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Numerical Analysis

Numerical analysis of the electric field formulation of an eddy current problem

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Abstract

In this paper we analyze a finite element method for the numerical solution of an eddy current problem in a bounded conducting domain. We use a weak formulation in terms of the electric field and impose non-local non-standard boundary conditions. The unique data are the input current intensities which are imposed by means of some special curves lying on the boundary of the domain. Optimal error estimates are shown and implementation issues are discussed. ***To cite this article: A. Bermúdez et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Analyse numérique de la formulation en champ électrique d'un problème de courants de Foucault. Dans cet article on analyse une méthode d'éléments finis pour la résolution numérique d'un problème de courants de Foucault dans un domaine conducteur borné. On utilise une formulation en champ électrique et on impose des conditions aux limites non-standard et non-locales. Les seules données sont les intensités d'entrée à travers la frontière du domaine qui sont imposées à l'aide de courbes contenues dans celle-ci. On démontre des estimations d'erreur optimales et on donne quelques indications sur l'implementation de la méthode sur ordinateur. ***Pour citer cet article : A. Bermúdez et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

On s'intéresse à un domaine conducteur borné qui est traversé par un courant alternatif de fréquence angulaire ω. Dans ce cas le modèle est constitué des Éqs. (1)–(3). Dans [3] et [4] nous avons étudié ce problème avec des conditions aux limites réalistes d'un point de vue pratique. Plus précisement, associé à la décomposition de la frontière donnée à la Section 2 nous considérons les conditions aux limites (4)–(8) où les seules données sont les intensités du courant, I_n, $n = 1, \ldots, N$. Ce problème a été analysé dans [4] en faisant une formulation en termes du champ magnétique (voir le Problème MP à la Section 2). Dans cet article nous étudions le même problème mais en utilisant une formulation en *champ électrique* introduite dans [7].

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Nous appelons $\partial\Omega = \overline{\Gamma}_{\mathbf{E}}^0 \cup \overline{\Gamma}_{\mathbf{J}}^0$: $\Gamma_{\mathbf{E}}^0$ les parties de la frontière du domaine qui correspondent aux entrées et sortie du courant tandis que l'on note par $\Gamma_{\mathbf{J}}^0$ celle qui n'est pas traversée par le courant. Soit $\mathcal{E} := \{\mathbf{G} \in \mathrm{H}(\mathbf{curl}, \Omega)\colon\ \mathbf{G} \times \mathbf{n} = \mathbf{0}$ dans $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathbf{E}}^0)$ et $\mathbf{curl}\,\mathbf{G} \cdot \mathbf{n} = 0$ dans $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathbf{J}}^0)\}$ et $a^E : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ la forme sesquilinéaire et continue définie par

$$a^E(\mathbf{E}, \mathbf{G}) := \int_\Omega \sigma \mathbf{E} \cdot \overline{\mathbf{G}} + \int_\Omega \frac{1}{\mathrm{i}\omega\mu} \mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{G}}.$$

Soit $\mathbf{H_I}$ la solution du Problème MP (voir Section 2). On considère le

Problem PE. Trouver $\mathbf{E} \in \mathcal{E}$ tel que

$$a^E(\mathbf{E}, \mathbf{G}) = L^{\mathbf{I}}(\mathbf{G}) \quad \forall \mathbf{G} \in \mathcal{E},$$

où $L^{\mathbf{I}}(\mathbf{G}) := \int_\Omega \mathbf{curl}\,\mathbf{H_I} \cdot \overline{\mathbf{G}} - \int_\Omega \mathbf{H_I} \cdot \mathbf{curl}\,\overline{\mathbf{G}}$.

Théorème 0.1. *Le Problème* PE *admet une solution unique* $\mathbf{E_I} = \frac{1}{\sigma}\mathbf{curl}\,\mathbf{H_I}$.

Pour résoudre ce problème numériquement nous considérons une famille de maillages de tetraèdres regulière de Ω, $\{\mathcal{T}_h\}$, où, comme d'habitude, h représente la taille du maillage correspondant. Alors le champ électrique est discrétisé par des éléments finis d'arête de Nédélec. Plus precisément, l'espace discret pour approcher $\mathcal{E}$ est défini par (9). Pour traiter la contrainte $\mathbf{curl}\,\mathbf{E}_h \cdot \mathbf{n} = 0$ qui apparaît dans la définition de $\mathcal{E}_h$, on introduit un multiplicateur de Lagrange discret défini sur $\Gamma_{\mathbf{J}}^0$. Alors, avec les espaces discrets $\mathfrak{F}_h$ et $\mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$ définis dans (12) et (11), respectivement, on obtient le problème approché suivant :

Problem MPED. Trouver $\mathbf{E}_h \in \mathfrak{F}_h$ et $\lambda_h \in \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$ tels que

$$a^E(\mathbf{E}_h, \mathbf{G}_h) + b^E(\overline{\mathbf{G}}_h, \lambda_h) = L^{\mathbf{I}}(\mathbf{G}_h) \quad \forall \mathbf{G}_h \in \mathfrak{F}_h,$$
$$b^E(\mathbf{E}_h, \nu_h) = 0 \quad \forall \nu_h \in \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0),$$

où b^E est la forme sesquilinéaire définie dans $\mathfrak{F}_h \times \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$ par $b^E(\mathbf{G}_h, \nu_h) := \int_{\Gamma_{\mathbf{J}}^0} \mathbf{curl}\,\mathbf{G}_h \cdot \mathbf{n}\,\bar{\nu}_h$. Nous remarquons que $L^{\mathbf{I}}(\mathbf{G}_h)$ est calculé à l'aide de (10) qui ne depend que des intensités, les seules données du problème.

Théorème 0.2. *Le Problème* MPED *admet une solution unique* $(\mathbf{E}_h, \lambda_h) \in \mathfrak{F}_h \times \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$. *Par ailleurs, si la solution* $\mathbf{E}$ *du Problème* PE *appartient à* $\mathrm{H}^r(\mathbf{curl}, \Omega)$, *avec* $r > 1/2$, *alors on a l'estimation d'erreur*

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} \leqslant Ch^r \|\mathbf{E}\|_{\mathrm{H}^r(\mathbf{curl},\Omega)}.$$

1. Introduction

The aim of this paper is to introduce and analyze a finite element method to solve an eddy current problem in a bounded conducting domain. The work is motivated by the need of a three-dimensional model to simulate metallurgical electrodes and to improve the axisymmetric models already developed to this goal (see for instance [1] and [2]). However the method can be applied to more general eddy current problems. The mathematical and numerical analysis developed in this work completes somehow the results obtained in [3] and [4]. In those papers, we also solve the eddy current problem in a bounded conducting domain by using non-standard boundary conditions which include the input current intensities as unique boundary data. In [4] the problem is formulated and then analyzed by using the magnetic field as the only unknown. In the present work, we study a similar problem but making a weak formulation in terms of the electric field. This formulation has been introduced in [7] by using some topological concepts.

2. The eddy current model

We are interested in a bounded conducting domain Ω (the electrode) crossed by an alternating electric current of "low" angular frequency ω. In this case the model reduces to the low-frequency harmonic Maxwell equations

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J} \quad \text{in } \Omega, \tag{1}$$

$$\mathrm{i}\omega\mu\mathbf{H} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{J} = \sigma\mathbf{E} \quad \text{in } \Omega, \tag{3}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{J}$ the current density and $\mathbf{E}$ the electric field, all of them complex valued (see, for instance, [6]). Coefficients μ and σ are the magnetic permeability and the electric conductivity, respectively. We assume that $\mu, \sigma \in \mathrm{L}^\infty(\Omega)$ and there exist constants $\underline{\mu}$ and $\underline{\sigma}$ such that $\mu(\mathbf{x}) \geqslant \underline{\mu} > 0$ and $\sigma(\mathbf{x}) \geqslant \underline{\sigma} > 0$, a.e. in Ω.

The 3D domain Ω is assumed to have a Lipschitz-continuous and connected boundary $\partial\Omega$. However, it is not necessary that Ω be simply connected. This boundary splits into two surfaces of non-zero 2D-measure $\Gamma_\mathbf{E}$ and $\Gamma_\mathbf{J}$: $\partial\Omega = \overline{\Gamma}_\mathbf{E} \cup \overline{\Gamma}_\mathbf{J}$. The (open) surface $\Gamma_\mathbf{E}$ corresponds to the tip of the electrode where the electric arc arises. In its turn, the rest of the electrode boundary splits as follows: $\overline{\Gamma}_\mathbf{J} = \overline{\Gamma}_\mathbf{J}^0 \cup \overline{\Gamma}_\mathbf{J}^1 \cup \ldots \cup \overline{\Gamma}_\mathbf{J}^N$, where $\Gamma_\mathbf{J}^n$, $n = 1, \ldots, N$, are the (open) parts of the boundary connected to the wires supplying electric current to the electrode and $\Gamma_\mathbf{J}^0 = \Gamma_\mathbf{J} \setminus (\overline{\Gamma}_\mathbf{J}^1 \cup \ldots \cup \overline{\Gamma}_\mathbf{J}^N)$ is the remaining part. Finally, we also assume that $\Gamma_\mathbf{E}$ and $\Gamma_\mathbf{J}^0$ are connected, $\overline{\Gamma}_\mathbf{J}^n \cap \overline{\Gamma}_\mathbf{E} = \emptyset$ and $\overline{\Gamma}_\mathbf{J}^n \cap \overline{\Gamma}_\mathbf{J}^m = \emptyset$, $m, n = 1, \ldots, N$, $m \neq n$ (see Fig. 1).

In order to solve the problem (1)–(3) by using realistic boundary data from the point of view of applications, we have introduced the following conditions in [4]:

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_\mathbf{E}, \tag{4}$$

$$\int_{\Gamma_\mathbf{J}^n} \mathbf{J} \cdot \mathbf{n} = I_n \quad \text{on } \Gamma_\mathbf{J}^n, \quad n = 1, \ldots, N, \tag{5}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_\mathbf{J}^n, \quad n = 1, \ldots, N, \tag{6}$$

$$\mathbf{J} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_\mathbf{J}^0, \tag{7}$$

$$\mu\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{8}$$

where I_n, $n = 1, \ldots, N$, are the current intensities through the wires.

Some of the previous boundary conditions are neither *essential* nor *natural*. Therefore, in order to solve the problem, we have introduced in [4] a Lagrange multiplier defined on the boundary. The resulting mixed formulation involving the magnetic field and the Lagrange multiplier has been analyzed in detail in both the continuous and the discrete cases.

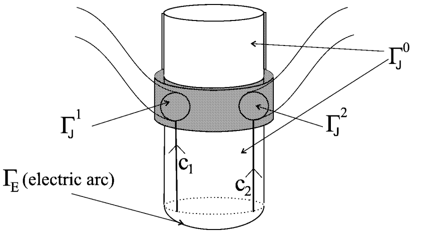

Fig. 1. Sketch of an electrode.

Fig. 1. Une électrode.

Throughout this paper, we use standard notation for function spaces $\mathrm{H}(\mathrm{div},\Omega)$ and $\mathrm{H}(\mathbf{curl},\Omega)$. We also use the space $\mathrm{H}^r(\mathbf{curl},\Omega) := \{\mathbf{G} \in \mathrm{H}^r(\Omega)^3\colon\ \mathbf{curl}\,\mathbf{G} \in \mathrm{H}^r(\Omega)^3\}$, for each real number $r > 0$. Each of these spaces is endowed with its natural norm, v.g., $\|\mathbf{G}\|^2_{\mathrm{H}^r(\mathbf{curl},\Omega)} = \|\mathbf{G}\|^2_{\mathrm{H}^r(\Omega)^3} + \|\mathbf{curl}\,\mathbf{G}\|^2_{\mathrm{H}^r(\Omega)^3}$.

Let us denote by $\mathrm{H}^{1/2}_{00}(\Gamma_{\mathbf{J}})$ the space of functions defined on $\Gamma_{\mathbf{J}}$ that extended by 0 to $\partial\Omega \setminus \Gamma_{\mathbf{J}}$ belong to $\mathrm{H}^{1/2}(\partial\Omega) := \{\nu|_{\partial\Omega}\colon\ \nu \in \mathrm{H}^1(\Omega)\}$. Let $\mathrm{H}^{-1/2}_{00}(\Gamma_{\mathbf{J}})$ be the dual space of $\mathrm{H}^{1/2}_{00}(\Gamma_{\mathbf{J}})$ and $\langle\cdot,\cdot\rangle_{\Gamma_{\mathbf{J}}}$ the corresponding duality pairing. Before introducing a weak formulation in terms of the electric field, we need to recall some questions concerning the mixed problem studied in [4]. For this purpose, we define the spaces

$$\mathcal{X} := \mathrm{H}(\mathbf{curl},\Omega),$$
$$\mathcal{L} := \left\{\nu \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathbf{J}})\colon\ \nu|_{\Gamma^n_{\mathbf{J}}} = \text{constant},\ n = 1,\dots,N\right\}.$$

Let $a^H : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be the sesquilinear continuous and elliptic form defined by

$$a^H(\mathbf{H},\mathbf{G}) := \mathrm{i}\omega \int_\Omega \mu \mathbf{H} \cdot \overline{\mathbf{G}} + \int_\Omega \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\overline{\mathbf{G}},$$

and b^H be the sesquilinear continuous form defined on $\mathcal{X} \times \mathcal{L}$ by $b^H(\mathbf{G},\nu) := \langle \mathbf{curl}\,\mathbf{G}\cdot\mathbf{n}, \nu\rangle_{\Gamma_{\mathbf{J}}}$. Given (complex) input intensities $I_1,\dots,I_N$ through each wire, we denote $\mathbf{I} = (I_1,\dots,I_N) \in \mathbb{C}^N$. In [4] we make the following mixed formulation in terms of the magnetic field and a Lagrange multiplier λ_I:

Problem MP. Given $\mathbf{I} \in \mathbb{C}^N$, find $\mathbf{H}_{\mathbf{I}} \in \mathcal{X}$ and $\lambda_{\mathbf{I}} \in \mathcal{L}$ such that

$$a^H(\mathbf{H}_{\mathbf{I}},\mathbf{G}) + b^H(\overline{\mathbf{G}},\lambda_{\mathbf{I}}) = 0 \quad \forall \mathbf{G} \in \mathcal{X},$$
$$b^H(\mathbf{H}_{\mathbf{I}},\nu) = \sum_{n=1}^{\mathrm{N}} I_n \bar{\nu}|_{\Gamma^n_{\mathbf{J}}} \quad \forall \nu \in \mathcal{L}.$$

We have proved that this problem has a unique solution which satisfies the Maxwell equations (1)–(3) and the boundary conditions (4)–(8) in a weak sense. Moreover, we have shown that the Lagrange multiplier $\lambda_{\mathbf{I}}$ is actually the electrical potential on the boundary. Numerical solution to this problem has been also discussed in that paper.

3. A weak formulation in terms of the electric field

The aim of this work is to analyze the problem (1)–(3) with boundary conditions (4)–(8) by using a weak formulation in terms of the *electric field* instead of the magnetic field.

Since $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_{\mathbf{E}}$ and on $\Gamma^n_{\mathbf{J}}$, $n = 1,\dots,N$, we denote by $\Gamma^0_{\mathbf{E}} = \Gamma_{\mathbf{E}} \cup \Gamma^1_{\mathbf{J}} \cup \dots \cup \Gamma^n_{\mathbf{J}}$, the subset of the boundary where $\mathbf{E} \times \mathbf{n} = \mathbf{0}$. Therefore $\partial\Omega = \overline{\Gamma}^0_{\mathbf{E}} \cup \overline{\Gamma}^0_{\mathbf{J}}$ and we define

$$\mathcal{E} := \left\{\mathbf{G} \in \mathrm{H}(\mathbf{curl},\Omega)\colon\ \mathbf{G} \times \mathbf{n} = \mathbf{0} \text{ in } \mathrm{H}^{-1/2}_{00}\big(\Gamma^0_{\mathbf{E}}\big) \text{ and } \mathbf{curl}\,\mathbf{G}\cdot\mathbf{n} = 0 \text{ in } \mathrm{H}^{-1/2}_{00}\big(\Gamma^0_{\mathbf{J}}\big)\right\}.$$

From now on $\mathbf{H}_{\mathbf{I}}$ denotes the solution of Problem MP. We can prove the following result:

Theorem 3.1. *Let* $\mathbf{E}_{\mathbf{I}} := \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}_{\mathbf{I}}$. *Then* $\mathbf{E}_{\mathbf{I}} \in \mathcal{E}$ *and it satisfies*

$$\int_\Omega \sigma \mathbf{E}_{\mathbf{I}} \cdot \overline{\mathbf{G}} + \int_\Omega \frac{1}{\mathrm{i}\omega\mu}\,\mathbf{curl}\,\mathbf{E}_{\mathbf{I}} \cdot \mathbf{curl}\,\overline{\mathbf{G}} = L^{\mathbf{I}}(\mathbf{G}) \quad \forall \mathbf{G} \in \mathcal{E},$$

where $L^{\mathbf{I}} \in \mathcal{E}'$ *is defined by* $L^{\mathbf{I}}(\mathbf{G}) := \int_\Omega \mathbf{curl}\,\mathbf{H}_{\mathbf{I}} \cdot \overline{\mathbf{G}} - \int_\Omega \mathbf{H}_{\mathbf{I}} \cdot \mathbf{curl}\,\overline{\mathbf{G}}$.

Let $a^E : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ be the sesquilinear and continuous form defined by

$$a^E(\mathbf{E}, \mathbf{G}) := \int_\Omega \sigma \mathbf{E} \cdot \overline{\mathbf{G}} + \int_\Omega \frac{1}{\mathrm{i}\omega\mu} \mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{G}}.$$

We introduce the

Problem PE. Find $\mathbf{E} \in \mathcal{E}$ such that $a^E(\mathbf{E}, \mathbf{G}) = L^{\mathbf{I}}(\mathbf{G})\ \forall \mathbf{G} \in \mathcal{E}$.

Corollary 3.2. *Problem* PE *has a unique solution* $\mathbf{E_I} = \frac{1}{\sigma} \mathbf{curl}\,\mathbf{H_I}$.

Remark 1. If $\mathbf{H_I}$ and $\mathbf{G}$ are smooth, the linear functional $L^{\mathbf{I}}$ can be written directly in terms of the input current intensities without the need of computing the magnetic field $\mathbf{H_I}$ (see (10) below). We refer the reader to [7] for the definition of $\mathrm{L}^{\mathbf{I}}$ by means of some homology notions.

4. Numerical solution

We consider a family of regular tetrahedral meshes $\{\mathcal{T}_h\}$ of Ω where, as usual, h denotes the corresponding mesh-size. The electric field is a function of $\mathrm{H}(\mathbf{curl}, \Omega)$ which will be discretized by using Nédélec edge finite elements (see [8]). More precisely, $\mathrm{H}(\mathbf{curl}, \Omega)$ will be approximated by the finite dimensional space

$$\mathcal{N}_h(\Omega) := \{\mathbf{G}_h \in \mathrm{H}(\mathbf{curl}, \Omega)\colon\ \mathbf{G}_h|_K \in \mathcal{N}(K)\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{N}(K) := \{\mathbf{G}_h \in \mathcal{P}_1(K)^3\colon\ \mathbf{G}_h(\mathbf{x}) = \mathbf{a} \times \mathbf{x} + \mathbf{b},\ \mathbf{a}, \mathbf{b} \in \mathbb{C}^3,\ \mathbf{x} \in K\}$. Then, we consider the following discrete space to approximate $\mathcal{E}$ and write the corresponding discrete problem:

$$\mathcal{E}_h := \{\mathbf{G}_h \in \mathcal{N}_h(\Omega)\colon\ \mathbf{G}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathbf{E}}^0 \text{ and } \mathbf{curl}\,\mathbf{G}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathbf{J}}^0\}. \tag{9}$$

Problem PED. Find $\mathbf{E}_h \in \mathcal{E}_h$ such that

$$a^E(\mathbf{E}_h, \mathbf{G}_h) = L^{\mathbf{I}}(\mathbf{G}_h) \quad \forall \mathbf{G}_h \in \mathcal{E}_h.$$

Remark 2. The term $L^{\mathbf{I}}(\mathbf{G}_h)$ above can be written in terms of the input current intensities as follows:

$$L^{\mathbf{I}}(\mathbf{G}_h) = \sum_{n=1}^{N} I_n \int_{c_n} \overline{\mathbf{G}}_h \cdot \mathbf{t}, \tag{10}$$

where c_n is a curve joining each input current surface $\Gamma_{\mathbf{J}}^n$ to $\Gamma_{\mathbf{E}}$ (see Fig. 1). This is the form actually used for the computer implementation.

Theorem 4.1. *Problem* PED *has a unique solution* $\mathbf{E}_h$. *Furthermore, if the solution* $\mathbf{E}$ *of Problem* PE *belongs to* $\mathrm{H}^r(\mathbf{curl}, \Omega)$ *with* $r > 1/2$, *then*

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathrm{H}(\mathbf{curl}, \Omega)} \leqslant C h^r \|\mathbf{E}\|_{\mathrm{H}^r(\mathbf{curl}, \Omega)}.$$

Notice that Problem PED is actually a "theoretical" method because its implementation requires to impose somehow the constraint $\mathbf{curl}\,\mathbf{E} \cdot \mathbf{n} = 0$ on $\Gamma_{\mathbf{J}}^0$ in the definition of $\mathcal{E}_h$ to trial and test functions. In order to deal with this condition, we introduce below a discrete Lagrange multiplier defined on the boundary $\Gamma_{\mathbf{J}}^0$.

Let $\mathcal{T}_h^{\Gamma_{\mathbf{J}}^0}$ be the triangular mesh induced by $\mathcal{T}_h$ on the polyhedral surface $\Gamma_{\mathbf{J}}^0$. We consider the following finite-dimensional space:

$$\mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0) := \left\{ q_h \in \mathrm{L}^2(\Gamma_{\mathbf{J}}^0) \colon q_h|_T \in \mathcal{P}_0(T)\ \forall T \in \mathcal{T}_h^{\Gamma_{\mathbf{J}}^0} \quad \text{and} \int_{\Gamma_{\mathbf{J}}^0} q_h = 0 \right\}. \tag{11}$$

Let $\mathfrak{F}_h$ be the subspace of $\mathcal{N}_h(\Omega)$ defined by

$$\mathfrak{F}_h := \left\{ \mathbf{G}_h \in \mathcal{N}_h(\Omega) \colon \mathbf{G}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathbf{E}}^0 \right\}. \tag{12}$$

We introduce the mixed discrete problem:

Problem MPED. Find $\mathbf{E}_h \in \mathfrak{F}_h$ and $\lambda_h \in \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$ such that

$$a^E(\mathbf{E}_h, \mathbf{G}_h) + b^E(\overline{\mathbf{G}}_h, \lambda_h) = L^{\mathbf{I}}(\mathbf{G}_h) \quad \forall \mathbf{G}_h \in \mathfrak{F}_h,$$
$$b^E(\mathbf{E}_h, \nu_h) = 0 \quad \forall \nu_h \in \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0),$$

where b^E is the sesquilinear form defined on $\mathfrak{F}_h \times \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$ by $b^E(\mathbf{G}_h, \nu_h) := \int_{\Gamma_{\mathbf{J}}^0} \mathbf{curl}\,\mathbf{G}_h \cdot \mathbf{n}\, \bar{\nu}_h$.

Theorem 4.2. *Problem* MPED *has a unique solution* $(\mathbf{E}_h, \lambda_h) \in \mathfrak{F}_h \times \mathcal{Q}_h^0(\Gamma_{\mathbf{J}}^0)$. *Furthermore, if the solution* $\mathbf{E}$ *of Problem* PE *belongs to* $\mathrm{H}^r(\mathbf{curl}, \Omega)$*, with* $r > 1/2$*, then the following error estimate holds true*:

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} \leqslant C h^r \|\mathbf{E}\|_{\mathrm{H}^r(\mathbf{curl},\Omega)}.$$

We have implemented the method described above in a MATLAB code. In order to test the performance and convergence properties of the method, we have solved a particular problem with known analytical solution. Numerical results corresponding to this problem and simulations of real electrodes will be described in the forthcoming paper [5].

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