+\mathrm{O}\big(\delta^{2/3}\big)
\end{aligned}\right\}. \tag{4}$$

Here $a(x,t)$ are the inverse Lagrangian coordinates associated to the velocity u, given by:

$$a_{,t}+(u\cdot\nabla)a=0,\quad a(x,0)=x\quad \text{in } \mathbb{R}^3\times\mathbb{R}. \tag{5}$$

The inclusion of the variable a in the expansions (4) is a formalization of the Taylor hypothesis, which states that the turbulent perturbation is transported by the mean flow.

Furthermore, $w(y,\tau;x,t)$, $\pi(y,\tau;x,t)$ and the other high-order terms in the expansions, are smooth functions defined in $\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}$. In preceding MPP models, these functions are assumed to be Y-periodic in the variable y. Here we weaken this condition, and we assume them to be (AY)-periodic, where A is a regular 3×3 matrix that depends on the mean field, that we shall define later. From the physical point of view this formalizes the fact that the mean field deforms the period cell of the perturbation.

All (AY)-periodic functions are almost-periodic (the space of the almost-periodic functions is the closure of trigonometric polynomials in the L^2 norm (cf. [1])). Then, for example, the mean value of w is given by

$$\langle w\rangle=\frac{\int_{AY}w(y)\,\mathrm{d}y}{\int_{AY}\mathrm{d}y}=\lim_{R\to\infty}\frac{\int_{B(R)}w(y)\,\mathrm{d}y}{\int_{B(R)}\mathrm{d}y},$$

where $B(R)$ is the closed ball of $\mathbb{R}^3$ of radius R and center the origin. This mean value does not depend on the period cell of w. Moreover, it is invariant under translations and rotations.

We furthermore assume that the high-order terms in the expansions (4) are uniformly almost-periodic in the fast time variable τ. This ensures that the (y,τ) mean value given, by example for w, as

$$\langle\langle w\rangle\rangle(x,t)=\lim_{R\to\infty,\tau\to\infty}\frac{\int_{-\tau}^{\tau}\int_{B(R)}w(y,\sigma;x,t)\,\mathrm{d}y\,\mathrm{d}\sigma}{2\tau\int_{B(R)}\mathrm{d}y},$$

exists for all the terms in the expansions. This space–time averaged operator is translation-invariant (homogeneous) in space and time, and rotation-invariant (isotropic) in space.

Following the MPP modelling technique, we deduce the following averaged equations set (see [4] for details):

$$\left.\begin{aligned} &u_{,t} + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0 && \text{in } \mathbb{R}^3\times\mathbb{R},\\ &k_{,t} + (u\cdot\nabla)k + \delta^{2/3}\mathbf{R}:\nabla u + \mu\delta^{2/3}\psi_k = 0 && \text{in } \mathbb{R}^3\times\mathbb{R},\\ &h_{,t} + (u\cdot\nabla)h + \mu\delta^{2/3}\psi_h = 0 && \text{in } \mathbb{R}^3\times\mathbb{R},\end{aligned}\right\} \tag{6}$$

and also a system of partial differential equations "in microstructure" (in (y,τ), fast variables), for the perturbation (w,π): These are the 3D Euler equations,

$$\widetilde{w}_{,\tau} + (\widetilde{w}\cdot\nabla_y)\widetilde{w} + C\nabla_y\pi = 0, \quad \nabla_y\cdot\widetilde{w} = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{7}$$

where $\widetilde{w} = G^t w$ and $C = G^t G$ with $G = \nabla a$.

This model appears as a two equations turbulence model, including the mean turbulent kinetic energy k and the mean turbulent helicity h, respectively defined by:

$$k = \delta^{2/3}\langle|w|^2\rangle/2; \qquad h = \delta^{2/3}\langle w\cdot r\rangle/2, \quad \text{with } r = (G\nabla_y)\times w. \tag{8}$$

The closure terms of the model (6) are the Reynolds stress tensor $\mathbf{R} = \langle w\otimes w\rangle$ and the scalar functions, $\psi_k = \langle\!\langle |r|^2\rangle\!\rangle$ and $\psi_h = \langle\!\langle r\cdot((G\nabla_y)\times r)\rangle\!\rangle$, which represent the dissipation of mean turbulent kinetic energy k and mean turbulent helicity h, respectively.

In view of (7) and (8), it is reasonable to look for a perturbation w completely determined by the mean-field data G, h and k. To do this, we propose the following initial condition for Euler equations (7):

$$\widetilde{w}(y,0) = \delta^{-1/3}\sqrt{k}G^t w^0\left(\frac{h}{k}\big(G^{-t}y\big)\right), \tag{9}$$

and look for a solution $\widetilde{w}$ (AY)-periodic in space, with $A = \frac{h}{k}G^{-t}$. With these initial and boundary condition, we may recover $\widetilde{w}$ from a unique canonical fluctuation w^*, as states the following result (cf. [4]):

Theorem 2.1. *Assume that the canonical Euler problem*

$$\left.\begin{aligned} &w^*_{,\tau} + (w^*\cdot\nabla_y)w^* + \nabla_y\pi^* = 0, \quad \nabla_y\cdot w^* = 0 && \textit{in } \mathbb{R}^3\times\mathbb{R},\\ &w^*(y,0) = w^0(y), && \textit{in } \mathbb{R}^3,\\ &w^*,\ \pi^*\ Y\textit{-periodic in } y, \textit{ uniformly almost-periodic in } \tau \end{aligned}\right\} \tag{10}$$

admits a unique solution. Then, the problem

$$\left.\begin{aligned} &\widetilde{w}_{,\tau} + (\widetilde{w}\cdot\nabla_y)\widetilde{w} + C\nabla_y\pi = 0, \quad \nabla_y\cdot\widetilde{w} = 0 && \textit{in } \mathbb{R}^3\times\mathbb{R},\\ &\widetilde{w}(y,0) = \delta^{-1/3}\sqrt{k}G^t w^0\left(\frac{h}{k}\big(G^{-t}y\big)\right) && \textit{in } \mathbb{R}^3,\\ &\widetilde{w},\ \pi\ AY\textit{-periodic in } y, \textit{ uniformly almost-periodic in } \tau; \end{aligned}\right\} \tag{11}$$

with $A = \frac{h}{k}G^{-t}$, admits a unique solution given by

$$\left.\begin{aligned} &\widetilde{w}(y,\tau) = \delta^{-1/3}\sqrt{k}\,G^t w^*\left(\frac{h}{k}\big(G^{-t}y\big), \delta^{-1/3}\frac{h}{\sqrt{k}}\tau\right),\\ &\pi(y,\tau) = \delta^{-2/3}k\pi^*\left(\frac{h}{k}\big(G^{-t}y\big), \delta^{-1/3}\frac{h}{\sqrt{k}}\tau\right). \end{aligned}\right\} \tag{12}$$

In addition, this solution verifies

$$\big\langle w(\cdot,\tau)\big\rangle = 0, \quad \frac{1}{2}\delta^{2/3}\big\langle\big|w(\cdot,\tau)\big|^2\big\rangle = k, \quad \frac{1}{2}\delta^{2/3}\big\langle(w\cdot r)(\cdot,\tau)\big\rangle = h; \tag{13}$$

where $w = G^{-t}\widetilde{w}$, *at any time* $\tau \in \mathbb{R}$.

Notice that the effect of the mean velocity field on the perturbation is to change the shape of the period cell, while kinetic energy and helicity modify the size of this cell and of the perturbation itself.

The expression (12) for $\widetilde{w}$, together with its almost-periodic character allows us to rigorously determine the structure of the closure terms of our model. This structure is given by the following result:

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *we assume in addition that* $w^* \in C^2(\mathbb{R}^3 \times \mathbb{R})$ *and that* $r^* = \nabla_y \times w^*$ *and* $\nabla_y \times r^*$ *are uniformly almost-periodic in time. Then, the closure terms* $\mathbf{R}$, ψ_k *and* ψ_h *of the model* (6) *verify*

$$\mathbf{R} = \delta^{-2/3} k\,\mathbf{R}^*, \qquad \psi_k = \delta^{-2/3}\frac{h^2}{k}\psi_k^*, \qquad \psi_h = \delta^{-2/3}\frac{h^3}{k^2}\psi_h^*; \tag{14}$$

where

$$\mathbf{R}^* = \langle\langle w^* \otimes w^*\rangle\rangle, \qquad \psi_q^* = \langle\!\langle |r^*|^2 \rangle\!\rangle, \qquad \psi_h^* = \langle\!\langle r^* \cdot (\nabla_y \times r^*)\rangle\!\rangle, \tag{15}$$

with $r^* = \nabla_y \times w^*$.

Proof. To simplify the notation, let us denote $q = \delta^{-2/3}k$, $l = \delta^{-2/3}h$. We shall prove, for instance, the second identity in (14), the remaining follow from the same basic arguments.

We observe at first that $r(y,\tau) = \frac{l}{\sqrt{q}} r^*(\frac{l}{q}G^{-t}y, \frac{l}{\sqrt{q}}\tau)$. Then,

$$\begin{aligned}
\psi_k &= \lim_{(R,\tau)\to\infty} \frac{1}{2\tau|B(R)|}\int_{-\tau}^{\tau}\int_{B(R)} |r|^2(y,\sigma)\,\mathrm{d}y\,\mathrm{d}\sigma \\
&= \frac{l^2}{q}\lim_{(R,\tau)\to\infty}\frac{1}{2\tau\,|B(R)|}\int_{-\tau}^{\tau}\int_{B(R)} |r^*|^2\left(\frac{l}{q}G^{-t}y, \frac{l}{\sqrt{q}}\sigma\right)\mathrm{d}y\,\mathrm{d}\sigma.
\end{aligned}$$

Now, we write $2\tau|B(R)| = \int_{-\tau}^{\tau}\int_{B(R)} \mathrm{d}y\,\mathrm{d}\sigma$, and make the change of variables $z = \frac{l}{q}G^{-t}y$ and $\alpha = \frac{l}{\sqrt{q}}\sigma$. Then, since $\mathrm{Det}(G^t) = 1$ (because $\nabla \cdot u = 0$),

$$\psi_k = \frac{l^2}{q}\lim_{(R,\tau)\to\infty}\left(\int_{-\frac{l}{\sqrt{q}}\tau}^{\frac{l}{\sqrt{q}}\tau}\int_{K_R} |r^*|^2(z,\alpha)\,\mathrm{d}z\,\mathrm{d}\alpha\right)\left(\int_{-\frac{l}{\sqrt{q}}\tau}^{\frac{l}{\sqrt{q}}\tau}\int_{K_R}\mathrm{d}z\,\mathrm{d}\alpha\right)^{-1},$$

where $K_R = G^t B(|\frac{l}{q}|R) = RK_1$.

Now, K_1 is a bounded set, and r^* is Y-periodic in space, uniformly almost-periodic in time. Then, we may use the properties of almost-periodic functions (cf. [1]) to ensure that

$$\psi_k = \frac{l^2}{q}\langle\!\langle |r^*|^2\rangle\!\rangle,$$

and the conclusion follows. □

With respect to MPP turbulence models, here we drop the dependence of the closure terms upon the matrix G, and we explicit its dependence upon k and h. We stress that the main difference between MPP models and more classical turbulence models has been the dependence upon the Lagrangian co-ordinates (through this matrix G) of the closure terms. Our analysis shows that the perturbation itself depends upon G, while the closure terms do not.

Let us remark that the structure of ψ_k and ψ_h in terms of k and h is just the same as that obtained by dimensional analysis.

2.3. A k–ε-MPP incompressible model

Model (6) is equivalent to a two equations model for the statistics k and ε, with $\varepsilon = \mu\psi_k = \delta^{-2/3}\mu\frac{h^2}{k}\psi_k^*$, with $\psi_k^* = \psi_k^*(I)$. Indeed, we may combine the equations in (6) to replace the equation for h by an equation for ε. The pair of equations for (k, h) in (6) is found to be equivalent to

$$\left.\begin{aligned} &k_{,t} + (u\cdot\nabla)k + k\,\mathbf{R}^*:\nabla u + \varepsilon = 0 \qquad \text{in } \mathbb{R}^3\times\mathbb{R},\\ &\varepsilon_{,t} + (u\cdot\nabla)\varepsilon + d\frac{\varepsilon^2}{k} - \varepsilon\,\mathbf{R}^*:\nabla u = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}; \end{aligned}\right\} \quad \text{with } d = 2\frac{\psi_h^*}{\psi_k^*} - 1. \tag{16}$$

In view of the equation for k, we identify ε as the rate of viscous dissipation of the turbulent kinetic energy. The above model turns out to be the standard k–ε model for locally homogeneous turbulence.

This model is simplified if the turbulence is isotropic. This follows from the result:

Theorem 2.3. *Assume that the initial perturbation w^0 is invariant under all the rotations that leave invariant the cube Y: $w^0(Qy) = Qw^0(y),\ \forall y\in\mathbb{R}^3$, for all matrix rotations Q such that $QY = Y$. Then, the perturbation w^* also is invariant in the same sense.*

This result follows from the invariance under rotations of Euler equations (10).

As a consequence, the canonical tensor $\mathbf{R}^*$ is also invariant under the same rotations:

$\mathbf{R}^* = Q^{\mathrm{T}}\mathbf{R}^*Q$ for all matrix rotations Q such that $QY = Y$. In turn, a careful choice of some matrices Q satisfying this property allows to prove that this implies $\mathbf{R}^* = \frac{2}{3}I$. Then, model (16) is reduced to:

$$\left.\begin{aligned} &k_{,t} + (u\cdot\nabla)k + \varepsilon = 0 \qquad \text{in } \mathbb{R}^3\times\mathbb{R},\\ &\varepsilon_{,t} + (u\cdot\nabla)\varepsilon + d\frac{\varepsilon^2}{k} = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}. \end{aligned}\right\} \tag{17}$$

We identify these equations as the standard k–ε model for locally homogeneous and isotropic turbulence. In the classical turbulence modelling theory, this model is obtained by dimensional analysis, and the constant d is calculated from experimental measurements. In our case, d is obtained from w^*. A numerical solution of (10) with $64\times 64\times 64$ degrees of freedom provides the value $d \simeq 1.58$ (cf. [4]), to be compared to the experimental values, ranging from 1.72 to 2.01 (cf. [6]). Notice that from our analysis this constant d depends upon the initial perturbation w^0.

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