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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 381–385

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

On an open problem for Jacobians raised by Bourgain, Brezis and Mironescu

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Received and accepted 4 July 2003

Presented by Haïm Brezis

Abstract

We establish a Jacobian estimate in the context of Ginzburg–Landau theory, which was conjectured in a recent work of Bourgain, Brezis and Mironescu. ***To cite this article: F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Un problème ouvert sur les Jacobiens soulevé par Bourgain, Brezis et Mironescu. Nous démontrons une estimée pour des Jacobiens dans le contexte de la fonctionnelle de Ginzburg–Landau. Cela répond à une conjecture dans un travail récent de Bourgain, Brezis et Mironescu. ***Pour citer cet article : F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit $N \geqslant 3$ et B^N la boule unité de $\mathbb{R}^N$. Nous considérons des applications $u_\varepsilon : B^N \to \mathbb{C}$ et leur énergie de Ginzburg–Landau définie par (1). Ici $0 < \varepsilon$ représente un petit paramètre, et nous supposons que u_ε se situe dans le régime énergétique défini par (H_0). Sous cette hypothèse, nous établissons pour $N = 3$, dans le Théorème 1.2, une estimation uniforme pour le Jacobien de u_ε défini par (2). Il s'agit d'une estimation dans le dual de l'espace de Sobolev $W^{1,3}$ conjecturée par Bourgain, Brezis et Mironescu dans [3]. Notre preuve utilise de manière cruciale une nouvelle inégalité (voir Théorème 1.3 ci-dessous) qu'ils ont obtenue dans [3], combinée à une procédure d'approximation (Théorème 2.1) qui reprend des constructions de [1,8,9,4].

Rappelons que dans le même esprit, une estimation similaire, dans le dual de $\mathcal{C}^{0,\alpha}$, avait auparavant été obtenue par Jerrard et Soner [8].

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1631-073X/$ – see front matter
doi:10.1016/S1631-073X(03)00367-4

1. Introduction

Let $N \geqslant 3$ and B^N be the unit ball in $\mathbb{R}^N$. We consider maps $u_\varepsilon : B^N \to \mathbb{C}$ and the Ginzburg–Landau energy of u_ε defined by

$$E_\varepsilon(u_\varepsilon) = \int_{B^N} e_\varepsilon(u_\varepsilon) = \int_{B^N} \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}, \tag{1}$$

where $0 < \varepsilon < 1$ represents a small parameter. We assume throughout that the map u_ε verifies the bound

$$E_\varepsilon(u_\varepsilon) \leqslant M_0 |\log \varepsilon|, \tag{H$_0$}$$

where M_0 is a fixed positive constant. In order to analyze possible concentration phenomena in the asymptotic limit $\varepsilon \to 0$ of maps $(u_\varepsilon)_{0<\varepsilon<1}$ verifying (H$_0$), an important quantity is the Jacobian

$$J u_\varepsilon = \sum_{i<j} \partial_i u_\varepsilon \times \partial_j u_\varepsilon \, \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{2}$$

In particular, the following important estimate was derived in [8].

Theorem 1.1 [8]. *Let $0 < \alpha < 1$. There exists a constant $C(\alpha)$ depending only on α (but not on ε) such that, if u_ε verifies* (H$_0$), *then we have*

$$\left| \int_{B^N} J u_\varepsilon \wedge \phi \right| \leqslant C(\alpha) M_0 \|\phi\|_{\mathcal{C}_c^{0,\alpha}(B^N)}, \quad \forall \phi \in \mathcal{C}_c^{0,\alpha}\big(B^N; \Lambda^{N-2}\mathbb{R}^N\big). \tag{3}$$

It is also known (see, e.g., [8,1]) that (3) does not hold in general in the case $\alpha = 0$ (i.e., assuming $\phi \in \mathcal{C}_c^0(B^N; \Lambda^{N-2}\mathbb{R}^N)$), that is, there is no estimate (uniform in ε) for the Jacobian in L^1. However, this is essentially true, up to correction terms which are small (in appropriate weaker norms), see, e.g., [8,1]. In connection with Theorem 1.1, and in view of a new linear estimate (see Theorem 1.3 below) which holds in dimension $N = 3$, Bourgain, Brezis and Mironescu raised the following question:

Conjecture [3]. *Is it true that, for every compact subset $K \subset B^3$, and every u_ε verifying* (H$_0$),

$$\left| \int_{B^3} J u_\varepsilon \wedge \phi \right| \leqslant C(K) \|\phi\|_{W^{1,3}(B^3)}, \quad \forall \phi \in \mathcal{C}_0^\infty\big(K; \Lambda^1\mathbb{R}^3\big)? \tag{4}$$

In this Note, we establish (4) under the additional assumption that u_ε is bounded in L^∞ (see however Remark 1(b) below). More precisely, we have

Theorem 1.2. *Assume u_ε verifies* (H$_0$) *and*

$$\|u_\varepsilon\|_{L^\infty(B^3)} \leqslant M_1. \tag{H$_1$}$$

Then, for every compact subset $K \subset B^3$ there exists a constant $C(K, M_1)$ depending only on K and M_1, but neither on M_0 nor on ε, such that for ε sufficiently small,

$$\left| \int_{B^3} J u_\varepsilon \wedge \phi \right| \leqslant C(K, M_1) M_0 \|\phi\|_{W^{1,3}(B^3)}, \quad \forall \phi \in \mathcal{C}_0^\infty\big(K; \Lambda^1\mathbb{R}^3\big). \tag{5}$$

Remark 1. (a) If we allow the constant in (5) to depend on M_0, then the same conclusion holds without smallness assumption on ε.

(b) Although assumption (H_1) is presumably not optimal, conclusion (5) is not true assuming only (H_0), as the following counterexample shows: let $\phi \in \mathcal{C}_0^\infty(B^3;\Lambda^1\mathbb{R}^3)$, $u \in \mathcal{C}_0^\infty(B^3)$ be such that

$$\int_{B^3} Ju \wedge \phi = 1.$$

Define $v_{\varepsilon,R}(x) := 1 + \sqrt{R|\log\varepsilon|}\,u(Rx)$, and $\phi_R(x) = \phi(Rx)$, for $R > 1$. By scaling,

$$\left|\int_{B^3} Jv_{\varepsilon,R} \wedge \phi_R\right| = |\log\varepsilon|, \qquad \|\phi_R\|_{W^{1,3}(B^3)} = \|\phi\|_{W^{1,3}(B^3)}$$

and

$$\int_{B^3} |\nabla v_{\varepsilon,R}|^2 = |\log\varepsilon| \int_{B^3} |\nabla u|^2, \qquad \frac{1}{\varepsilon^2}\int_{B^3}\bigl(1-|v_{\varepsilon,R}|^2\bigr)^2 \leqslant \frac{C(u)|\log\varepsilon|^2}{R\varepsilon^2}.$$

Therefore, (4) does not hold choosing R arbitrarily large.

As mentioned, inequalities (4), (5) where motivated by a new beautiful inequality derived in [3] (see also [2] for further developments). More precisely, we have

Theorem 1.3 [3]. *There exists a universal constant C_0 such that, for any closed oriented rectifiable curve Γ in $\mathbb{R}^3$, we have*

$$\left|\int_\Gamma \vec{\phi}\cdot\vec{t}\right| \leqslant C_0\,\mathcal{H}^1(\Gamma)\|\vec{\phi}\|_{W^{1,3}(\mathbb{R}^3)}, \quad \forall\vec{\phi} \in \mathcal{C}_0^\infty\bigl(\mathbb{R}^3;\mathbb{R}^3\bigr), \tag{6}$$

where $\vec{t}$ is the tangent vector to Γ.

Our proof of Theorem 1.2 relies on estimate (6) in an essential way. Two other important ingredients are:

- an approximation of Ju_ε by the Jacobian of a "canonical" function v_ε on which we have better control;
- the oriented and unoriented coarea formulae.

In the next sections we describe these methods.

2. Approximation

As mentioned, the Jacobian of u_ε is not necessarily uniformly bounded in L^1. However, in this section we will see that there is a map v_ε whose Jacobian Jv_ε is suitably close to Ju_ε, and is uniformly bounded in L^1. More precisely

Theorem 2.1. *Assume u_ε verifies* (H_0). *Then, for ε sufficiently small, there exists a smooth function $v_\varepsilon : B^N \to \mathbb{C}$ such that*

$$|v_\varepsilon| \leqslant 1, \tag{7}$$

$$E_\varepsilon(v_\varepsilon) \leqslant CM_0|\log\varepsilon|, \tag{8}$$

$$\|Ju_\varepsilon - Jv_\varepsilon\|_{[\mathcal{C}_c^{0,1}(B^N)]^*} \leqslant C\varepsilon^\alpha E_\varepsilon(u_\varepsilon), \tag{9}$$

$$\|Jv_\varepsilon\|_{L^1} \leqslant CM_0, \tag{10}$$

where $C > 0$ and $0 < \alpha < 1$ depend only on N.

Remark 2. One may also require in addition that $\|u_\varepsilon - v_\varepsilon\|_{L^2} \leqslant C\varepsilon^\alpha E_\varepsilon(u_\varepsilon)$ (see [4]).

The proof of Theorem 2.1 is not straightforward and borrows ideas from the analysis in [5,9,7,1,8,10]. However, since it cannot be deduced directly from those works, a detailed proof will be given in [4].

3. Interpolation

We write $Ju_\varepsilon = Jv_\varepsilon + \kappa_\varepsilon$, so that $\kappa_\varepsilon = Ju_\varepsilon - Jv_\varepsilon$.

Lemma 3.1. *For every $\delta > 0$ we have*

$$\|\kappa_\varepsilon\|_{[W_0^{1,2+\delta}(B^N)]^*} \leqslant CM_0\,\varepsilon^\sigma|\log\varepsilon|, \tag{11}$$

where $\sigma = \frac{\alpha\delta}{2+\delta}$, and C depends only on δ and M_1.

Proof. Let $\phi \in \mathcal{C}_0^\infty(B^N; \Lambda^{N-2}\mathbb{R}^N)$. Since $Ju = \frac{1}{2}\,\mathrm{d}(u \times \mathrm{d}u)$, we have

$$\kappa_\varepsilon = Ju_\varepsilon - Jv_\varepsilon = \frac{1}{2}\,\mathrm{d}(u_\varepsilon \times \mathrm{d}u_\varepsilon - v_\varepsilon \times \mathrm{d}v_\varepsilon) = \frac{1}{2}\,\mathrm{d}\big((u_\varepsilon - v_\varepsilon) \times (\mathrm{d}u_\varepsilon + \mathrm{d}v_\varepsilon)\big), \tag{12}$$

so that,

$$\begin{aligned}\left|\int_{B^N} \kappa_\varepsilon \wedge \phi\right| &= \frac{1}{2}\left|\int_{B^N} (u_\varepsilon - v_\varepsilon) \times (\mathrm{d}u_\varepsilon + \mathrm{d}v_\varepsilon) \wedge \mathrm{d}\phi\right| \\ &\leqslant C\big(\|u_\varepsilon\|_{L^\infty} + \|v_\varepsilon\|_{L^\infty}\big)\big(\|\mathrm{d}u_\varepsilon\|_{L^2} + \|\mathrm{d}v_\varepsilon\|_{L^2}\big)\,\|\mathrm{d}\phi\|_{L^2} \\ &\leqslant CM_1M_0|\log\varepsilon|\|\mathrm{d}\phi\|_{L^2}.\end{aligned} \tag{13}$$

Therefore,

$$\|\kappa_\varepsilon\|_{[W_0^{1,2}(B^N)]^*} \leqslant CM_1M_0|\log\varepsilon|. \tag{14}$$

Combining (9) and (14) we obtain, by interpolation,

$$\|\kappa_\varepsilon\|_{[W_0^{1,2+\delta}(B^N)]^*} \leqslant C(\delta)M_1^{2/(2+\delta)}M_0|\log\varepsilon|\varepsilon^\sigma. \tag{15}$$

The lemma is proved. □

4. The coarea formula

Since v_ε is smooth, by Sard's Theorem for a.e. $z \in \mathbb{C}$, $v_\varepsilon^{-1}(z)$ is a a smooth oriented $(N-2)$-submanifold of B^N. The (oriented) coarea formula (see [6], 3.2.22) applied to v_ε yields

$$\int_{B^N} Jv_\varepsilon \wedge \phi = \int_{\mathbb{C}}\left[\int_{v_\varepsilon^{-1}(z)} \phi\right]\mathrm{d}z, \tag{16}$$

for any $\phi \in \mathcal{C}_0^\infty(B^N; \Lambda^{N-2}\mathbb{R}^N)$.

5. Proof of Theorem 1.2

Here $N = 3$. By (16) we have

$$\left|\int_{B^N} Jv_\varepsilon \wedge \phi\right| \leqslant \int_{\mathbb{C}} \left|\int_{v_\varepsilon^{-1}(z)} \phi\right| \mathrm{d}z. \tag{17}$$

Let $0 < R < 1$ be such that $K \subset B(R)$ and let $z \in \mathbb{C}$ be a regular value of v_ε. If (each component of) $v_\varepsilon^{-1}(z) \cap B(R)$ is a closed loop, we simply set $\gamma(z) := v_\varepsilon^{-1}(z) \cap B(R)$. Otherwise, we may always extend $v_\varepsilon^{-1}(z) \cap B(R)$ to a (union of) closed loop(s) $\gamma(z)$ verifying

$$\mathcal{H}^1(\gamma(z)) \leqslant \pi\, \mathcal{H}^1(v_\varepsilon^{-1}(z) \cap B(R)) \quad \text{and} \quad \gamma(z) \cap B(R) = v_\varepsilon^{-1}(z) \cap B(R).$$

By Theorem 1.3, we therefore have

$$\left|\int_{v_\varepsilon^{-1}(z)} \phi\right| = \left|\int_{\gamma(z)} \phi\right| \leqslant C\mathcal{H}^1(v_\varepsilon^{-1}(z) \cap B(R)) \|\phi\|_{W_0^{1,3}(K)}. \tag{18}$$

Integrating with respect to z, we obtain by (17), the unoriented coarea formula, and estimate (10),

$$\begin{aligned} \left|\int_{B^N} Jv_\varepsilon \wedge \phi\right| &\leqslant C\|\phi\|_{W_0^{1,3}(K)} \int_{\mathbb{C}} \mathcal{H}^1(v_\varepsilon^{-1}(z) \cap B(R))\, \mathrm{d}z \\ &\leqslant C\|\phi\|_{W_0^{1,3}(K)} \int_{B^N \cap B(R)} |Jv_\varepsilon| \leqslant C(K)M_0\|\phi\|_{W_0^{1,3}(K)}. \end{aligned} \tag{19}$$

On the other hand, applying Lemma 3.1 with $\delta = 1$, we obtain

$$\left|\int_{B^N} \kappa_\varepsilon \wedge \phi\right| \leqslant CM_0\varepsilon^{\alpha/3}|\log\varepsilon| \|\phi\|_{W_0^{1,3}(K)}. \tag{20}$$

Combining (19) and (20) the proof is completed.

Acknowledgements

The authors wish to thank J. Bourgain, H. Brezis and P. Mironescu for providing them with an early version of the manuscript [3], as well as for useful discussions.

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