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A super-stable motion with infinite mean branching ☆

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Abstract

A class of finite measure-valued càdlàg superprocesses X with Neveu's (1992) continuous-state branching mechanism is constructed. To this end, we start from certain supercritical (α, d, β)-superprocesses $X^{(\beta)}$ with symmetric α-stable motion and $(1+\beta)$-branching and prove convergence on path space as $\beta \downarrow 0$. The log-Laplace equation related to X has the locally non-Lipschitz function $u \log u$ as non-linear term (instead of $u^{1+\beta}$ in the case of $X^{(\beta)}$). It can nevertheless be shown to be well-posed. X has infinite expectation, is immortal in all finite times, propagates mass instantaneously everywhere in space, and has locally countably infinite biodiversity.

Résumé

Nous construisons une classe de processus de branchement à valeurs mesures finis qui sont une extension naturelle de processus de branchement à valeurs réelles positives étudiés par Neveu (1992). Pour arriver a ce résultat nous commençons avec des (α, d, β)-superprocessus $X^{(\beta)}$ qui correspondent aux systèmes des particules dans lesquels le déplacement de masse est décrit par une loi α-stable et le branchement par une loi $(1+\beta)$-stable, et nous prouvons la convergence dans l'espace des trajectoires càdlàg larsque $\beta \downarrow 0$. L'équation log-Laplace qui est associée au le processus limite X comparteun terme non lineaireen u logu, qui n'est pas lipschitzien. Nous pouvons neamains démontrer que cette équation est bien-posée. X est d'ésperance infinie, est immortel à temp fini, propage sa masse instantanement dans tout l'espace, et a une diversité biologique localement infinie.

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1. Introduction

1.1. Motivation, background, and purpose

Bertoin and Le Gall (2000) established in [2] a connection between a particular continuous-state branching process $\bar{X} = (\bar{X}_t)_{t \geqslant 0}$ and a coalescent process investigated by Bolthausen and Sznitman (1998) in [3], by Pitman (1999) in [28] and recently by Bovier and Kurkova (2003) in [4]. This process $\bar{X}$ was actually introduced in connection with Ruelle's (1987) [29] probability cascades by Neveu (1992) in the preprint [25], so we call it henceforth *Neveu's continuous state branching process.* It is indeed a strange branching process: Its (individual) branching mechanism is given by the function $u \log u$, hence belongs to the domain of attraction of a stable law of index 1. On the other hand, the state at time $t > 0$ has a stable law of index $\mathrm{e}^{-t} < 1$ varying in time and tending to 0 as $t \uparrow \infty$. This process is at the borderline of processes with finite/infinite expectations and with explosion/non-explosion. Actually, it has infinite expectations, but it does not explode in finite time.

Fascinated by this process, we asked the question whether this model can be enriched by a spatial motion component. Indeed, imagine the "infinitesimally small parts" of Neveu's process move in $\mathbb{R}^d$ according to independent Brownian motions. Can this be made mathematically rigorous? In other words, *does a super-Brownian motion $X = (X_t)_{t \geqslant 0}$ exist with Neveu's branching mechanism,* and what properties does it have? Clearly, via log-Laplace transition functionals, such a superprocess X would be related to the Cauchy problem

$$\left.\begin{array}{c} \dfrac{\partial}{\partial t} u_t(x) = \Delta u_t(x) - u_t(x) \log u_t(x) \text{ on } (0,\infty) \times \mathbb{R}^d \\ \text{with initial condition } u_{0+} = \varphi \geqslant 0 \end{array}\right\} \tag{1}$$

(where Δ is the d-dimensional Laplacian and φ is an appropriate function on $\mathbb{R}^d$). Note that this diffusion-reaction equation is interesting in itself since the reaction term does not satisfy a local Lipschitz condition (the derivative has a singularity at 0).

1.2. Approach, sketch of the main results

As Neveu's process $\bar{X}$ can be approximated by a family $(\bar{X}^{(\beta)})_{0<\beta\leqslant 1}$ of supercritical continuous-state branching processes $\bar{X}^{(\beta)}$ of index $1+\beta$ by letting $\beta \downarrow 0$, we try to approximate the desired process X by a family of super-Brownian motions $X^{(\beta)}$ with $(1+\beta)$-branching mechanism. More precisely, we assume that $X^{(\beta)}$ is a supercritical super-Brownian motion related to the log-Laplace equation

$$\left.\begin{array}{c} \dfrac{\partial}{\partial t} u_t^{(\beta)}(x) = \Delta u_t^{(\beta)}(x) - \dfrac{1}{\beta}\big(u_t^{(\beta)}(x)\big)^{1+\beta} + \dfrac{1}{\beta} u_t^{(\beta)}(x) \text{ on } (0,\infty) \times \mathbb{R}^d \\ \text{with initial condition } u_{0+}^{(\beta)} = \varphi \geqslant 0. \end{array}\right\} \tag{2}$$

Of course, the relation between $X^{(\beta)}$ and $u^{(\beta)}$ from (2) is realized via log-Laplace transition functionals:

$$-\log \mathbb{E}_\mu\big[\exp\langle X_t^{(\beta)}, -\varphi\rangle\big] = \langle \mu, u_t^{(\beta)}\rangle. \tag{3}$$

Here $\langle \mu, f\rangle$ denotes the integral $\int_{\mathbb{R}^d} f(x)\mu(dx)$, and the expectation symbol $\mathbb{E}_\mu$ refers to the law $\mathbb{P}_\mu$ of $X^{(\beta)}$ starting from the finite measure $X_0^{(\beta)} = \mu$. We note that

$$\frac{1}{\beta}\big(v^{1+\beta} - v\big) \underset{\beta\downarrow 0}{\longrightarrow} v \log v, \quad v \geqslant 0, \tag{4}$$

therefore such set-up seems to be reasonable if X exists non-trivially at all.

Our *purpose* is to verify that the family $(X^{(\beta)})_{0<\beta\leqslant 1}$ of superprocesses is tight in law as $\beta \downarrow 0$ on the Skorohod space of càdlàg finite measure-valued paths, and that each limit point is identified as the unique process related to the log-Laplace equation (1). This then gives convergence to the desired process X (see Theorem 2 below) with total mass process $\bar{X} = X(\mathbb{R}^d)$. Actually, in the superprocesses we will replace the Brownian migration by a symmetric α-stable migration $(0 < \alpha \leqslant 2)$.

Note that many of the standard tools are not available for this route, since the local Lipschitz constants related to the non-linear term in the log-Laplace equation (2) blow up along (4), or – viewed in probabilistic terms – the expectations of $X^{(\beta)}$ become infinite as $\beta \downarrow 0$. On the other hand, a variety of monotonicity properties are available and serve as a substitute. For the well-posedness of equations as in (1), see Theorem 1 below.

1.3. First properties of X

Since Neveu's process $\bar{X}$ has very special properties, one expects also that X has interesting new properties compared with usual superprocesses. For instance, we suspect that X has absolutely continuous states at almost all times in *all* dimensions. (This conjecture will be confirmed in a forthcoming paper, Fleischmann and Mytnik (2004) [16].) Recall that the (α, d, β)-superprocesses $X^{(\beta)}$ have absolutely continuous states at almost all times in dimensions $d < \alpha/\beta$ (see the appendix of Fleischmann (1988) [13] for the case of critical (α, d, β)-superprocesses starting from Lebesgue measures), and we let $\beta \downarrow 0$. In this paper, however, we will content ourself with more modest properties of X.

Starting from a non-zero (deterministic) state, for each t fixed, $\bar{X}_t$ has a stable distribution with index e^{-t}. Therefore, $\bar{X}_t > 0$ almost surely, meaning that the total mass process $t \mapsto X_t(\mathbb{R}^d) = \bar{X}_t$ is immortal. Moreover, the underlying α-stable mass flow – more specifically the semigroup with generator Δ_α applied to measures – propagates mass instantaneously everywhere in space. Thus, our superprocess X is expected to be *immortal* and its mass should *propagate instantaneously* in space (see Proposition 16 below). This is in sharp contrast to the approximating supercritical $X^{(\beta)}$ processes for which $\bar{X}_t^{(\beta)} = 0$ with positive probability, for all α, β and $t > 0$. Moreover, if $\alpha = 2$, then $X^{(\beta)}$ has the compact support property.

As a further consequence of this, we obtain that X has *locally countably infinite biodiversity*, a notion introduced in Fleischmann and Klenke (2000) [15]. Roughly speaking, this means that, for fixed $t > 0$, in the clustering representation of the infinitely divisible random measure X_t, infinitely many clusters contribute to each given region (see Corollary 18 below). Putting it differently, at time $t > 0$, in every region there are infinitely many families originating from distinct ancestors at time 0. Again, in the case $\alpha = 2$, this contrasts with the (locally) finite biodiversity of the random states of the approximating superprocesses $X^{(\beta)}$.

The further *layout* of the paper is as follows: We first introduce some notation in Section 2.1, before in Section 2.2 we rigorously define the process X and its approximations $X^{(\beta)}$. There we also state Theorem 1 concerning the solutions u of equations as in (1). The main results concerning existence of and convergence to X are given in Theorem 2. The proofs are worked out in the remaining parts of Section 2 after the concept is explained in 2.3. In Section 3 we are concerned with immortality and infinite biodiversity of the constructed process X. The appendix gives the proof of an almost sure scaling limit on $\bar{X}_t$ as $t \uparrow \infty$ (see Proposition 10). This follows a sketch of proof in Neveu's unpublished work [25], which uses ideas of Grey (1977) [18] regarding the Galton–Watson case.

For background on superprocesses we refer to Dawson (1993) [5], Dynkin (1994) [7], Le Gall (1999) [23], Etheridge (2000) [9] and Perkins (2002) [27].

2. Construction

2.1. Preliminaries

For any metric space E, let $D(\mathbb{R}_+, E)$ and $C(\mathbb{R}_+, E)$ denote the space of functions $\mathbb{R}_+ := [0, \infty) \to E$, which are càdlàg and continuous, respectively. The former space is endowed with the Skorohod topology, the latter with the topology of uniform convergence on compact sets. By $C(\mathbb{R}^d)$ we denote the class of continuous real valued functions on $\mathbb{R}^d$ endowed with the supremum norm $\|\cdot\|_\infty$. We use $C_\ell(\mathbb{R}^d)$ for the subspace of functions which possess a finite limit as $|x|\uparrow\infty$, and $C_{\mathrm{com}}(\mathbb{R}^d)$ for the subspace of functions with compact support. The subspaces of functions whose derivatives up to order n exist and are also in $C_\ell(\mathbb{R}^d)$ are denoted by $C_\ell^n(\mathbb{R}^d)$. The superscripts "$+$" and "$++$" indicate the respective subspaces of non-negative functions and functions with positive infimum. We write $M_{\mathrm{f}} := M_{\mathrm{f}}(\mathbb{R}^d)$ for the finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. Throughout, c denotes generic positive constants, whose dependencies we sometimes cite in parentheses. The arrow $\Rightarrow$ is used to indicate convergence in law.

Fix a constant $\alpha\in(0,2]$. The semigroup associated with the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ is denoted by T^α,

$$T_t^\alpha\varphi(x)=\int_{\mathbb{R}^d} p_t^\alpha(x-y)\varphi(y)\,dy,\quad t>0,\ x\in\mathbb{R}^d, \tag{5}$$

where p^α is the (jointly continuous) kernel on $(0,\infty)\times\mathbb{R}^d$ of the symmetric α-stable motion in $\mathbb{R}^d$ related to Δ_α, see for example the appendix of Fleischmann and Gärtner (1986) [14]. For $\alpha=2$ we write $T:=T^2$ and $p:=p^2$, which are simply the heat semigroup and the heat kernel corresponding to the Laplacian Δ:

$$p_t(x)=(4\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{4t}\right),\quad t>0,\ x\in\mathbb{R}^d. \tag{6}$$

Let q^η denote the continuous transition density function of a stable process on $\mathbb{R}_+$ with index $\eta\in(0,1)$, so normalized that we have for the Laplace transform

$$\int_0^\infty q_t^\eta(s)\mathrm{e}^{-s\theta}\,ds=\exp(-t\theta^\eta),\quad t>0,\ \theta\geqslant 0. \tag{7}$$

Then, in the case $\alpha<2$, the subordination formula

$$p_t^\alpha(x)=\int_0^\infty q_t^{\alpha/2}(s)p_s(x)\,ds,\quad t>0,\ x\in\mathbb{R}^d \tag{8}$$

is well-known. Note that T^α from (5) is a strongly continuous, positive and conservative contraction semigroup on $C_\ell^+(\mathbb{R}^d)$, which follows via subordination (8) from the corresponding properties of T.

2.2. Main results

The construction of our process X is based on the well-posedness of the following *integral equation:*

$$u_t(x)=T_t^\alpha\varphi(x)-\int_0^t T_{t-s}^\alpha\big(g(u_s)\big)(x)\,ds, \tag{9}$$

for $t\geqslant 0$, $x\in\mathbb{R}^d$, $\varphi\in C_\ell^+(\mathbb{R}^d)$. Here,

$$g(v):=\rho v\log v,\quad v\geqslant 0, \tag{10}$$

is a continuous function on $\mathbb{R}_+$, and $\rho>0$ is an additional constant (for eventual scaling purposes). For a plot of g in the case $\rho=1$, see the dotted curve in Fig. 1. Note that Eq. (9) is the *mild form* of the following function-valued Cauchy problem analogous to (1):

$$\left.\begin{aligned}\frac{\partial}{\partial t}u_t=\Delta_\alpha u_t-g(u_t)\ \text{on}\ (0,\infty)\\ \text{with initial condition}\ u_{0+}=\varphi.\end{aligned}\right\} \tag{11}$$

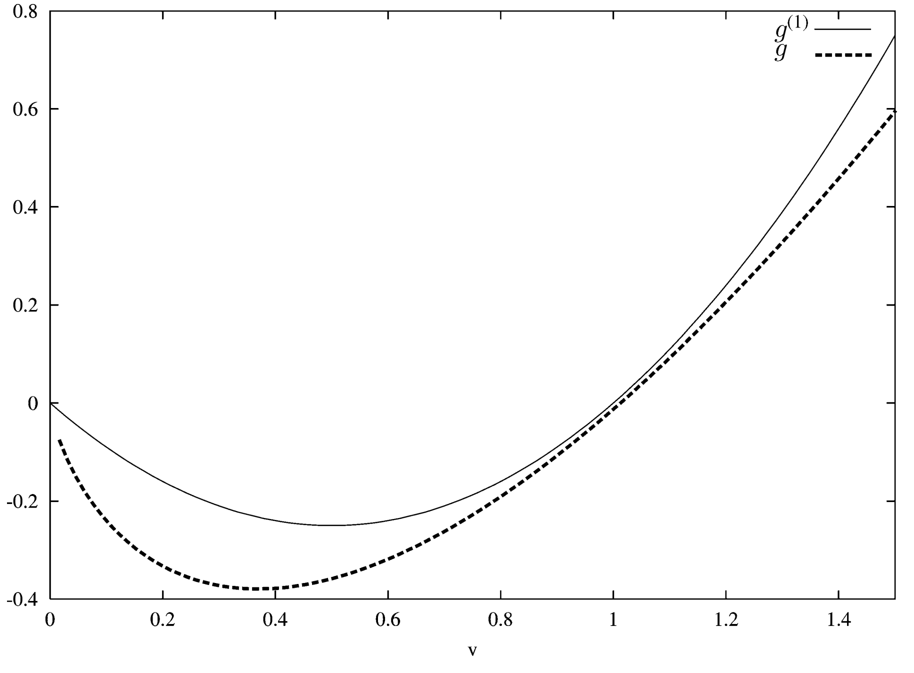

Fig. 1. Branching mechanisms $g^{(1)}(v) = v^2 - v$ and $g(v) = v \log v$.

Here a little care has to be taken since Δ_α is not a differential operator. A mapping $u : \mathbb{R}_+ \to C_\ell^+(\mathbb{R}^d)$ is called *a solution to* (11), if u is continuously differentiable in $C_\ell(\mathbb{R}^d)$ on $(0, \infty)$ (that is, the derivatives $\frac{\partial}{\partial t} u_t$ exist in $C_\ell(\mathbb{R}^d)$ for all $t > 0$ and the mapping $\frac{\partial}{\partial t} u : (0, \infty) \to C_\ell(\mathbb{R}^d)$ is continuous), u_t is in the domain of Δ_α for all $t \in [0, \infty)$, and (11) holds. In Sections 2.5 and 2.8 we will prove the following result. (Recall that the index ℓ in spaces as $C_\ell^{++}(\mathbb{R}^d)$ refers to existence of some finite limit.)

Theorem 1 (Well-posedness of log-Laplace equation).

(a) (Unique existence in the local Lipschitz region). *To every φ in $C_\ell^{++}(\mathbb{R}^d)$, there is a unique solution $u = u(\varphi)$ in $C(\mathbb{R}_+, C_\ell^{++}(\mathbb{R}^d))$ to Eq.* (9). *It satisfies*

$$\inf_{y \in \mathbb{R}^d} \varphi(y) \wedge 1 \leqslant u_t(\varphi)(x) \leqslant \|\varphi\|_\infty \vee 1, \quad t \geqslant 0, \ x \in \mathbb{R}^d. \tag{12}$$

Furthermore, if $\varphi \in C_\ell^{2,++}(\mathbb{R}^d)$, then $u(\varphi)$ is a solution to the function-valued Cauchy problem (11).

(b) (Extension to the non-Lipschitz region). *If $\varphi_n \in C_\ell^{++}(\mathbb{R}^d)$, $n \geqslant 1$, such that pointwise $\varphi_n \downarrow \varphi \in C_\ell^+(\mathbb{R}^d)$ as $n \uparrow \infty$, then pointwise $u(\varphi_n) \downarrow$ some $u(\varphi) \in C(\mathbb{R}_+, C_\ell^+(\mathbb{R}^d))$ as $n \uparrow \infty$, and the limit $u = u(\varphi)$ solves Eq.* (9), *satisfies* (12), *and is independent of the choice of the sequence $(\varphi_n)_{n \geqslant 1}$ converging to φ.*

We remark that the bounds in (12) are a direct consequence of the fact that $\log u$ changes sign at $u = 1$. Here we leave the question open whether or not solutions to (9) or (11) with non-negative initial conditions exist other than the ones constructed via monotone limits in (b) of the theorem. We also remark that the theorem implies the semigroup property for u, meaning that $u_{t+s}(\varphi) = u_t(u_s(\varphi))$ for $s, t \geqslant 0$ (see Dawson (1993) [5], p. 68). The semigroup property is tantamount to the log-Laplace relation (17) below describing a time-homogeneous Markov process X.

The proof of Theorem 1 will start from the well-posedness of the Cauchy problem analogous to (2) in the mild sense for $\beta \in (0,1]$ fixed:

$$u_t^{(\beta)}(x) = T_t^\alpha \varphi(x) - \int_0^t T_{t-s}^\alpha \big(g^{(\beta)}(u_s^{(\beta)})\big)(x)\,ds, \tag{13}$$

$t \geqslant 0, x \in \mathbb{R}^d, \varphi \in C_\ell^+(\mathbb{R}^d)$. Here,

$$g^{(\beta)}(v) := \frac{\rho}{\beta}(v^{1+\beta} - v), \quad v \geqslant 0. \tag{14}$$

For a plot of $g^{(\beta)}$ in the case $\rho = 1 = \beta$, see Fig. 1. To each φ in $C_\ell^+(\mathbb{R}^d)$, there is a unique solution $u^{(\beta)} = u^{(\beta)}(\varphi) \in C(\mathbb{R}_+, C_\ell^+(\mathbb{R}^d))$ to (13). Each $\varphi \in C_\ell^{++}(\mathbb{R}^d)$ is bounded away from 0 and ∞, implying that the solutions $u^{(\beta)}(\varphi)$ are also bounded away from 0 and ∞, uniformly in β (see Lemma 11 below). Therefore, passing to the limit as $\beta \downarrow 0$ for such initial condition φ, we end up in a local Lipschitz region of the function g of (10). This idea is behind part (a) of Theorem 1. (We learned this trick from Watanabe (1968) [31] who worked however in the simpler case of a compact phase space.)

Theorem 1(a) is sufficient for the construction of the desired process X. In Section 2.8 we then use probabilistic arguments using the log-Laplace transition functionals of X in order to derive part (b) of the theorem. The extension in (b) is needed in Section 3 for studying some properties of X.

As a starting point for the construction of the process X, for each $0 < \beta \leqslant 1$ we consider the (unique) time-homogeneous càdlàg strong Markov process $(X^{(\beta)}, \mathbb{P}^{(\beta)}_{\mu^{(\beta)}}, \mu^{(\beta)} \in M_\mathrm{f})$ with log-Laplace transition functional

$$-\log \mathbb{E}_{\mu^{(\beta)}}\big[\exp\langle X_t^{(\beta)}, -\varphi\rangle\big] = \langle \mu^{(\beta)}, u_t^{(\beta)}\rangle, \tag{15}$$

$t \geqslant 0$, $\varphi \in C_\ell^+(\mathbb{R}^d)$, with $u^{(\beta)}$ the unique solution to (13). The construction of $X^{(\beta)}$ is nowadays standard; for references see, for instance, Iscoe (1986) [20], Fitzsimmons (1988 and 1991) [11,12] and Chapter 4 of Dawson (1993) [5]. Note that $X^{(\beta)}$ is a supercritical (α, d, β)-superprocess. Properties of (α, d, β)-superprocesses have been widely studied in the critical case where the branching mechanism $g^{(\beta)}$ in (13) is replaced by

$$g_{\mathrm{crit}}^{(\beta)}(v) := b v^{1+\beta}, \quad v \geqslant 0, \tag{16}$$

with $b > 0$ a constant, see for example Iscoe (1986), Fleischmann (1988), Dawson and Vinogradov (1994) and Mytnik and Perkins (2003) [20,13,6,24]. These processes have finite mean for $\beta \leqslant 1$ but infinite variance for $\beta < 1$. More precisely, $\mathbb{E}_{\mu^{(\beta)}}[\langle X_t^{(\beta)}, \varphi\rangle^\theta] < \infty$ for all $t \geqslant 0$, $\varphi \in C_\ell^+(\mathbb{R}^d)$ with $\varphi \neq 0$, and $\mu^{(\beta)} \in M_\mathrm{f}$ with $\mu^{(\beta)} \neq 0$, if and only if $0 < \theta < 1 + \beta \leqslant 2$ (see also Lemma 9). The case we are interested in corresponds to $\beta = 0$ in the sense that the branching mechanism is in the domain of attraction of a stable law of index 1, see also Remark 4.

Our *main result* can now be formulated as follows:

Theorem 2 (Existence, uniqueness and approximation).

(a) (Unique existence of X). *For each $\mu \in M_\mathrm{f}$ there exists a unique time-homogeneous Markov process $X \in D(\mathbb{R}_+, M_\mathrm{f})$ with log-Laplace transition functional*

$$-\log \mathbb{E}_\mu\big[\exp\langle X_t, -\varphi\rangle\big] = \langle \mu, u_t\rangle, \quad t \geqslant 0,\ \varphi \in C_\ell^+(\mathbb{R}^d), \tag{17}$$

with u the unique solution to (9) *in the setting of Theorem* 1(a) *and* (b).

(b) (Approximation theorem). *Suppose that $X_0^{(\beta)} \Rightarrow X_0$ in M_f as $\beta \downarrow 0$, as well as* $\sup_{0<\beta\leqslant 1} \mathbb{E}[\langle X_0^{(\beta)}, 1\rangle^{\theta_0}] < \infty$, *for some* $0 < \theta_0 \leqslant 1$. *Then in law on $D(\mathbb{R}_+, M_\mathrm{f})$,*

$$X^{(\beta)} \Rightarrow X \quad \text{as } \beta \downarrow 0. \tag{18}$$

Furthermore, we have $\mathbb{E}[\sup_{0\leqslant t\leqslant T} \langle X_t, 1\rangle^\theta] < \infty$ *for all* $T \geqslant 0$ *and* $0 < \theta < \theta_0 \mathrm{e}^{-\rho T}$.

We call X the *super-α-stable motion with Neveu's branching mechanism* (and branching rate ρ). We would like to point out that the process X is related to a class of superprocesses considered by El Karoui and Roelly (1991) [8] who extend the original work by Watanabe (1968) [31]. However, these papers are restricted to a compact phase space, and existence, uniqueness and appropriate regularity of the log-Laplace equation (9) is *assumed* in [8], but rigorously established in the present work.

The proof of the approximation theorem proceeds via tightness in law and convergence of the finite dimensional distributions of subsequences combined with the uniqueness of the limit, which follows from the unique existence of log-Laplace solutions according to Theorem 1(a). This then also establishes the existence of X.

Remark 3 (*Critical processes degenerate*). Note that the "highly supercritical" process X cannot be attained as the limit of critical ones. Observe that setting $\beta = 0$ for the branching mechanism $g^{(\beta)}_{\mathrm{crit}}$ from (16) implies the linear log-Laplace equation

$$\frac{\partial}{\partial t} u_t^{(0,\mathrm{crit})} = \Delta_\alpha u_t^{(0,\mathrm{crit})} - b u_t^{(0,\mathrm{crit})}. \tag{19}$$

Hence, the corresponding measure-valued process is deterministic in this case.

Remark 4 (*Index convergence in canonical measures*). The process X that we derive here can also be seen as the appropriate one to be considered as the limiting case $\beta = 0$ in the following sense. Recall that the branching mechanism Ψ of a general continuous state branching process can be written as

$$\Psi(v) = c_1 v + c_2 v^2 + \int\limits_{(0,\infty)} \big(\mathrm{e}^{-xv} - 1 + xv 1_{\{x \leqslant 1\}}\big)\pi(dx), \tag{20}$$

with constants $c_1 \in \mathbb{R}$, $c_2 \geqslant 0$, and where the canonical measure π is a Radon measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge x^2)\pi(dx) < \infty$ (see for example Theorem 1 in Chapter II of Le Gall (1999) [23]). In the case of the branching mechanisms $g^{(\beta)}$, $0 < \beta < 1$, we have

$$c_1^{(\beta)} = c_1(\beta)\rho, \qquad c_2 = 0, \qquad \pi^{(\beta)}(dx) = c_3(\beta)\rho x^{-2-\beta}\,dx, \tag{21}$$

with some constants $c_1(\beta) \in \mathbb{R}$ and $c_3(\beta) > 0$, whereas for the limiting branching mechanism g,

$$c_1 = c\rho, \qquad c_2 = 0, \qquad \pi(dx) = \rho x^{-2}\,dx, \tag{22}$$

with some constant $c > 0$.

Remark 5 (*Approximation by particle systems*). X is also expected to be the *high density limit of suitable branching particle systems* as the number of initial particles N tends to infinity. Indeed, consider particles that move independently according to α-stable motions in $\mathbb{R}^d$, leaving a random number of offspring after their exponentially distributed lifetime with mean $1/\rho(1 + \log N)$. Let the number of offspring be described by a random variable with probability generating function

$$h_N(r) := (1 + \log N)^{-1}\big(\log N + r + (1-r)\log(1-r)\big), \quad 0 \leqslant r \leqslant 1. \tag{23}$$

The empirical measures of the particle system are given by $\frac{1}{N}\sum_i \delta_{\xi_t^{\alpha,i}}$, where $\xi_t^{\alpha,i}$ are the positions of the particles alive at time t and the sum is taken over all these particles. We note that $N \cdot \rho(1 + \log N)(h_N(1 - \frac{v}{N}) - (1 - \frac{v}{N})) \equiv \rho v \log v$. Heuristics drawn from Chapter II of Le Gall (1999) [23] (although there the non-Lipschitz branching mechanism considered here is excluded) identifies the left-hand side of the identity as the expression that should converge to the nonlinearity of the log-Laplace equation describing the limit process. One expects therefore that the aforementioned empirical measures converge in law to X_t as $N \uparrow \infty$ (provided that the initial states converge).

2.3. Concept of proof of Theorem 2

In preparation of the proof, we consider in Section 2.4 properties of Neveu's continuous state branching process $\bar{X}$ and its approximations $\bar{X}^{(\beta)}$. We prove some (monotone) convergence of the related branching mechanisms and log-Laplace functions, and show uniform boundedness of lower order moments, see Lemmas 7–9.

The log-Laplace equations (9) and (13) are studied in Section 2.5. We will deal with uniform convergences, comparisons, and solutions starting from "runaway" functions.

In order to show tightness in law of $X^{(\beta)}$ in $D(\mathbb{R}_+, M_\mathrm{f})$ we use Jakubowski's (1986) criterion (see Theorem 4.3 of [21]). Since $\{\langle\cdot,\varphi\rangle;\, \varphi\in C_\ell^{++}(\mathbb{R}^d)\}$ is a family of continuous functions on M_f that separates points and is closed under addition, Jakubowski's criterion states in the present case that just properties (a) and (b) in the following claim are sufficient for tightness in law on $D(\mathbb{R}_+, M_\mathrm{f})$.

Proposition 6 (Tightness of the $X^{(\beta)}$). *Let θ_0, $X_0^{(\beta)}$, and X_0 be as in Theorem* 2(b). *Then the following statements hold, implying tightness in law on $D(\mathbb{R}_+, M_\mathrm{f})$ of the family $(X^{(\beta)})_{0<\beta\leqslant 1}$.*

(a) (Tightness of one-dimensional processes). *For each $\varphi\in C_\ell^{++}(\mathbb{R}^d)$, the family $(\langle X^{(\beta)},\varphi\rangle)_{0<\beta\leqslant 1}$ is tight in law on $D(\mathbb{R}_+,\mathbb{R})$.*

(b) (Compact containment). *For any $T\geqslant 0$ and $\epsilon>0$, there exists a compact set $K_{\epsilon,T}\subset M_\mathrm{f}$ such that*

$$\inf_{0<\beta\leqslant 1}\mathbb{P}\big[X_t^{(\beta)}\in K_{\epsilon,T} \text{ for } 0\leqslant t\leqslant T\big]\geqslant 1-\epsilon. \tag{24}$$

Part (a) is shown in Section 2.6. Compact containment (b) is verified in Section 2.7.

2.4. Neveu's continuous state branching process

We begin with studying the total mass $\bar{X}^{(\beta)} = X^{(\beta)}(\mathbb{R}^d)$ and $\bar{X} = X(\mathbb{R}^d)$ of the superprocesses that we are considering. Their log-Laplace functions $\bar{u}^{(\beta)}$ and $\bar{u}$, both independent of a spatial variable, can be calculated explicitly. Indeed, define for $\lambda\geqslant 0$,

$$\bar{u}_t^{(\beta)}(\lambda) := \left(\lambda^{-\beta}\mathrm{e}^{-\rho t} + 1 - \mathrm{e}^{-\rho t}\right)^{-1/\beta}, \quad t\geqslant 0, \tag{25a}$$

$$\bar{u}_t(\lambda) := \lambda^{(\mathrm{e}^{-\rho t})}, \quad t\in\mathbb{R}, \tag{25b}$$

reading the right-hand side of (25a) as 0 for $\lambda=0$. Then $\bar{u}_t^{(\beta)}(\lambda)$ and $\bar{u}_t(\lambda)$ restricted to $t\geqslant 0$ are the respective unique non-negative solutions of (13) and (9) for $\varphi\equiv\lambda$. The uniqueness follows in the former case by the local Lipschitz continuity of $g^{(\beta)}$. The latter case can equivalently be written as in (11), or more generally as

$$\frac{\partial}{\partial t}w_t = -g(w_t) \quad \text{on } \mathbb{R} \text{ with } w_{t_0} = \lambda\geqslant 0, \tag{26}$$

where $t_0\in\mathbb{R}$ is fixed. Although g is not locally Lipschitz, (26) has a unique solution. In fact, the function g is locally Lipschitz on the locally compact space $(0,\infty)$, hence in a sufficiently small neighborhood of t_0 the solution w with $w_{t_0}=\lambda>0$ is unique, thus coincides with the corresponding $\bar{u}$. Repeating the argument, we get $w=\bar{u}$ on $\mathbb{R}$ in this case $\lambda>0$. Indeed, $\bar{u}$ maps $\mathbb{R}$ into $(0,\infty)$, thus the boarders 0 and ∞ cannot be reached during the extensions.

Assume now that w is a non-zero non-negative solution to (26) with $w_{t_0}=0$. Then there is a $t>t_0$ such that $w_t =: \theta>0$. But from the previously mentioned uniqueness, we necessarily obtain $w_s = \bar{u}_{-(t-s)}(\theta)$, $s\leqslant t$. Thus, $w_{t_0}>0$, which is a contradiction.

We thus have for $t, \lambda, \bar{X}_0^{(\beta)}, \bar{X}_0\geqslant 0$,

$$\mathbb{E}\big[\exp(-\bar{X}_t^{(\beta)}\lambda)\big] = \mathbb{E}\big[\exp\big(-\bar{X}_0^{(\beta)}\bar{u}_t^{(\beta)}(\lambda)\big)\big], \tag{27a}$$

$$\mathbb{E}\big[\exp(-\bar{X}_t\lambda)\big] = \mathbb{E}\big[\exp\big(-\bar{X}_0\bar{u}_t(\lambda)\big)\big]. \tag{27b}$$

We can right away verify the following properties of the branching mechanisms $g^{(\beta)}$ and g [introduced in (14) and (10)].

Lemma 7 (Properties of branching mechanisms). *For all* $v \in \mathbb{R}_+$ *we have* $g^{(\beta)}(v) \downarrow g(v)$ *as* $1 \geqslant \beta \downarrow 0$. *Furthermore,* $g^{(\beta)}$ *and* g *are negative on* $(0,1)$ *and positive on* $(1,\infty)$, *with the only intersection points* $g(v) = g^{(\beta)}(v) = 0$ *for* $v = 0$ *and* $v = 1$.

Proof. Let us start by showing that

$$\frac{\partial}{\partial\beta} g^{(\beta)}(v) = \rho\frac{v^{1+\beta}}{\beta^2}(\beta\log v - 1 + v^{-\beta}) \geqslant 0. \tag{28}$$

To see the non-negativity, we note that for $v = 0$ the derivative is zero. Otherwise we observe that $\beta\log v - 1 + v^{-\beta} \geqslant 0$ is equivalent to $1 + \log v^{-\beta} \leqslant v^{-\beta} = \exp(\log v^{-\beta})$, which is true. Thus, $g^{(\beta)}$ is monotonically non-increasing as $\beta \downarrow 0$. Actually, $g^{(\beta)} \downarrow g$ as $\beta \downarrow 0$. In order to show that the only intersection points of $g^{(\beta)}$ and g are at 0 and 1, where both functions are zero, we observe that for $v \neq 0$, $g^{(\beta)}(v) = g(v)$ is equivalent to $\exp(v') = 1 + v'$ where $v' = v^\beta - 1$. The only solution is therefore $v' = 0$, which is equivalent to $v = 1$. To see that both functions are negative on $(0,1)$ and positive on $(1,\infty)$, consider the derivatives of the two functions,

$$\frac{\partial}{\partial v} g^{(\beta)}(v) = \frac{\rho}{\beta}\big((1+\beta)v^\beta - 1\big) \quad \text{and} \quad \frac{\partial}{\partial v} g(v) = \rho(1 + \log v). \tag{29}$$

Thus, the derivative at $v = 0$ is $-\frac{\rho}{\beta}$ for $g^{(\beta)}$ and $-\infty$ for g. Likewise, at $v = 1$ the derivatives are all 1. □

From the monotone convergence of the branching mechanisms (Lemma 7) we obtain the following monotone convergence result for the solutions to the corresponding ordinary differential equations.

Lemma 8 (Monotone convergence of solutions). *For all* $\lambda \in \mathbb{R}_+$ *and* $t \geqslant 0$ *we have* $\bar{u}_t^{(\beta)}(\lambda) \uparrow \bar{u}_t(\lambda)$ *as* $\beta \downarrow 0$.

Proof. By Lemma 7, $g^{(\beta_1)} \geqslant g^{(\beta_2)}$ on $\mathbb{R}_+$ for $1 \geqslant \beta_1 \geqslant \beta_2 > 0$. Thus, by a standard comparison result (see for example Theorem 6.1 of Hale (1969) [19]), we obtain that $\bar{u}_t^{(\beta_1)}(\lambda) \leqslant \bar{u}_t^{(\beta_2)}(\lambda)$ for $\lambda \in \mathbb{R}_+$. Hence, $\bar{u}_t^{(\beta)}(\lambda)$ is non-decreasing as $\beta \downarrow 0$. For $\lambda > 0$ we now rewrite (25a) as

$$\bar{u}_t^{(\beta)}(\lambda) = \left[\left(1 + \beta\left(\mathrm{e}^{-\rho t}\frac{1}{\beta}(\lambda^{-\beta} - 1)\right)\right)^{1/\beta}\right]^{-1}. \tag{30}$$

Since $\frac{1}{\beta}(\lambda^{-\beta} - 1) \to -\log\lambda$, it converges to $\mathrm{e}^{\mathrm{e}^{-\rho t}\log\lambda} = \bar{u}_t(\lambda)$. □

As an immediate consequence, since the log-Laplace transforms converge, for each $t \geqslant 0$ fixed, $\bar{X}_t^{(\beta)}$ converges in law to $\bar{X}_t$ as $\beta \downarrow 0$, provided that $\bar{X}_0^{(\beta)} \to \bar{X}_0$ in law. We can also prove the following uniform moment bound.

Lemma 9 (Uniformly bounded lower order moments). *Suppose*

$$\sup_{0<\beta\leqslant 1} \mathbb{E}\big[(\bar{X}_0^{(\beta)})^{\theta_0}\big] < \infty \quad \textit{for some } 0 < \theta_0 \leqslant 1. \tag{31}$$

Then, for all $T \geqslant 0$ *and* $0 < \theta < \theta_0\mathrm{e}^{-\rho T}$,

$$\sup_{0<\beta\leqslant 1} \mathbb{E}\big[\sup_{t\leqslant T}(\bar{X}_t^{(\beta)})^\theta\big] < \infty. \tag{32}$$

Proof. Fix θ, θ_0, T as in the lemma, and

$$\eta > 1 \text{ such that } \theta e^{\rho T} + (\eta - 1) < \theta_0 \text{ and } \theta e^{\rho T} \eta < \theta_0. \tag{33}$$

Write $\|\cdot\|_\eta$ for the norm in the Lebesgue space $L^\eta(\mathbb{P})$. We use the following identity (see (2.1.11) of Zolotarev (1986) [33]),

$$x^\theta \theta^{-1} \Gamma(1-\theta) = \int_0^\infty \lambda^{-\theta-1}(1 - e^{-x\lambda})\, d\lambda, \tag{34}$$

which holds for any $x \geqslant 0$ (and $0 < \theta < 1$) and follows from a scaling of Euler's Gamma function Γ. Thus, for constants $c = c(\theta, \eta)$,

$$\begin{aligned}(\bar{X}_t^{(\beta)})^\theta &\leqslant c \int_{1/\eta}^\infty \lambda^{-\theta-1} d\lambda + c \int_0^{1/\eta} \lambda^{-\theta-1}(1 - e^{-\bar{X}_t^{(\beta)}\lambda})\, d\lambda \\ &\leqslant c + c \int_0^{1/\eta} \lambda^{-\theta-1}\left[\left|M_t^{(\beta)}(\lambda)\right| + \int_0^T e^{-\bar{X}_s^{(\beta)}\lambda} \bar{X}_s^{(\beta)} \left|g^{(\beta)}(\lambda)\right| ds\right] d\lambda,\end{aligned} \tag{35}$$

where

$$t \mapsto M_t^{(\beta)}(\lambda) = 1 - e^{-\bar{X}_t^{(\beta)}\lambda} + \int_0^t e^{-\bar{X}_s^{(\beta)}\lambda} \bar{X}_s^{(\beta)} g^{(\beta)}(\lambda)\, ds \tag{36}$$

is a martingale, as can be seen by differentiating the Laplace functional in representation (27a). Now,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\leqslant T}\left|M_t^{(\beta)}(\lambda)\right|\Big] &\leqslant \Big\|\sup_{t\leqslant T}\left|M_t^{(\beta)}(\lambda)\right|\Big\|_\eta \leqslant c\left\|M_T^{(\beta)}(\lambda)\right\|_\eta \\ &\leqslant c\|1 - e^{-\bar{X}_T^{(\beta)}\lambda}\|_\eta + c\left\|\int_0^T e^{-\bar{X}_s^{(\beta)}\lambda} \bar{X}_s^{(\beta)} \left|g^{(\beta)}(\lambda)\right| ds\right\|_\eta\end{aligned} \tag{37}$$

by Doob's L^η-inequality and the definition of the martingale. Next we apply the elementary inequality

$$(1-r)^\eta \leqslant 1 - r^\eta, \quad 0 \leqslant r \leqslant 1. \tag{38}$$

In fact, since $\eta > 1$, both sides coincide at $r = 0, 1$, but the left-hand function is convex whereas the right-hand one is concave. This gives

$$\mathbb{E}\big[(1 - e^{-\bar{X}_T^{(\beta)}\lambda})^\eta\big] \leqslant \mathbb{E}[1 - e^{-\bar{X}_T^{(\beta)}\lambda\eta}] = \mathbb{E}\big[1 - \exp\big(-\bar{X}_0^{(\beta)} \bar{u}_T^{(\beta)}(\lambda\eta)\big)\big], \tag{39}$$

where we exploited the Laplace relation (27a). By Lemma 8 and (25b),

$$\bar{u}_s^{(\beta)}(\lambda\eta) \leqslant \bar{u}_s(\lambda\eta) = (\lambda\eta)^{(e^{-\rho s})} \leqslant (\lambda\eta)^{(e^{-\rho T})}, \quad 0 \leqslant s \leqslant T, \tag{40}$$

provided that $0 \leqslant \lambda\eta \leqslant 1$.Thus, by (39) and (40),

$$\int_0^{1/\eta} \lambda^{-\theta-1}\|1 - e^{-\bar{X}_T^{(\beta)}\lambda}\|_\eta\, d\lambda \leqslant \int_0^{1/\eta} \lambda^{-\theta-1}\big(\mathbb{E}\big[1 - \exp\big(-\bar{X}_0^{(\beta)}(\lambda\eta)^{(e^{-\rho T})}\big)\big]\big)^{1/\eta}\, d\lambda. \tag{41}$$

By the substitution $(\lambda\eta)^{(\mathrm{e}^{-\rho T})} =: \tilde\lambda$ the latter integral can be written as

$$c(\eta,\rho,T)\int_0^1 \tilde\lambda^{-\theta \mathrm{e}^{\rho T}-1}\bigl(\mathbb{E}[1-\mathrm{e}^{-\bar X_0^{(\beta)}\tilde\lambda}]\bigr)^{1/\eta}\,d\tilde\lambda = c\int_0^1 \tilde\lambda^{-\theta\mathrm{e}^{\rho T}}\bigl(\tilde\lambda^{-\eta}\mathbb{E}[1-\mathrm{e}^{-\bar X_0^{(\beta)}\tilde\lambda}]\bigr)^{1/\eta}\,d\tilde\lambda. \tag{42}$$

Moreover, since $\theta\mathrm{e}^{\rho T} < \theta_0 \leqslant 1$, the measure $\tilde\lambda^{-\theta\mathrm{e}^{\rho T}}\,d\tilde\lambda$ on $[0,1]$ is finite, and by Jensen's inequality the integral can be bounded from above by

$$c\left(\int_0^1 \tilde\lambda^{-\theta\mathrm{e}^{\rho T}-\eta}\mathbb{E}[1-\mathrm{e}^{-\bar X_0^{(\beta)}\tilde\lambda}]\,d\tilde\lambda\right)^{1/\eta} \leqslant c\left(\mathbb{E}\left[\int_0^\infty \tilde\lambda^{-(\theta\mathrm{e}^{\rho T}+\eta-1)-1}[1-\mathrm{e}^{-\bar X_0^{(\beta)}\tilde\lambda}]\,d\tilde\lambda\right]\right)^{1/\eta}. \tag{43}$$

Using again (34), the latter expectation equals

$$c\mathbb{E}\bigl[(\bar X_0^{(\beta)})^{\theta\mathrm{e}^{\rho T}+\eta-1}\bigr] \tag{44}$$

and is bounded in β by our first assumption on η in (33), and by (31) concerning the initial states $\bar X_0^{(\beta)}$.

Since the expectation of the integral in (35) is bounded from above by the second norm expression in (37), to finish the proof it remains to show that

$$\sup_{0<\beta\leqslant 1}\int_0^{1/\eta}\lambda^{-\theta-1}\left\|\int_0^T \mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}\bigl|g^{(\beta)}(\lambda)\bigr|\,ds\right\|_\eta d\lambda < \infty. \tag{45}$$

First of all, by Lemma 7,

$$\bigl|g^{(\beta)}(\lambda)\bigr| \leqslant |g(\lambda)| = \lambda|\log\lambda|, \quad \text{since } \lambda\leqslant 1/\eta<1. \tag{46}$$

Next,

$$\left\|\int_0^T \mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}\,ds\right\|_\eta \leqslant \int_0^T \bigl\|\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}\bigr\|_\eta\,ds. \tag{47}$$

Clearly,

$$(\mathrm{e}^{-r\lambda}r)^\eta \leqslant c(\eta)\lambda^{-(\eta-1)}\mathrm{e}^{-r\lambda}r, \quad r,\lambda\geqslant 0. \tag{48}$$

Therefore,

$$\mathbb{E}\bigl[(\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)})^\eta\bigr] \leqslant c\lambda^{-(\eta-1)}\mathbb{E}[\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}]. \tag{49}$$

But by (27a),

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}] &= -\frac{\partial}{\partial\lambda}\mathbb{E}[\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}] = -\frac{\partial}{\partial\lambda}\mathbb{E}\bigl[\exp\bigl(-\bar X_0^{(\beta)}\bar u_s^{(\beta)}(\lambda)\bigr)\bigr]\\ &= \mathbb{E}\left[\exp\bigl(-\bar X_0^{(\beta)}\bar u_s^{(\beta)}(\lambda)\bigr)\bar X_0^{(\beta)}\frac{\partial}{\partial\lambda}\bar u_s^{(\beta)}(\lambda)\right],\end{aligned} \tag{50}$$

and by (25a),

$$\frac{\partial}{\partial\lambda}\bar u_s^{(\beta)}(\lambda) = c\bigl(\bar u_s^{(\beta)}(\lambda)\bigr)^{1+\beta}\lambda^{-(1+\beta)}\mathrm{e}^{-\rho s}. \tag{51}$$

Combining (49)–(51) gives

$$\|\mathrm{e}^{-\bar X_s^{(\beta)}\lambda}\bar X_s^{(\beta)}\|_\eta \leqslant c\lambda^{-(\eta-1)/\eta}\lambda^{-(1+\beta)/\eta}\bigl(\mathbb{E}\bigl[\exp\bigl(-\bar X_0^{(\beta)}\bar u_s^{(\beta)}(\lambda)\bigr)\bar X_0^{(\beta)}\bigl(\bar u_s^{(\beta)}(\lambda)\bigr)^{1+\beta}\bigr]\bigr)^{1/\eta}. \tag{52}$$

Using $e^{-r} r^{1-\theta_0} \leqslant c$ for all $r \geqslant 0$, we obtain

$$\|e^{-\bar{X}_s^{(\beta)}\lambda} \bar{X}_s^{(\beta)}\|_\eta \leqslant c\lambda^{-1-\beta/\eta} \big(\bar{u}_s^{(\beta)}(\lambda)\big)^{(\beta+\theta_0)/\eta} \big(\mathbb{E}\big[(\bar{X}_0^{(\beta)})^{\theta_0}\big]\big)^{1/\eta}. \tag{53}$$

By (31), the latter norm expression is bounded in β. Moreover,

$$\lambda^{-\beta/\eta}\big(\bar{u}_s^{(\beta)}(\lambda)\big)^{\beta/\eta} \leqslant e^{\rho s/\eta} \leqslant e^{\rho T/\eta} = c. \tag{54}$$

Going back to (45), inserting (46), (47), (53), and (54), it remains to consider

$$\int_0^{1/\eta} \lambda^{-\theta} |\log \lambda| \lambda^{-1} \int_0^T \big(\bar{u}_s^{(\beta)}(\lambda)\big)^{\theta_0/\eta} \, ds \, d\lambda \leqslant c \int_0^{1/\eta} \lambda^{-\theta-1} |\log \lambda| \lambda^{(\theta_0/\eta)e^{-\rho T}} \, d\lambda, \tag{55}$$

where we used (40). But $-\theta - 1 + (\theta_0/\eta)e^{-\rho T} > -1$ by our second assumption on η in (33). Hence, the integral in (55) is finite. This gives (45), finishing the proof. □

Asymptotic properties as $t \uparrow \infty$ of the total mass process $\bar{X}$ have been explored in the Galton–Watson setting, amongst others by Grey (1977) [18]. This led Neveu (1992) [25] to sketch the following proposition, whose proof is given in our appendix:

Proposition 10 (Almost sure limit of total mass process). *For all (deterministic) initial states $\bar{X}_0 = m > 0$, there exists an exponentially distributed random variable V with mean $1/m$, so that as $t \uparrow \infty$,*

$$e^{-\rho t} \log(\bar{X}_t) \to \log\left(\frac{1}{V}\right) \quad a.s. \tag{56}$$

An interesting *open problem* is the long-term behaviour of the spatial process X constructed here. (An answer will be given in the forthcoming paper Fleischmann and Vakhtel (2004) [17].)

2.5. Log-Laplace equations

In this section we construct solutions to Eq. (9) as the limit of solutions to (13), and investigate properties needed in the proof of Theorem 2, as well as in Section 3.

Lemma 11 (Approximating solutions). *Fix $\beta \in (0,1]$. For each φ in $C_\ell^+(\mathbb{R}^d)$, there is a unique solution $u^{(\beta)} = u^{(\beta)}(\varphi) \in C(\mathbb{R}_+, C_\ell^+(\mathbb{R}^d))$ to the integral equation (13). If additionally $\varphi \in C_\ell^{2,+}(\mathbb{R}^d)$ (contained in the domain of Δ_α), then u is continuously differentiable in $C_\ell(\mathbb{R}^d)$ on $(0,\infty)$ with $u_t^{(\beta)}$ in the domain of Δ_α for every $t \geqslant 0$, and it solves the related function-valued Cauchy problem*

$$\left.\begin{aligned} &\frac{\partial}{\partial t} u_t = \Delta_\alpha u_t - g^{(\beta)}(u_t) \text{ on } (0,\infty) \\ &\qquad\text{with initial condition } u_{0+} = \varphi. \end{aligned}\right\} \tag{57}$$

All solutions $u^{(\beta)}$ satisfy

$$0 \leqslant \inf_{y\in\mathbb{R}^d} \varphi(y) \wedge 1 \leqslant u_t^{(\beta)}(\varphi)(x) \leqslant \|\varphi\|_\infty \vee 1, \quad t \geqslant 0,\ x \in \mathbb{R}^d. \tag{58}$$

Also, monotonicity in the initial conditions holds, meaning that for φ_1, φ_2 in $C_\ell^+(\mathbb{R}^d)$,

$$\varphi_1 \leqslant \varphi_2 \quad \text{implies} \quad u_t^{(\beta)}(\varphi_1) \leqslant u_t^{(\beta)}(\varphi_2), \quad t \geqslant 0. \tag{59}$$

Furthermore,

$$\lim_{\delta\downarrow 0} \sup_{0<\beta\leqslant 1} \sup_{0\leqslant s\leqslant\delta} \|u_s^{(\beta)}(\varphi) - \varphi\|_\infty = 0, \quad \varphi \in C_\ell^+(\mathbb{R}^d). \tag{60}$$

Proof. Let us first observe that $g^{(\beta)}$ interpreted as a mapping $C_\ell^+(\mathbb{R}^d) \to C_\ell(\mathbb{R}^d)$, is locally Lipschitz continuous, indeed it is continuously differentiable. Since T^α is strongly continuous on $C_\ell^+(\mathbb{R}^d)$, Theorem 6.1.4 of Pazy (1983) [26] then implies that for any $\varphi \in C_\ell^+(\mathbb{R}^d)$ there exists a unique solution $u^{(\beta)} \in C([0,t_0), C_\ell^+(\mathbb{R}^d))$ to (13) up to a possible "explosion time" $t_0 \leqslant \infty$. Because $g^{(\beta)}$ is continuously differentiable we may further apply Theorem 6.1.5 of Pazy (1983) [26] in order to conclude that if additionally $\varphi \in C_\ell^{2,+}(\mathbb{R}^d)$, then $u^{(\beta)}$ is continuously differentiable in $C_\ell(\mathbb{R}^d)$ on $(0,t_0)$, all $u_t^{(\beta)}$ belong to the domain of Δ_α for $0 \leqslant t < t_0$, and $u^{(\beta)}$ solves the Cauchy problem (57) up to the explosion time t_0.

By a probabilistic argument, we show next the bound on the solutions $u^{(\beta)}$ as claimed in (58). The boundedness of the solutions uniformly in $t \geqslant 0$ implies in particular that the explosion time $t_0 = \infty$. Here, we use the monotonicity in the initial condition stated in (59), which follows from the log-Laplace representation (15). Thus, we may estimate $u^{(\beta)}$ with the $\bar{u}^{(\beta)}$ given in (25a), related to the total mass process. We obtain for all $x \in \mathbb{R}^d$ and $t \geqslant 0$,

$$\bar{u}_t^{(\beta)}\Big(\inf_{y\in\mathbb{R}^d} \varphi(y)\Big) \leqslant u_t^{(\beta)}(\varphi)(x) \leqslant \bar{u}_t^{(\beta)}\big(\|\varphi\|_\infty\big). \tag{61}$$

Since as $t \uparrow \infty$, $\bar{u}_t^{(\beta)}(\lambda) \downarrow 1$ for $\lambda \geqslant 1$ and $\bar{u}_t^{(\beta)}(\lambda) \uparrow 1$ for $0 < \lambda \leqslant 1$, the bounds on $u^{(\beta)}$ as in (58) follow.

In order to prove relation (60) we use (13) and obtain

$$\big\|u_s^{(\beta)}(\varphi) - \varphi\big\|_\infty \leqslant \|T_s^\alpha \varphi - \varphi\|_\infty + \Bigg\|\int_0^s T_{s-r}^\alpha g^{(\beta)}\big(u_r^{(\beta)}(\varphi)\big)\,dr\Bigg\|_\infty \leqslant \|T_s^\alpha\varphi - \varphi\|_\infty + c(\varphi)s, \tag{62}$$

where the second term at the right-hand side of (62) has been estimated by noting that $g^{(\beta)}(v)$ is bounded uniformly over all $0 < \beta \leqslant 1$ and $v \in [0, 1 \vee \|\varphi\|_\infty]$. The result now follows since $\sup_{0\leqslant s\leqslant\delta} \|T_s^\alpha\varphi - \varphi\|_\infty \to 0$ as $\delta \downarrow 0$, by the strong continuity of the semigroup T^α acting on $C_\ell^+(\mathbb{R}^d)$. □

Lemma 12 (Convergence to a limiting solution). *Take $\varphi \in C_\ell^{++}(\mathbb{R}^d)$. Then there exists a unique solution $u(\varphi) \in C(\mathbb{R}_+, C_\ell^{++}(\mathbb{R}^d))$ to* (9)*, which satisfies for any $T > 0$,*

$$\lim_{\beta\downarrow 0} \sup_{0\leqslant t\leqslant T} \big\|u_t^{(\beta)}(\varphi) - u_t(\varphi)\big\|_\infty = 0. \tag{63}$$

For all $t \geqslant 0$, the solution u fulfills

$$0 < \inf_{y\in\mathbb{R}^d} \varphi(y) \wedge 1 \leqslant u_t(\varphi)(x) \leqslant \|\varphi\|_\infty \vee 1, \quad t \geqslant 0,\ x \in \mathbb{R}^d, \tag{64}$$

and is monotone in the initial condition (*see* (59))*. Furthermore, for φ in $C_\ell^{2,++}(\mathbb{R}^d)$, u is continuously differentiable in $C_\ell(\mathbb{R}^d)$ on $(0,\infty)$ with u_t in the domain of Δ_α for every $t \geqslant 0$, and it solves* (11).

Proof. Solutions to (13) with initial condition $\varphi \in C_\ell^{++}(\mathbb{R}^d)$ are bounded away from zero and infinity according to (58) of Lemma 11. We can therefore estimate for $0 < \beta_1 \leqslant \beta_2 \leqslant 1$,

$$\begin{aligned}|u_t^{(\beta_1)} - u_t^{(\beta_2)}|(x) &= \left|\int_0^t T_{t-s}^\alpha\big(g^{(\beta_2)}(u_s^{(\beta_2)}) - g^{(\beta_1)}(u_s^{(\beta_1)})\big)(x)\,ds\right| \\ &\leqslant \int_0^t T_{t-s}^\alpha\big|g^{(\beta_2)}(u_s^{(\beta_2)}) - g^{(\beta_2)}(u_s^{(\beta_1)})\big|(x)\,ds + \int_0^t T_{t-s}^\alpha\big|g^{(\beta_2)}(u_s^{(\beta_1)}) - g^{(\beta_1)}(u_s^{(\beta_1)})\big|(x)\,ds\end{aligned}$$

$$\leqslant C(\beta_2,\varphi)\int_0^t \|u_s^{(\beta_2)}-u_s^{(\beta_1)}\|_\infty\,ds+\delta(\beta_1,\beta_2,\varphi)t. \tag{65}$$

Here, we have set

$$C(\beta,\varphi):=\sup_{\inf_{y\in\mathbb{R}^d}\varphi(y)\wedge 1\leqslant v\leqslant\|\varphi\|_\infty\vee 1}\left|\frac{\partial g^{(\beta)}}{\partial v}(v)\right|<\infty, \tag{66a}$$

$$\delta(\beta_1,\beta_2,\varphi):=\sup_{\inf_{y\in\mathbb{R}^d}\varphi(y)\wedge 1\leqslant v\leqslant\|\varphi\|_\infty\vee 1}\left|g^{(\beta_2)}(v)-g^{(\beta_1)}(v)\right|<\infty. \tag{66b}$$

We now note that $\frac{\partial g^{(\beta)}}{\partial v}$ converges to $\frac{\partial g}{\partial v}$, uniformly on compact intervals in $(0,\infty)$ as $\beta\downarrow 0$ [recall (29)], and hence $\sup_{0<\beta\leqslant 1}C(\beta,\varphi)<\infty$. Likewise, $g^{(\beta)}$ tends to g, uniformly on compact sets in $(0,\infty)$, and thus $\sup_{\beta_1,\beta_2\leqslant\epsilon}\delta(\beta_1,\beta_2,\varphi)\to 0$ as $\epsilon\downarrow 0$. But by Gronwall's Inequality,

$$\sup_{t\leqslant T}\|u_t^{(\beta_2)}-u_t^{(\beta_1)}\|_\infty\leqslant\delta(\beta_1,\beta_2,\varphi)T\mathrm{e}^{C(\beta_2,\varphi)T}, \tag{67}$$

and so $(u^{(\beta_n)})_{n\geqslant 1}$ with $\beta_n\downarrow 0$ form a Cauchy sequence on $C([0,T],C_\ell^{++}(\mathbb{R}^d))$. Of course, the limit, which we call u, fulfills (64) as well as monotonicity in the initial condition as in (59). We can therefore repeat essentially the same arguments as in the array (65) to show that

$$\lim_{\beta\downarrow 0}\sup_{t\leqslant T}\left\|\int_0^t T_{t-s}^\alpha\left|g(u_s)-g^{(\beta)}(u_s^{(\beta)})\right|ds\right\|_\infty=0. \tag{68}$$

Hence, u satisfies (9). Because of the boundedness away from 0 we are securely in the local Lipschitz region of g. Thus, the same arguments concerning further regularity for initial conditions $\varphi\in C_\ell^{2,++}(\mathbb{R}^d)$ as detailed in the proof of Lemma 11 apply. This concludes the *existence* part of the lemma.

It remains to show *uniqueness* of solutions. We first note that for any solution $u(\varphi)$ to (9) with $\varphi\in C_\ell^{++}(\mathbb{R}^d)$ there exists a $t_0>0$ so that $u_t(\varphi)(x)\geqslant\frac{1}{2}\inf_{y\in\mathbb{R}^d}\varphi(y)>0$ for all $t\leqslant t_0$ and $x\in\mathbb{R}^d$. Indeed, for $T>0$ fixed, u is bounded above, $u_t(x)\leqslant\|\varphi\|_\infty+t\sup_{v\in\mathbb{R}_+}(-g(v))\leqslant C(T)$ for $t\leqslant T$, where we choose $C(T)>1$. Thus, on $[0,T]$, we can bound u from below, $u_t(x)\geqslant\inf_y\varphi(y)-g(C(T))t$, so that we can find a $t_0\in(0,T]$ satisfying $t_0\leqslant(g(C(T)))^{-1}(\frac{1}{2}\inf_y\varphi(y))$ and having the desired property.

The branching mechanism g is Lipschitz continuous on compact intervals of $(0,\infty)$ so that uniqueness on $[0,t_0]$ follows by Gronwall's Inequality. Thus, the solution on $[0,t_0]$ must be the one that we constructed above, which is in fact bounded below by $\inf_{y\in\mathbb{R}^d}\varphi(y)$. Hence, we can reiterate the same argument to see that uniqueness must hold on any arbitrary time interval, and that $u\in C(\mathbb{R}_+,C_\ell^{++}(\mathbb{R}^d))$. □

Lemma 13 (Comparison of solutions). *Fix $0<\beta_1\leqslant\beta_2\leqslant 1$, and φ in $C_\ell^+(\mathbb{R}^d)$ so that $\varphi(x)=1$ for all $|x|>c$, for some constant $c>0$. We obtain $u^{(\beta_1)}(\varphi)\geqslant u^{(\beta_2)}(\varphi)$ on $\mathbb{R}_+\times\mathbb{R}^d$. In particular, if additionally $\varphi\in C_\ell^{++}(\mathbb{R}^d)$ we have $\sup_{0<\beta\leqslant 1}u_t^{(\beta)}(\varphi)\leqslant u_t(\varphi)$.*

Proof. The proof is an adaptation of standard arguments, see for example Theorem 10.1 of Smoller (1983) [30]. Let us first additionally assume that φ belongs to $C_\ell^{2,+}(\mathbb{R}^d)$. We define the (at this stage possibly signed) function $v_t:=u_t^{(\beta_1)}-u_t^{(\beta_2)}$, $t\geqslant 0$, which then satisfies according to Lemma 11,

$$\left.\begin{aligned}&\frac{\partial}{\partial t}v_t=\Delta_\alpha v_t-g^{(\beta_1)}(u_t^{(\beta_1)})+g^{(\beta_2)}(u_t^{(\beta_2)}),\\&v_{0+}=0,\end{aligned}\right\} \tag{69}$$

on $\mathbb{R}_+$. Let $f \in C(\mathbb{R}_+, C_\ell^+(\mathbb{R}^d))$ be defined by $f_t = -g^{(\beta_1)}(u_t^{(\beta_2)}) + g^{(\beta_2)}(u_t^{(\beta_2)}) \geqslant 0$ (recall that by Lemma 7 the $g^{(\beta)}$ are non-decreasing in β). Then, for some $\xi(t,x)$ between $u_t^{(\beta_1)}(x)$ and $u_t^{(\beta_2)}(x)$,

$$-g^{(\beta_1)}\big(u_t^{(\beta_1)}(x)\big) + g^{(\beta_2)}\big(u_t^{(\beta_2)}(x)\big) = -(g^{(\beta_1)})'\big(\xi(t,x)\big)v_t(x) + f_t(x) \tag{70}$$

(with $(g^{(\beta)})'$ denoting the derivative of $g^{(\beta)}$). Note that the following double supremum

$$\sup_{t\in\mathbb{R}_+}\sup_{x\in\mathbb{R}^d}\big|(g^{(\beta_1)})'\big(\xi(t,x)\big)\big|$$

is finite because $g^{(\beta_1)}$ is locally Lipschitz and, by (58), ξ is bounded uniformly over all t and x. Also, by (70) $(g^{(\beta_1)})'(\xi(\cdot,\cdot)) \in C(\mathbb{R}_+, C_\ell(\mathbb{R}^d))$ as a difference of functions in this space. Thus, we can find some constant $R < 0$ so that $-(g^{(\beta_1)})'(\xi(t,x)) + R < 0$ for all t and x. Therefore, $\tilde{v}_t := \mathrm{e}^{Rt} v_t$ satisfies

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\tilde{v}_t = \Delta_\alpha \tilde{v}_t + \big(-(g^{(\beta_1)})'\big(\xi(t,\cdot)\big) + R\big)\tilde{v}_t + \mathrm{e}^{Rt} f_t,\\ &\tilde{v}_{0+} = 0, \end{aligned}\right\} \tag{71}$$

on $\mathbb{R}_+$. Fix $T > 0$. Suppose that $\tilde{v}_t(x) < 0$ for some $(t,x) \in (0,T] \times \mathbb{R}^d$. Then $\tilde{v}$ must attain a negative minimum on $(0,T] \times \mathbb{R}^d$ in some point $(t_{\min}, x_{\min})$. This follows from the fact that for any $t \in [0,T]$, we have $\tilde{v}_t(x) \to 0$ for $|x| \uparrow \infty$. To see this, note that for the initial conditions φ considered here, $u_t^{(\beta_i)}(x) \to 1$ for $|x| \uparrow \infty$ $(i = 1,2)$. In fact, using the mild form (13) of the equations, Lemma 7, and the monotonicity (59) in the initial condition,

$$T_t^\alpha(\varphi \wedge 1)(x) \leqslant u_t^{(\beta_i)}(\varphi \wedge 1)(x) \leqslant u_t^{(\beta_i)}(\varphi)(x) \leqslant u_t^{(\beta_i)}(\varphi \vee 1)(x) \leqslant T_t^\alpha(\varphi \vee 1)(x), \tag{72}$$

and the lower and upper bounds converge appropriately to 1 as $|x| \uparrow \infty$.

At the minimum $(t_{\min}, x_{\min})$ we would have that $\frac{\partial}{\partial t}\tilde{v}_{t_{\min}}(x_{\min}) \leqslant 0$ as well as $\Delta_\alpha \tilde{v}_{t_{\min}}(x_{\min}) \geqslant 0$ by the positive maximum principle (cf. Theorem 4.2.2 of Ethier and Kurtz (1986) [10]). Recalling the choice of R and f we obtain a contradiction to the equality in (71), and therefore may conclude that $\tilde{v}$, hence v, is indeed non-negative on $[0,T] \times \mathbb{R}^d$ and so also on $\mathbb{R}_+ \times \mathbb{R}^d$. Finally, to remove the additional requirement that $\varphi \in C_\ell^{2,+}(\mathbb{R}^d)$, we use the fact that there exists a sequences $\varphi_n \in C_\ell^{2,+}(\mathbb{R}^d)$ such that $\|\varphi - \varphi_n\|_\infty \to 0$ as $n \uparrow \infty$. Arguments analogous to those in (65) to (67) then show immediately that $\|u_t^{(\beta_i)}(\varphi) - u_t^{(\beta_i)}(\varphi_n)\|_\infty \to 0$, and so we are done. The convergence statement (63) of Lemma 12 now finishes the proof. □

In order to show that the mass of the processes $X^{(\beta)}$ does not escape to infinity as $\beta \downarrow 0$, we need to consider the behaviour of u started from *"runaway" test functions* r_k, $k \geqslant 1$. We first define an auxiliary function $r_k^{(\epsilon)}$ for some fixed $0 < \epsilon < \frac{1}{2}$ by

$$r_k^{(\epsilon)}(x) := \begin{cases} \dfrac{1}{k} & \text{for } |x| \leqslant k+\epsilon,\\[2mm] \dfrac{1-k^{-1}}{1-2\epsilon}|x| + \dfrac{-k+1-\epsilon+(1-\epsilon)k^{-1}}{1-2\epsilon} & \text{for } k+\epsilon < |x| \leqslant k+1-\epsilon,\\[2mm] 1 & \text{for } |x| > k+1-\epsilon. \end{cases} \tag{73}$$

In short, $r_k^{(\epsilon)}$ is radially symmetric and linearly increasing in $|x|$ between its two constant values $\frac{1}{k}$ and 1. Note also that $r_k^{(\epsilon)}$ is monotonically non-increasing in k. Now let $\Phi \in C^{\infty,+}(\mathbb{R}^d)$ with support in $B(0,\epsilon)$, the open ball around 0 with ϵ radius, and so that $\int_{\mathbb{R}^d}\Phi(x)\,dx = 1$. We then define

$$r_k(x) := \int_{\mathbb{R}^d} \Phi(x-y) r_k^{(\epsilon)}(y)\,dy, \tag{74}$$

as the mollification of $r_k^{(\epsilon)}$. As an immediate consequence of the properties of $r_k^{(\epsilon)}$, we obtain that r_k belongs to $C_\ell^{\infty,++}(\mathbb{R}^d)$, is also radially symmetric, monotonically non-increasing in k, and that it is constantly $\frac{1}{k}$ (respectively 1) for $|x| \leqslant k$ (respectively $|x| \geqslant k+1$).

Lemma 14 (Runaway solutions). *We have $u_t(r_k)(x) \downarrow 0$ as $k \uparrow \infty$, for any $0 \leqslant t < 1/\rho$ and $x \in \mathbb{R}^d$. The same statement holds for r_k replaced by $|\Delta_\alpha r_k| \vee r_k$ and $|g(r_k)| \vee r_k$.*

Proof. Let $t \geqslant 0$. We note that $u_t(r_k)(x)$ is monotonically non-increasing in k for every x, and bounded below by zero, so that a pointwise limit exists, which we call $u_t(r_\infty)(x)$. From the radial symmetry in the definition of r_k as well as in Eq. (9) we can immediately observe that, for all k, $u_t(r_k)(0) = \min_{x\in\mathbb{R}^d} u_t(r_k)(x)$.

Now consider a test function $\psi \in C_\ell^{2,++}(\mathbb{R}^d)$ with $\psi(x) = \exp(-|x|)$ for $|x| \geqslant 1$. We will first show that there exists a constant $\kappa = \kappa(\alpha) > 0$, such that

$$\Delta_\alpha \psi(x) \leqslant \kappa \psi(x) \tag{75}$$

for all $x \in \mathbb{R}^d$. Indeed, for $\alpha = 2$ this follows from the fact that $\Delta\psi(x) = (1 - \frac{d-1}{|x|})\psi(x) \leqslant \psi(x)$ for all $|x| \geqslant 1$. For $0 < \alpha < 2$, we use the well-known representation (see, for example, (5) of Section IX.11 in Yosida (1980) [32]),

$$\Delta_\alpha \psi(x) = \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty s^{-1-\alpha/2}\big[\psi(x) - T_s\psi(x)\big]\,ds, \tag{76}$$

where once more Γ is Euler's Gamma function. Thus, we obtain

$$\Delta_\alpha \psi(x) \leqslant c \int_0^1 s^{-1-\alpha/2}\big[\psi(x) - T_s\psi(x)\big]\,ds + c\psi(x) \int_1^\infty s^{-1-\alpha/2}\,ds. \tag{77}$$

Here, the integral of the second term is finite. The first term can be estimated by Taylor's Formula,

$$\int_0^1 s^{-1-\alpha/2}\big[\psi(x) - T_s\psi(x)\big]\,ds \leqslant \sup_{0\leqslant s<1} T_s\Delta\psi(x) \int_0^1 s^{-1-\alpha/2} s\,ds \leqslant c\psi(x), \tag{78}$$

where, in the second inequality, we have used (75) for $\alpha = 2$ together with the well known fact that

$$\sup_{0\leqslant s\leqslant 1} T_s\psi \leqslant c\psi. \tag{79}$$

It is also well known that the mild solution u to (11) is also a solution in the weak form for an appropriate class of test functions including our ψ. Thus, we obtain for any $t \geqslant 0$,

$$\begin{aligned}\big\langle u_t(r_k), \psi\big\rangle &= \langle r_k, \psi\rangle + \int_0^t \big\langle u_s(r_k), \Delta_\alpha\psi - \big(\rho \log u_s(r_k)\big)\psi\big\rangle\,ds \\ &\leqslant \langle r_k, \psi\rangle + (\kappa + \rho\log k) \int_0^t \big\langle u_s(r_k), \psi\big\rangle\,ds.\end{aligned} \tag{80}$$

Here, we have used that $\frac{1}{k} = u_0(r_k)(0) \leqslant u_t(r_k)(x) \leqslant 1$ implies $-\log u_t(r_k)(x) \leqslant \log k$. We also used (75). We can now apply Gronwall's Inequality in order to obtain for all $t \geqslant 0$,

$$\langle u_t(r_k),\psi\rangle \leqslant \left(\int_{\mathbb{R}^d} r_k(x)\psi(x)\,dx\right)\mathrm{e}^{(\kappa+\rho\log k)t} \leqslant \left(\int_{|x|<k} \frac{1}{k}\psi(x)\,dx + \int_{|x|\geqslant k}\psi(x)\,dx\right)\mathrm{e}^{\kappa t}k^{\rho t}$$
$$\leqslant \left(c(\psi)k^{\rho t-1} + c(d)k^{\rho t}\mathrm{e}^{-k/2}\right)\mathrm{e}^{\kappa t}. \tag{81}$$

Now restrict to $t<1/\rho$. Then the latter expression converges to zero as $k\uparrow\infty$. This implies that $\langle u_t(r_\infty),\psi\rangle = 0$, that is $u_t(r_\infty)(x)=0$ for almost all x. Taken together with the monotonicity in $|x|$, we obtain $u_t(r_\infty)=0$.

The statement of the lemma for $|\Delta_\alpha r_k|\vee r_k$ and $|g(r_k)|\vee r_k$ in $C^{++}(\mathbb{R}^d)$ follows by repeating the same line of arguments. The estimates of (80) hold true unchanged since both initial conditions are still bounded below by $\frac{1}{k}$ which is hence also true for the solutions u. The only changes in the calculations given in (81) occur thus in the estimates of the initial condition. Since $\sup_k |\Delta_\alpha r_k|\vee r_k \leqslant c<\infty$, we now estimate

$$\int_{\mathbb{R}^d} \left(|\Delta_\alpha r_k|\vee r_k\right)(x)\psi(x)\,dx \leqslant \int_{|x|<k}\frac{1}{k}\psi(x)\,dx + c\int_{|x|\geqslant k}\psi(x)\,dx, \tag{82}$$

with the additional constant c being inconsequential in the concluding calculations. Because $\sup_x |g(r_k(x))|\vee r_k = \sup_{0\leqslant a\leqslant 1}|g(a)|\leqslant c<\infty$, we estimate in this case,

$$\int_{\mathbb{R}^d} \left(\left|g(r_k)\right|\vee r_k\right)(x)\psi(x)\,dx \leqslant \int_{|x|<k}\frac{1}{k}(\log k)\psi(x)\,dx + c\int_{|x|\geqslant k}\psi(x)\,dx. \tag{83}$$

The constant in the second integral on the right-hand side is once again unimportant. The first term now leads to $k^{\rho t-1}\log k$ (instead of $k^{\rho t-1}$), which still converges to zero (for $t<1/\rho$). □

2.6. Tightness of the one-dimensional processes

In order to show *part* (a) *of Proposition* 6, we use Aldous' criterion of tightness (see [1]) in a version stated as Theorem 3.8.6(c) in Ethier and Kurtz [10]:

Lemma 15 (Aldous' criterion). *Let $(Y^{(\beta)})_{\beta\in I}$ be a family of processes with sample paths in $D(\mathbb{R}_+,\mathbb{R})$, and assume that $(Y^{(\beta)}_t)_{\beta\in I}$ is tight in law on $\mathbb{R}$, for any fixed time $t>0$. Let $S^{(\beta)}_T$ denote the collection of all $Y^{(\beta)}$-stopping times bounded by $T>0$. Then $(Y^{(\beta)})_{\beta\in I}$ is tight in law on $D(\mathbb{R}_+,\mathbb{R})$, if for some $\eta>0$,*

$$\lim_{\delta\to 0}\sup_{\beta\in I}\sup_{\tau\in S^{(\beta)}_T}\sup_{0\leqslant s\leqslant\delta}\mathbb{E}\left[\left(|Y^{(\beta)}_{\tau+s}-Y^{(\beta)}_\tau|\wedge 1\right)^\eta\right]=0. \tag{84}$$

So fix φ and θ_0 as in Proposition 6. First note that $(\langle X^{(\beta)}_t,\varphi\rangle)_{0<\beta\leqslant 1}$ is tight in law for any given time t as a consequence of Lemma 9. According to Lemma 15, it now suffices to verify that for $1>\delta\downarrow 0$,

$$\sup_{0<\beta\leqslant 1}\sup_{\tau\in S^{(\beta)}_T}\sup_{0\leqslant s\leqslant\delta}\mathbb{E}\left[\left(\left|\langle X^{(\beta)}_{\tau+s},\varphi\rangle - \langle X^{(\beta)}_\tau,\varphi\rangle\right|\wedge 1\right)^2\right]\to 0. \tag{85}$$

For each $m>0,\beta,t$ we define the event $A^{m,\beta,t} := \{\sup_{0\leqslant s\leqslant t}\langle X^{(\beta)}_s,\varphi\rangle\geqslant m\}$. We then bound the quantity in (85) by

$$\sup_{0<\beta\leqslant 1}\mathbb{P}[A^{m,\beta,T+1}] \tag{86a}$$
$$+\,c(m)\sup_{0<\beta\leqslant 1}\sup_{\tau\in S^{(\beta)}_T}\sup_{0\leqslant s\leqslant\delta}\mathbb{E}\left[\left|\exp\langle X^{(\beta)}_{\tau+s},-\varphi\rangle - \exp\langle X^{(\beta)}_\tau,-\varphi\rangle\right|^2\right]. \tag{86b}$$

Note that the term in (86a) converges to zero as $m \uparrow \infty$ due to Lemma 9. Using conditioning at time τ, the strong Markov property, time-homogeneity, as well as the log-Laplace relation (15), we bound the expectation in (86b) by

$$\begin{aligned}&\bigl|\mathbb{E}\bigl[\exp\langle X^{(\beta)}_{\tau+s}, -2\varphi\rangle - \exp\bigl(\langle X^{(\beta)}_{\tau+s}, -\varphi\rangle + \langle X^{(\beta)}_{\tau}, -\varphi\rangle\bigr)\bigr]\bigr| \\ &\quad + \bigl|\mathbb{E}\bigl[\exp\langle X^{(\beta)}_{\tau}, -2\varphi\rangle - \exp\bigl(\langle X^{(\beta)}_{\tau+s}, -\varphi\rangle + \langle X^{(\beta)}_{\tau}, -\varphi\rangle\bigr)\bigr]\bigr| \\ &\leqslant \bigl|\mathbb{E}\bigl[\exp\bigl\langle X^{(\beta)}_{\tau}, -u^{(\beta)}_s(2\varphi)\bigr\rangle - \exp\bigl\langle X^{(\beta)}_{\tau}, -u^{(\beta)}_s(\varphi) - \varphi\bigr\rangle\bigr]\bigr| \\ &\quad + \bigl|\mathbb{E}\bigl[\exp\langle X^{(\beta)}_{\tau}, -2\varphi\rangle - \exp\bigl\langle X^{(\beta)}_{\tau}, -u^{(\beta)}_s(\varphi) - \varphi\bigr\rangle\bigr]\bigr|. \end{aligned} \tag{87}$$

Now take θ such that $0 < \theta < \theta_0 \mathrm{e}^{-\rho T}$. Observe that there exists a constant $c(\theta)$ so that, for all $x, y \geqslant 0$, we have

$$|\mathrm{e}^{-x} - \mathrm{e}^{-y}| \leqslant c(\theta)|x-y|^{\theta}. \tag{88}$$

Therefore inequality (87) can be continued by

$$\begin{aligned}&\leqslant c(\theta)\bigl(\mathbb{E}\bigl[\bigl\langle X^{(\beta)}_{\tau}, \bigl|u^{(\beta)}_s(2\varphi) - u^{(\beta)}_s(\varphi) - \varphi\bigr|\bigr\rangle^{\theta}\bigr] + \mathbb{E}\bigl[\bigl\langle X^{(\beta)}_{\tau}, \bigl|u^{(\beta)}_s(\varphi) - \varphi\bigr|\bigr\rangle^{\theta}\bigr]\bigr) \\ &\leqslant c(\theta)\bigl(\bigl\|u^{(\beta)}_s(2\varphi) - 2\varphi\bigr\|^{\theta}_{\infty} + \bigl\|u^{(\beta)}_s(\varphi) - \varphi\bigr\|^{\theta}_{\infty}\bigr)\mathbb{E}\bigl[\sup_{0\leqslant t\leqslant T}\langle X^{(\beta)}_t, 1\rangle^{\theta}\bigr]. \end{aligned} \tag{89}$$

Since (89) is independent of the stopping times and converges to zero uniformly over $0 < \beta \leqslant 1$ and $0 \leqslant s \leqslant \delta$ as $\delta \downarrow 0$ by Lemmas 9 and 11, we obtain (85). This finishes the proof of Proposition 6(a). □

2.7. *Compact containment and convergence*

In this section, we show Proposition 6(b), thus establishing tightness in law. The convergence stated in Theorem 2(b) then follows by identifying the unique limit of any convergent subsequence. This also verifies the existence of the process X stated in Theorem 2(a).

Proof of Proposition 6(b). According to the characterisation of compact sets in M_{f} (see Kallenberg (1976) [22] A 7.5), claim (b) is implied by the following two statements:

(i) For all $\epsilon > 0$ there exists an $N_\epsilon \geqslant 1$ so that

$$\sup_{0<\beta\leqslant 1}\mathbb{P}\bigl[\sup_{0\leqslant t\leqslant T}\bar{X}^{(\beta)}_t > N_\epsilon\bigr] < \epsilon. \tag{90}$$

(ii) For all $\epsilon > 0$ there exists a k_ϵ such that for the Borel set $A_{k_\epsilon} := \{x \in \mathbb{R}^d\colon |x| > k_\epsilon + 1\}$,

$$\sup_{0<\beta\leqslant 1}\mathbb{P}\bigl[\sup_{0\leqslant t\leqslant T}X^{(\beta)}_t(A_{k_\epsilon}) > \epsilon\bigr] < \epsilon. \tag{91}$$

We remark that (i) is satisfied according to Lemma 9. For (ii) consider the test function $r_k \in C^{\infty,++}_{\ell}(\mathbb{R}^d)$ defined in (74), which has been chosen so that $r_k \geqslant 1_{A_k}$. Thus, it suffices to show (91) with A_{k_ϵ} replaced by r_{k_ϵ}.

1°(*Proof of* (91) *on a small time interval*). We will first show this statement for $T =: \tilde{t} < 1/\rho$ since we want to use Lemma 14. For each $K \geqslant 1$, we define a stopping time $\tau_K = \tau_K(k,\beta) := \inf\{t \geqslant 0\colon \langle X^{(\beta)}_t, |\Delta_\alpha r_k| + |g(r_k)|\rangle \geqslant K\}$. For each sample ω, either $\tau_K \leqslant T$ or $\tau_K > T$, hence we can make the following estimate involving the process stopped at τ_K:

$$\mathbb{P}\bigl[\sup_{0\leqslant t\leqslant T}\langle X^{(\beta)}_t, r_k\rangle > \epsilon\bigr] \leqslant \mathbb{P}[\tau_K \leqslant T] + \mathbb{P}\bigl[\sup_{0\leqslant t\leqslant T}\langle X^{(\beta)}_{t\wedge\tau_K}, r_k\rangle > \epsilon\bigr]. \tag{92}$$

Since there is a constant c independent of k so that $|\Delta_\alpha r_k| + |g(r_k)| < c$, Lemma 9 implies that as $K \uparrow \infty$,

$$\sup_{0<\beta\leqslant 1}\sup_{k\geqslant 1}\mathbb{P}\bigl[\tau_K(k,\beta) \leqslant T\bigr] \leqslant \sup_{0<\beta\leqslant 1}\mathbb{P}\biggl[\sup_{0\leqslant t\leqslant T}\bar{X}^{(\beta)}_t \geqslant \frac{K}{c}\biggr] \to 0. \tag{93}$$

In order to deal with the second probability in (92), we define the martingale

$$t \mapsto M_t^{(\beta)}(r_k) := 1 - \exp\langle X_t^{(\beta)}, -r_k\rangle + \int_0^t \exp\langle X_s^{(\beta)}, -r_k\rangle \big\langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\big\rangle\, ds. \tag{94}$$

Thus, the stopped process $M^{(\beta,\tau_K)}(r_k)$, defined by

$$M_t^{(\beta,\tau_K)}(r_k) := 1 - \exp\langle X_{t\wedge\tau_K}^{(\beta)}, -r_k\rangle + \int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \big\langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\big\rangle\, ds, \tag{95}$$

is also a martingale. For some $\epsilon' > 0$, the second term in (92) is equal to

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{0\leqslant t\leqslant T}\big(1 - \exp\langle X_{t\wedge\tau_K}^{(\beta)}, -r_k\rangle\big) > \epsilon'\Big] \qquad (96)\\
&= \mathbb{P}\left[\sup_{0\leqslant t\leqslant T}\left(M_t^{(\beta,\tau_K)}(r_k) - \int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle\big\langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\big\rangle\, ds\right) > \epsilon'\right]\\
&\leqslant \mathbb{P}\left[\sup_{0\leqslant t\leqslant T} M_t^{(\beta,\tau_K)}(r_k) > \frac{\epsilon'}{2}\right] + \mathbb{P}\left[\sup_{0\leqslant t\leqslant T}\int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle\big\langle X_s^{(\beta)}, \big|-\Delta_\alpha r_k + g^{(\beta)}(r_k)\big|\big\rangle\, ds > \frac{\epsilon'}{2}\right]\\
&\leqslant \frac{2}{\epsilon'}\left(\mathbb{E}\big[|M_T^{(\beta,\tau_K)}(r_k)|\big] + \mathbb{E}\left[\int_0^{T\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle\big\langle X_s^{(\beta)}, \big|-\Delta_\alpha r_k + g^{(\beta)}(r_k)\big|\big\rangle\, ds\right]\right)\\
&\leqslant \frac{2}{\epsilon'}\left(\mathbb{E}\big[1 - \exp\langle X_T^{(\beta,\tau_K)}, -r_k\rangle\big] + 2\mathbb{E}\left[\int_0^{T\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle\big\langle X_s^{(\beta)}, |\Delta_\alpha r_k| + \big|g(r_k)\big|\big\rangle\, ds\right]\right).
\end{aligned}$$

Here, we have used the martingale as well as Markov's Inequality for the second inequality. Consider now the first expectation of the last expression. It is bounded by

$$\mathbb{P}[\tau_K \leqslant T] + \mathbb{E}\big[1 - \exp\big\langle X_0^{(\beta)}, -u_T^{(\beta)}(r_k)\big\rangle\big]. \tag{97}$$

By (93), the probability term becomes small as $K \uparrow \infty$, uniformly in β and k. The rest of the expression can be bounded by $\mathbb{E}[1 - \exp\langle X_0^{(\beta)}, -u_T(r_k)\rangle]$ by Lemma 13. As $u_T(r_k) \leqslant 1$, the expectation converges to zero as $k \uparrow \infty$ for each β, by Lemma 14 and Lebesgue's Dominated Convergence Theorem. Furthermore $X_0^{(\beta)} \Rightarrow X_0$ and the convergence of $u_T(r_k) \downarrow 0$ is monotone in k yielding convergence of the expectation uniformly over all $0 < \beta \leqslant 1$.

Using the fact that $\sup_{0\leqslant a\leqslant K} a(1 - \exp(-a))^{-1} =: c(K) < \infty$, the expectation in the last line of the array (96) is bounded by

$$\begin{aligned}
&c(K)\int_0^T \mathbb{E}\big[1 - \exp\big\langle X_s^{(\beta)}, -|\Delta_\alpha r_k| - \big|g(r_k)\big|\big\rangle\big]\, ds\\
&\quad \leqslant c(K)\int_0^T \mathbb{E}\big[1 - \exp\big\langle X_0^{(\beta)}, -u_s^{(\beta)}\big(\big(|\Delta_\alpha r_k| \vee r_k\big) + \big(\big|g(r_k)\big| \vee r_k\big)\big)\big\rangle\big]\, ds
\end{aligned}$$

$$\leqslant c(K)\int_0^T \mathbb{E}\big[1-\exp\big\langle X_0^{(\beta)}, -u_s\big(\big(|\Delta_\alpha r_k|\vee r_k\big)+\big(\big|g(r_k)\big|\vee r_k\big)\big)\big\rangle\big]\,ds. \tag{98}$$

Here, we have exploited the log-Laplace representation (15) and the monotonicity of $u_s^{(\beta)}$ in the initial condition in the first inequality, as well as Lemma 13 in the second inequality. Again, by Lemma 14 together with the convergence in law of $X_0^{(\beta)}$ to X_0 and the uniform boundedness of the solutions in k, we obtain β-uniform convergence of the integrand to zero as $k\uparrow\infty$ for each $s\leqslant T$. Since the integrand is bounded by 1 a further application of Lebesgue's Dominated Convergence Theorem leads to the appropriate convergence of the entire expression.

Thus, we can finally conclude that there exists a k_ϵ such that the left-hand side of (92) is smaller than ϵ for all β. First, choose K large enough keeping in mind (93) and then k_ϵ large enough. This concludes the proof of (91) and hence of claim (ii) for $T=\tilde t<1/\rho$.

2°(*Tightness on a small time interval*). Taken together with Proposition 6(a) we obtain tightness in law on the path space $D([0,\tilde t], M_{\mathrm f})$.

3°(*Convergence of the finite dimensional distributions on finite time intervals*). We show subsequently that any subsequence, denoted by $X^{(\beta_n)}$ where $\beta_n\downarrow 0$ as $n\uparrow\infty$, convergent in law on the space $D([0,T], M_{\mathrm f})$, tends to a unique limit X that satisfies the log-Laplace relation (17) on $[0,T]$. It suffices to identify the finite dimensional distributions of X. As $\{\langle\cdot,\varphi\rangle\colon \varphi\in C_\ell^{++}(\mathbb{R}^d)\}$ is separating in $M_{\mathrm f}$, any $X_t\in M_{\mathrm f}$ can be characterised by $\langle X_t,\varphi\rangle$ for $\varphi\in C_\ell^{++}(\mathbb{R}^d)$.

For $m\geqslant 1$, let $0\leqslant t_1\leqslant\cdots\leqslant t_m\leqslant T$, as well as $\varphi_i\in C_\ell^{++}(\mathbb{R}^d)$ $(1\leqslant i\leqslant m)$ and define recursively

$$u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)=u_{t_1,\dots,t_{m-1}}\big(\varphi_1,\dots,\varphi_{m-1}+u_{t_m-t_{m-1}}(\varphi_m)\big). \tag{99}$$

Analogously, we define $u^{(\beta)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)$ and note that by the Markov property and (15),

$$\mathbb{E}\left[\prod_{i=1}^m \exp\langle X_{t_i}^{(\beta)},-\varphi_i\rangle\right]=\mathbb{E}\big[\exp\big\langle X_0^{(\beta)},-u^{(\beta)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\big]. \tag{100}$$

We can further show that as $n\uparrow\infty$,

$$\big\|u^{(\beta_n)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)-u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\|_\infty\to 0. \tag{101}$$

This follows by induction using (58) and (64) upon noting that for any sequence $(\varphi_n)_{n\geqslant1}$ of continuous functions in $C_\ell^{++}(\mathbb{R}^d)$ with $0<c_1\leqslant\varphi_n\leqslant c_2<\infty$ and $\|\varphi_n-\varphi\|_\infty\to 0$ we have $\|u_t^{(\beta_n)}(\varphi_n)-u_t(\varphi)\|_\infty\to 0$ for any $t\geqslant 0$. To see this, consider that the expression is bounded by

$$\big\|u_t^{(\beta_n)}(\varphi_n)-u_t(\varphi_n)\big\|_\infty+\big\|u_t(\varphi_n)-u_t(\varphi)\big\|_\infty, \tag{102}$$

where the first term converges to zero as in (65) to (67) in the proof of Lemma 12. The convergence to 0 of the second term uses $\|T_t^\alpha\varphi_n-T_t^\alpha\varphi\|_\infty\to 0$ along with similar arguments. We may now conclude that

$$\begin{aligned}
&\left|\mathbb{E}\left[\prod_{i=1}^m \exp\langle X_{t_i},-\varphi_i\rangle-\exp\big\langle X_0,-u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\right]\right|\\
&\quad=\lim_{n\uparrow\infty}\left|\mathbb{E}\left[\prod_{i=1}^m \exp\langle X_{t_i}^{(\beta_n)},-\varphi_i\rangle-\exp\big\langle X_0,-u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\right]\right|\\
&\quad=\lim_{n\uparrow\infty}\big|\mathbb{E}\big[\exp\big\langle X_0^{(\beta_n)},-u^{(\beta_n)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle-\exp\big\langle X_0,-u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\big]\big|\\
&\quad\leqslant\lim_{n\uparrow\infty}\Big(c(\theta_0)\mathbb{E}\big[\big\langle X_0^{(\beta_n)},\big|u^{(\beta_n)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)-u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big|\big\rangle^{\theta_0}\big]
\end{aligned}$$

$$+\left|\mathbb{E}\left[\exp\left\langle X_0^{(\beta_n)}, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\right\rangle - \exp\left\langle X_0, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\right\rangle\right]\right|\Big), \tag{103}$$

where we have used (100) for the second equality and (88) for the third inequality. Both terms in the last expression converge to 0 as $n \uparrow \infty$, the first by the fact (101) since $\sup_n \mathbb{E}[\langle X_0^{\beta_n}, 1\rangle^{\theta_0}] < \infty$ by assumption. But $u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)$ is unique and so any limit point in law of the $(X^{(\beta)})_{0<\beta\leqslant 1}$ has the same distribution as the unique process X in $D([0,T], M_{\mathrm{f}})$ satisfying (17) for $\varphi \in C_\ell^{++}(\mathbb{R}^d)$.

4°(*Extension of convergence to* $\mathbb{R}_+$). We can now reiterate these arguments in order to lift the restriction of the assumption $T = \tilde{t} < 1/\rho$. From the above, we know that $X_{\tilde{t}}^{(\beta)} \Rightarrow X_{\tilde{t}}$, and from Lemma 9 we obtain $\sup_{0<\beta\leqslant 1} \mathbb{E}[\langle X_{\tilde{t}}^{(\beta)}, 1\rangle^{\theta}] < \infty$ for any $0 < \theta < \theta_0 \mathrm{e}^{\rho\tilde{t}}$. Thus, we can apply the same arguments to the process started at $\tilde{t}$ which converges again on the next interval of length $\tilde{t}$. This implies convergence of the processes in $D([0, 2\tilde{t}], M_{\mathrm{f}})$. Further reiteration yields convergence on an arbitrary finite time interval $[0,T]$, and therefore on $\mathbb{R}_+$. This completes the proof of Proposition 6. □

Note that with the previous proof we also verified Theorem 2(b). The completion of the proof of Theorem 2(a) is postponed to the end of Section 2.8.

2.8. Log-Laplace equations (continued)

Recall that Theorem 1(a) was proved with Lemma 12.

Completion of proof of Theorem 1(b). The uniqueness of the extension to non-negative initial conditions relies on the existence of the process X according to Theorem 2(a), constructed before. By Lemma 12, $u(\varphi_n)$ exists for all $n \geqslant 1$ and is bounded below by $\inf_{x\in\mathbb{R}^d} \varphi(x) \wedge 1$. From the log-Laplace representation (17) we see that the sequence is monotonically non-increasing as $n \uparrow \infty$ and that, for each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the limit $\lim_{n\uparrow\infty} u_t(\varphi_n)(x) := u_t(\varphi)(x)$ exists. Clearly, the limit is independent of the choice of the sequence $(\varphi_n)_{n\geqslant 1}$ since the left-hand side of (17) converges to a unique limit by the Dominated Convergence Theorem. This implies that $g(u_t(\varphi_n)(x))$ converges boundedly pointwise to $g(u_t(\varphi)(x))$. Thus by Lebesgue's Dominated Convergence Theorem,

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}^{\alpha}(x-y) g\big(u_s(\varphi_n)(y)\big)\,dy\,ds \to \int_0^t \int_{\mathbb{R}^d} p_{t-s}^{\alpha}(x-y) g\big(u_s(\varphi)(x)\big)\,dy\,ds$$

as $n \uparrow \infty$. Hence, $u(\varphi)$ fulfills (9) pointwise.

Like the approximating sequence, $(t,x) \mapsto u_t(\varphi)(x)$ is a uniformly bounded non-negative function on $\mathbb{R}_+ \times \mathbb{R}^d$. It only remains to show joint continuity in t and x. The right continuity at $t = 0$ follows immediately from the strong continuity of T_t^{α} as well as the boundedness of the solutions. Otherwise, we consider for some $0 < \epsilon < T$, $\epsilon < t \leqslant t' \leqslant T$ and $x, x' \in \mathbb{R}^d$,

$$\begin{aligned}\big|u_{t'}(x') - u_t(x)\big| &\leqslant \int_{\mathbb{R}^d} \big|p_{t'}^{\alpha}(x'-y) - p_t^{\alpha}(x-y)\big|\varphi(y)\,dy + \int_t^{t'} \int_{\mathbb{R}^d} p_{t'-s}^{\alpha}(x'-y)\big|g\big(u_s(y)\big)\big|\,dy\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \big|p_{t'-s}^{\alpha}(x'-y) - p_{t-s}^{\alpha}(x-y)\big|\big|g\big(u_s(y)\big)\big|\,dy\,ds \\ &\leqslant c\Bigg(\int_{\mathbb{R}^d} \big|p_{t'}^{\alpha}(x'-y) - p_t^{\alpha}(x-y)\big|\,dy + |t'-t|\end{aligned} \tag{104a}$$

$$+\int_0^t \int_{\mathbb{R}^d} \left|p^\alpha_{t'-s}(x'-y) - p^\alpha_{t-s}(x-y)\right| dy\,ds\Bigg). \tag{104b}$$

Now, let $|t'-t| \downarrow 0$ as well as $|x'-x| \downarrow 0$. We note that

$$\sup_{\epsilon < t \leqslant T} \sup_{x\in\mathbb{R}^d} p^\alpha_t(x) < \infty, \tag{105}$$

and that $p^\alpha_t(x)$ is jointly continuous on $(0,\infty) \times \mathbb{R}^d$ (see, for instance, Appendix in Fleischmann and Gärtner (1986) [14]). Thus, by Lebesgue's Dominated Convergence Theorem, the spatial integrals in (104a) and (104b) converge to zero, the latter for all $s < t$. Since the spatial integral in (104b) is further bounded by 2, another application of Lebesgue's Theorem concludes the proof of Theorem 1(b). □

Completion of proof of Theorem 2(a). It remains to verify that the uniquely constructed limit process X also satisfies the log-Laplace relation (17) with $\varphi \in C^+_\ell(\mathbb{R}^d)$ and $u(\varphi)$ the unique solution in the setting of Theorem 1(b). This can be seen by considering $\varphi_n \downarrow \varphi$ with $\varphi_n \in C^{++}_\ell(\mathbb{R}^d)$. In this case, both sides of the representation (17) converge appropriately due to Lebesgue's Dominated Convergence Theorem, and we are done. □

3. Immortality and infinite biodiversity

As already mentioned in Section 1.3, our process X is immortal and propagates instantaneously:

Proposition 16 (Immortality and instantaneous propagation). *Take* $\mu \in M_{\mathrm{f}}\backslash\{0\}$, $t > 0$, *and* $\varphi \in C^+_{\mathrm{com}}\backslash\{0\}$. *Then* $\langle X_t, \varphi\rangle > 0$, $\mathbb{P}_\mu$*-a.s.*

In other words, almost surely the Lebesgue measure is absolutely continuous with respect to X_t. Recall that in the case $\alpha = 2$ this is quite different from the behaviour of the approximating supercritical $X^{(\beta)}$ processes.

Proof. By the Markov property of X, we may fix $0 < t < 1/\rho$. Clearly,

$$\mathbb{P}_\mu\big[\langle X_t, \varphi\rangle = 0\big] = \lim_{\theta\uparrow\infty} \mathbb{E}_\mu[\mathrm{e}^{-\langle X_t, \theta\varphi\rangle}] = \exp\big[-\lim_{\theta\uparrow\infty}\langle \mu, u_t(\theta\varphi)\rangle\big]. \tag{106}$$

Hence, by Monotone Convergence it suffices to show that for each $x \in \mathbb{R}^d$,

$$u_t(\theta\varphi)(x) \uparrow \infty \quad \text{as } \theta \uparrow \infty. \tag{107}$$

Let us now consider a sequence $(\varphi_n)_{n\geqslant 1}$ with $\varphi_n \in C^{++}_\ell(\mathbb{R}^d)$ and $\varphi_n \downarrow \varphi$ pointwise as well as $\|\varphi_n\|_\infty \to \|\varphi\|_\infty$. By the Feynman–Kac representation of solutions to (9) in the local Lipschitz region,

$$u_t(\theta\varphi_n)(x) = \theta E_x\left[\varphi_n(\xi_t)\exp\left(-\int_0^t \rho \log\big[u_{t-s}(\theta\varphi_n)(\xi_s)\big]\right)\right], \tag{108}$$

where (ξ, P_x) is a motion with "generator" Δ_α started at x. Consequently, by the estimate (64) in Lemma 12, for θ large enough, so that $\theta\|\varphi\|_\infty \geqslant 1$,

$$u_s(\theta\varphi_n)(\xi_s) \leqslant \theta\|\varphi_n\|_\infty, \quad s, \theta \geqslant 0. \tag{109}$$

Therefore,

$$\begin{aligned} u_t(\theta\varphi_n)(x) &\geqslant \theta E_x\big[\varphi_n(\xi_t)\exp\big(-\rho t \log\big[\theta\|\varphi_n\|_\infty\big]\big)\big] \\ &= \theta\big(\theta\|\varphi_n\|_\infty\big)^{-\rho t} E_x\big[\varphi_n(\xi_t)\big] = \theta^{1-\rho t}\|\varphi_n\|_\infty^{-\rho t}\, T^\alpha_t \varphi_n(x). \end{aligned} \tag{110}$$

By Theorem 1(b) the left-hand side converges to $u_t(\theta\varphi)(x)$ as $n \uparrow \infty$. The right-hand side converges by assumption implying

$$u_t(\theta\varphi_n)(x) \geqslant \theta^{1-\rho t} \|\varphi\|_\infty^{-\rho t} T_t^\alpha \varphi(x), \tag{111}$$

which becomes infinite as $\theta \uparrow \infty$ since $\varphi \neq 0$ giving (107). This completes the proof. □

Proposition 16 implies that X has countably infinite biodiversity. This we want to make precise now. Recall that an infinitely divisible random measure $Y \in M_\mathrm{f}$ has a *clustering representation*

$$Y = \gamma + \sum_i \chi_i \tag{112}$$

(see, for instance, Lemma 6.5 in Kallenberg (1976) [22]). Here $\gamma \in M_\mathrm{f}$ is the deterministic component of Y (or the essential infimum of Y), and the clusters $\chi_i \in M_\mathrm{f}$ are the "points" of a Poissonian point measure on $\mathcal{M}_\mathrm{f}(\mathbb{R}^d) \setminus \{0\}$ with some intensity measure $\mathbf{Q}$, which is called the *canonical measure* of Y. We can reformulate (112) as the *classical Lévy–Hincin formula* for log-Laplace transforms,

$$-\log \mathbb{E}_\mu[\mathrm{e}^{-\langle Y,\varphi\rangle}] = \langle \gamma, \varphi\rangle + \int_{\mathcal{M}_\mathrm{f}(\mathbb{R}^d)} \mathbf{Q}(d\chi)(1 - \mathrm{e}^{-\langle \chi,\varphi\rangle}) \tag{113}$$

(see Theorem 6.1 of Kallenberg [22]). Let B be a bounded Borel subset of $\mathbb{R}^d$. If $\gamma = 0$, then the number $\#\{i\colon \chi_i(B) > 0\}$ of clusters in B has a Poisson distribution with expectation $\mathbf{Q}(\chi\colon \chi(B) > 0) \leqslant \infty$. If $\gamma(B) > 0$ then one could say a "continuum of clusters"contributes to $Y(B)$. Therefore in Fleischmann and Klenke (2000) [15] the following terminology was introduced:

Definition 17 (*Biodiversity*). We say that the (local) *biodiversity* of the infinitely divisible random measure Y is

- *finite*, if $\gamma = 0$ and $\mathbf{Q}(\chi\colon \chi(B) > 0) < \infty$ for every compact set B,
- *countably infinite*, if $\gamma = 0$ and $\mathbf{Q}(\chi\colon \chi(B) > 0) = \infty$ for every open set $B \neq \emptyset$,
- *uncountably infinite*, if $\gamma(B) > 0$ for every open set $B \neq \emptyset$.

Armed with this terminology, we can now prove the following result:

Corollary 18 (Countably infinite biodiversity). *For every fixed $\mu \neq 0$ and $t > 0$, the random measure X_t has* (*locally*) *countably infinite biodiversity.*

Recall that this is in contrast to the finite biodiversity of the random states of the approximating processes $X^{(\beta)}$.

Proof. For Y to have finite local biodiversity, it is necessary and sufficient that

$$\mathbb{P}_\mu\big[Y(B) = 0\big] > 0 \quad \text{for any compact set } B. \tag{114}$$

This follows from the simple observation that

$$\mathbf{Q}\big(\chi\colon \chi(B) > 0\big) = -\log \mathbb{P}_\mu\big[Y(B) = 0\big], \tag{115}$$

provided that $\gamma = 0$. Then from Proposition 16 it follows that the X_t have infinite biodiversity. Finally, the random measure X_t does not have a deterministic component, since $X_t(\mathbb{R}^d)$ has a stable distribution with index $\mathrm{e}^{-\rho t}$ [recall (25b) and (27b)]. This finishes the proof. □

Remark 19 (*Genealogy*). Recall that in genealogical terms Corollary 18 means that at time t in each bounded region the families of individuals have countably infinite many different ancestors at time 0. It would also be very interesting to study the more detailed genealogy of our superprocess X. The genealogy of Neveu's branching process $\bar{X}$ was worked out in Bertoin and Le Gall (2000) [2]. It is connected with the Bolthausen–Sznitman coalescent and the description of the generalized random energy model of spin glasses (see Neveu (1992) [25]).

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Appendix A

Proof of Proposition 10. We first note that $t \mapsto M_t(\lambda) := \exp(-\bar{u}_{-t}(\lambda)\bar{X}_t) = \exp(-\lambda^{(\mathrm{e}^{\rho t})}\bar{X}_t)$ is a martingale, for each $\lambda > 0$, since for $s \leqslant t$,

$$\mathbb{E}_m\big[\exp\big(-\bar{u}_{-t}(\lambda)\bar{X}_t\big) \mid \mathcal{F}_s\big] = \exp\big(-\bar{u}_{t-s}\big(\bar{u}_{-t}(\lambda)\big)\bar{X}_s\big) = \exp\big(-\bar{u}_{-s}(\lambda)\bar{X}_s\big) \tag{A.1}$$

by the Markov and branching property of the process $\bar{X}$ and the semigroup property of the solution $\bar{u}$. Since $M_t(\lambda)$ takes values in $[0,1]$ the limit as $t \uparrow \infty$ exists a.s., and we denote it by $W(\lambda)$. By Lebesgue's Dominated Convergence Theorem, for all $\theta > 0$,

$$\begin{aligned}\mathbb{E}_m\big[W^{\theta}(\lambda)\big] &= \lim_{t\uparrow\infty} \mathbb{E}_m\big[\exp\big(-\theta\bar{u}_{-t}(\lambda)\bar{X}_t\big)\big] \\ &= \lim_{t\uparrow\infty} \exp\big(-\bar{u}_t\big(\theta\bar{u}_{-t}(\lambda)\big)m\big) \\ &= \lim_{t\uparrow\infty} \exp\big(-(\theta\lambda^{(\mathrm{e}^{\rho t})})^{\mathrm{e}^{-\rho t}} m\big) = \mathrm{e}^{-\lambda m}.\end{aligned} \tag{A.2}$$

This implies that $W(\lambda)$ takes the value 1 with probability $\mathrm{e}^{-\lambda m}$ and is 0 otherwise. Since $M_t(\lambda)$ is monotonically non-increasing in λ for each $t \geqslant 0$, the limit $W(\lambda)$ is non-increasing in λ. Also note that $W(\lambda)$ is defined a.s. for all rational λ. With the exception of a null set, we can therefore define the threshold variable $V := \inf\{\text{rational}\ \lambda\colon\ W(\lambda) = 0\}$. From $\mathbb{P}_m[V < \lambda] = \lim_{\lambda'\uparrow\lambda} \mathbb{P}_m[W(\lambda') = 0] = 1 - \mathrm{e}^{-\lambda m}$, we obtain that V is exponentially distributed with mean $1/m$. It follows that a.s.

$$\lambda^{(\mathrm{e}^{\rho t})}\bar{X}_t \to \begin{cases} 0 & \text{for } \lambda < V, \\ \infty & \text{for } \lambda > V, \end{cases} \tag{A.3}$$

as $t \uparrow \infty$. This implies that for any random variables V_0 and V_1 with rational values so that $V_0 < V < V_1$,

$$V_1^{-\mathrm{e}^{\rho t}} \leqslant \bar{X}_t \leqslant V_0^{-\mathrm{e}^{\rho t}}, \tag{A.4}$$

a.s. for $t = t(\omega)$ large enough. Hence, we have

$$\log\left(\frac{1}{V_1}\right) \leqslant \liminf_{t\uparrow\infty} \mathrm{e}^{-\rho t}\log(\bar{X}_t) \leqslant \limsup_{t\uparrow\infty} \mathrm{e}^{-\rho t}\log(\bar{X}_t) \leqslant \log\left(\frac{1}{V_0}\right) \tag{A.5}$$

almost surely. The statement now follows by letting almost surely V_0 and V_1 tend to V.

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