

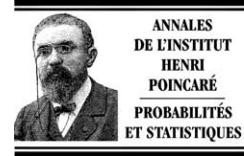


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Ann. I. H. Poincaré – PR 40 (2004) 309–336



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Random walks in random medium on \mathbb{Z} and Lyapunov spectrum

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Received 7 March 2003; received in revised form 10 October 2003; accepted 21 October 2003

Available online 18 February 2004

Abstract

We consider a one-dimensional random walk with bounded steps in a stationary and ergodic random medium. We show that the algebraic structure of the random walk is given by geometrical invariants related to the description of a space of harmonic functions. We then prove a recurrence criterion similar to Key's Theorem [E.S. Key, *Ann. Probab.* 12 (2) (1984) 529] in terms of the sign of an intermediate Lyapunov exponent of a random matrix. We show that this exponent is simple and we relate it to the dominant exponents of two non-negative matrices associated to the random walks of left and right records. We also give an algorithm to compute that exponent. In a last part, we deduce from [J. Brémont, *Ann. Probab.* 30 (3) (2002) 1266] that the Law of Large Numbers is always valid.

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Résumé

Nous considérons une marche aléatoire unidimensionnelle à pas bornés en milieu aléatoire stationnaire ergodique. Nous montrons que la structure algébrique de la marche aléatoire est donnée par des invariants géométriques liés à la description d'un espace de fonctions harmoniques. Nous donnons ensuite un critère de récurrence du même type que celui de Key [E.S. Key, *Ann. Probab.* 12 (2) (1984) 529], en fonction du signe d'un exposant de Lyapunov intermédiaire d'une matrice aléatoire. Nous prouvons que cet exposant est simple et nous le relierons aux exposants maximaux de deux matrices positives associées aux marches des records à gauche et à droite. Nous donnons aussi un algorithme pour calculer cet exposant. Dans une dernière partie, nous déduisons de [J. Brémont, *Ann. Probab.* 30 (3) (2002) 1266] que la Loi des Grands Nombres est toujours vérifiée.

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MSC: 60J10; 60K37

Keywords: Markov chain; Random walk in random environment; Lyapunov exponent; Cone

1. Introduction

Random media are frequently introduced in Physics to model properties of statistical homogeneity (see Bernasconi [3]). We consider in this paper a one-dimensional model of random walks with bounded steps in random medium. It corresponds to giving a stationary field of transition laws on \mathbb{Z} .

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doi:10.1016/j.anihpb.2003.10.006

1.1. Model

Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible dynamical system, that is a probability space $(\Omega, \mathcal{F}, \mu)$ with an invertible transformation T , measurable as well as its inverse. We assume the system to be ergodic. The space Ω will be considered as the space of the environments.

We now fix two integers $L \geq 1$ and $R \geq 1$ and introduce the set $\Lambda = \{-L, \dots, +R\}$ of consecutive integers. Consider then a family $(p_z)_{z \in \Lambda}$ of positive random variables on (Ω, \mathcal{F}) , indexed by Λ , satisfying a minoration condition, precisely there exists $\varepsilon > 0$ such that:

$$\forall z \in \Lambda, z \neq 0, \quad p_z \geq \varepsilon \text{ and } \sum_{z \in \Lambda} p_z = 1, \quad \mu\text{-ae.} \quad (1)$$

For any fixed environment $\omega \in \Omega$, introduce the Markov chain $(\xi_n(\omega))_{n \geq 0}$ on \mathbb{Z} such that $\xi_0(\omega) = 0$ and with the following transition laws:

$$\forall x \in \mathbb{Z}, \forall z \in \Lambda, \quad \mathcal{P}_0^\omega(\xi_{n+1}(\omega) = x + z \mid \xi_n(\omega) = x) := p_z(T^x \omega).$$

We write $(\mathcal{P}_k^\omega)_{k \in \mathbb{Z}}$ for the family of measures on the space of jumps $\Lambda^{\mathbb{N}}$ with such transition laws and conditional to a given departure point $k \in \mathbb{Z}$. The “quenched problem” is to describe the behaviour of the random walk $(\xi_n(\omega))_{n \geq 0}$ with \mathcal{P}_0^ω -probability one, for μ -ae medium ω .

Notations. The dependence in $\omega \in \Omega$ will always be implicit. Any expression of the form $f(T^k \omega)$ will simply be denoted by $T^k f$ or $f(k)$. In the sequel, we write P_k for \mathcal{P}_k^ω , $k \in \mathbb{Z}$.

1.2. Known results

We now give an overview on the study of the model, centered on the asymptotic properties of the random walk. We denote by (L, R, erg) the previous model where the environment is a general dynamical system. We also introduce the notation (L, R, iid) for the independent case, corresponding to the situation where Ω is a product space, μ is a product probability measure, T is the left shift on Ω and the $(p_z)_{z \in \Lambda}$ depend only on the first coordinate.

The case $(1, 1, \text{iid})$ has been intensively studied. The first result is due to Solomon [17] who showed a recurrence criterion in function of the sign of $\int \log(p_{-1}/p_1) d\mu$. The proof extends naturally to $(1, 1, \text{erg})$, see Alili [1] for example. For general L and R , the situation is more complex. Key [10] in 1984 proved a recurrence criterion for (L, R, iid) , using Oseledets’ Ergodic Multiplicative Theorem [15]. The recurrence or transience of the random walk is then given by the sign of the sum $\gamma_R(M_K, T^{-1}) + \gamma_{R+1}(M_K, T^{-1})$, involving the R th and $(R+1)$ th Lyapunov exponents with respect to T^{-1} of a random matrix M_K of dimension $(R+L) \times (R+L)$ built with the $(p_z)_{z \in \Lambda}$. The theorem also indicates that one of those two exponents is always zero.

A first remark is that Key’s Theorem extends to (L, R, erg) after a minor modification using conditional expectation in Theorem (17), p. 539 of Key [10]. See [5] for example. The form of the theorem can be simplified as one remarks that $M_K u = u$, where u is the vector in \mathbb{R}^{R+L} with all components equal to one. Considering ${}^t(M_K)$ restricted to u^\perp in a particular basis, one can deduce from Key’s result a recurrence criterion in terms of the sign of the R th Lyapunov exponent $\gamma_R(M, T)$ of a random matrix M of dimension $(L+R-1) \times (L+R-1)$. This was first noticed by Letchikov [13]. Another proof is given in [7]. The general study by the author in [6] and for the model $(L, 1, \text{erg})$, concerning for example the existence of the absolutely continuous invariant measure for the random walk of the “environments seen from the particle”, confirms the role of the matrix M , as well as the present work.

Studies in order to obtain a more “efficient” criterion were developed, first by Letchikov [14] for $(2, 1, \text{erg})$ under an hypothesis of density and then by Derriennic [8] who suppressed that hypothesis, using the theory of

representation of a Markov chain by cycles and weights. The result is a recurrence criterion expressed in terms of the sign of $\int \log f d\mu$, where f is the random continued fraction defined by the relation

$$f = \frac{p_{-1}}{p_1} + \frac{p_{-1} + p_{-2}}{p_1} \frac{1}{T^{-1}f}. \tag{2}$$

It is checked in [7] that the previous criterion and Key’s Theorem are equivalent, as it is shown that $\int \log f d\mu = \gamma_1(M, T)$.

We proved in [6], in the study of $(L, 1, erg)$, a generalization of the previous equality (2) with a similar function f . When $R = 1$ (or $L = 1$, taking M^{-1}), a very important property is that M has non-negative coefficients and one can then use the directional contraction properties of M in the positive cone of \mathbb{R}^L . In this case, there exists a unique positive random vector V with a norm equal to 1 and a unique positive random scalar λ , where $\log(\lambda)$ is a bounded function, such that

$$MV = \lambda TV \quad \text{and} \quad \int \log(\lambda) d\mu = \gamma_1(M, T).$$

The precise behaviour of the random walk is read on the properties of λ with respect to $(\Omega, \mathcal{F}, \mu, T)$. For example, a characterization of the Law of Large Numbers or the Central Limit Theorem can be given.

For the model $(2, 2, erg)$, let us note that [5] contains a simple proof by direct calculations of the recurrence criterion in terms of the sign of $\gamma_2(M, T)$. An important point is that when $L = R = 2$, the matrix M is deterministically and explicitly conjugated to a non-positive matrix. This fact is one of the motivations for the present work.

We finally mention another approach due to Bolthausen and Goldsheid [4] for the study of the general model with bounded steps in an *iid* environment. It consists in taking $m \geq \max\{L, R\}$ and in considering the Markov chain $(q_n(\omega), r_n(\omega))_{n \geq 0}$ on the strip $\mathbb{Z} \times \{0, \dots, m - 1\}$ defined by the Euclidian division $\xi_n(\omega) = mq_n(\omega) + r_n(\omega)$. Then for general random walks on a strip a recurrence criterion is given in terms of the sign of the top Lyapunov exponent of a non-negative matrix. However that matrix is of difficult access since it contains limit expressions built with the transition laws. Anyway, some tools and objects of [4] are similar to what we consider in the present work.

1.3. Content of the article

The aim of the present paper is to study the structure of the random walk in order to prove rather simply the previous recurrence criterion in terms of $\gamma_R(M, T)$ and then to show that this criterion is in fact reasonably explicit. This paper seems to be a prerequisite for the development of the same study as in [6] for the model (L, R, erg) .

This way, we show that the algebraic structure of the random walk is given by the geometry of a space of harmonic functions. To describe this space, we prove the existence of deterministic and explicit cones in the external powers of order R and L of the underlying space that are invariant by the corresponding external powers of the matrices M and M^{-1} . We then show that the central Lyapunov exponent $\gamma_R(M, T)$ of M with respect to T is simple and can be expressed as the difference of the dominant Lyapunov exponents of two non-negative matrices related to the random walks of the left and right records. We then deduce the recurrence criterion according to the sign of $\gamma_R(M, T)$.

Next we give a theoretical algorithm of calculation for $\gamma_R(M, T)$. The existence of the invariant cones implies exponential convergence of this algorithm. Rate and constants can also be made explicit.

We finally mention, as a corollary of [6], that the Law of Large Numbers is always valid for the model (L, R, erg) .

2. Harmonic functions and gradient-vectors

We consider an interval of integers $[a, b]$, with $a < b$, and we introduce quantities controlling the behaviour of the random walk in that interval, conditionally to a departure point. In the sequel, we will let a or b become infinite in order to deduce an asymptotic result.

Definition 2.1. For all $k \in [a - L + 1, b + R - 1]$, we define:

$$\begin{cases} P_k\{a, b, +\} = P_k\{\text{reach}[-\infty, a] \cup [b, +\infty[\text{ by the right side}, \\ P_k\{a, b, -\} = P_k\{\text{reach}[-\infty, a] \cup [b, +\infty[\text{ by the left side}. \end{cases}$$

For $\zeta \in \{a - l \mid 0 \leq l \leq L - 1\} \cup \{b + r \mid 0 \leq r \leq R - 1\}$, we also set:

$$P_k\{a, b, \zeta\} = P_k\{\text{reach}[-\infty, a] \cup [b, +\infty[\text{ at the point } \zeta\}.$$

From the Markov property, any function of the form $k \mapsto P_k\{a, b, \zeta\}$, where ζ belongs to the enlarged boundary of $[a, b]$ or to $\{\pm\}$, is harmonic and precisely is a barycenter of its L left neighbours and its R right neighbours.

The quantities of interest in the sequel are the “difference-vectors” or “gradient-vectors” derived from these functions. We introduce them now, as well as the matrix M .

Definition 2.2. We set $d = R + L - 1$.

Definition 2.3. Let $a \leq k < b$ be integers. For any $\zeta \in \{a - l \mid 0 \leq l \leq L - 1\} \cup \{b + r \mid 0 \leq r \leq R - 1\} \cup \{\pm\}$, we write $V_k(a, b, \zeta)$ for the “gradient-vector” in \mathbb{R}^d :

$$V_k(a, b, \zeta) = {}^t(g_{k+R-1}(a, b, \zeta), \dots, g_k(a, b, \zeta), \dots, g_{k-L+1}(a, b, \zeta)),$$

where we define $g_k(a, b, \zeta) = P_k\{a, b, \zeta\} - P_{k+1}\{a, b, \zeta\}$.

Definition 2.4. We write M for the following random matrix of dimensions $d \times d$, where all entries are equal to 0 except for the first line and a sub-diagonal of ones:

$$M = \begin{pmatrix} -a_1 & \dots & -a_{R-1} & b_L & \dots & b_1 \\ 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \end{pmatrix}, \tag{3}$$

with:

$$a_i = \left(\frac{p_{R-i} + \dots + p_R}{p_R} \right) \quad \text{and} \quad b_i = \left(\frac{p_{-L+i-1} + \dots + p_{-L}}{p_R} \right).$$

We begin with a lemma showing how the vectors $V_k(a, b, \zeta)$, with ζ as above, and the matrix M naturally appear.

Lemma 2.5. *Let $a < b$ be integers and let $\zeta \in \{a - l \mid 0 \leq l \leq L - 1\} \cup \{b + r \mid 0 \leq r \leq R - 1\} \cup \{\pm\}$. Then for any k such that $a < k < b$, one has:*

$$V_k(a, b, \zeta) = M(k)V_{k-1}(a, b, \zeta). \tag{4}$$

Proof. Fixing ζ as indicated in the statement of the lemma, we simplify $P_k\{a, b, \zeta\}$ into $f(k)$. Let now $a < k < b$. Using the Markov property, we get:

$$f(k) = \sum_{l=-L}^R p_l(k) f(k+l). \tag{5}$$

In factor of the left member $f(k)$, we write $1 = \sum_{l=-L}^R p_l(k)$. Equality (5) becomes:

$$\sum_{l=1}^R p_l(k)(f(k) - f(k+l)) = \sum_{l=1}^L p_{-l}(k)(f(k-l) - f(k)). \tag{6}$$

Introducing the successive differences of the function f , that is setting $g(k) = f(k) - f(k+1)$, from relation (6) we obtain:

$$\sum_{l=0}^{R-1} g(k+l)(p_{l+1}(k) + \dots + p_R(k)) = \sum_{l=1}^L g(k-l)(p_{-l}(k) + \dots + p_{-L}(k)), \tag{7}$$

which can be rewritten as

$$g(k+R-1) = - \sum_{l=0}^{R-2} g(k+l) \left(\frac{p_{l+1}(k) + \dots + p_R(k)}{p_R(k)} \right) + \sum_{l=1}^L g(k-l) \left(\frac{p_{-l}(k) + \dots + p_{-L}(k)}{p_R(k)} \right).$$

Using M and the definition of $V_k(a, b, \zeta)$, the previous relation is finally equivalent to

$$V_k(a, b, \zeta) = M(k)V_{k-1}(a, b, \zeta). \quad \square$$

The previous lemma indicates that the matrix M makes all the gradient-vectors $V_k(a, b, \zeta)$ “circulate” on the \mathbb{Z} -axis. Let us now study the linear dependence between those vectors.

Lemma 2.6. *Let $a < k < b$ be integers. Define the subspaces $E = \text{Vect}(V_k(a, b, \zeta) \mid \zeta \in \{a-l \mid l=0, \dots, L-1\})$ and $F = \text{Vect}(V_k(a, b, \zeta) \mid \zeta \in \{b+r \mid r=0, \dots, R-1\})$. Then:*

- (i) $E + F = \mathbb{R}^d$.
- (ii) $E \cap F = \mathbb{R}V_k(a, b, +)$ and $V_k(a, b, +) = -V_k(a, b, -)$ is a non-zero vector.

Proof. Let $(\alpha_i)_{1 \leq i \leq L}$ and $(\beta_i)_{1 \leq i \leq R}$ be real numbers such that

$$\alpha_L V_k(a, b, a-L+1) + \dots + \alpha_1 V_k(a, b, a) + \beta_1 V_k(a, b, b) + \dots + \beta_R V_k(a, b, b+R-1) = 0. \tag{8}$$

Applying to (8) on the one hand the matrices $M(b-1) \dots M(k+1)$ and on the other hand the matrices $(M(k) \dots M(a+1))^{-1}$ we get

$$\begin{cases} \alpha_L V_{b-1}(a, b, a-L+1) + \dots + \alpha_1 V_{b-1}(a, b, a) + \beta_1 V_{b-1}(a, b, b) + \dots \\ \quad + \beta_R V_{b-1}(a, b, b+R-1) = 0, \\ \alpha_L V_a(a, b, a-L+1) + \dots + \alpha_1 V_a(a, b, a) + \beta_1 V_a(a, b, b) + \dots + \beta_R V_a(a, b, b+R-1) = 0. \end{cases}$$

Let us consider for example the second equality. Projecting this relation orthogonally on the subspace $\text{Vect}(e_i \mid R+1 \leq i \leq d)$ and setting $e_{d+1} = 0$ this gives

$$\sum_{r=0}^{L-2} \alpha_{L-r}(e_{d-r} - e_{d-r+1}) - \alpha_1 e_{R+1} = 0.$$

We thus obtain $\alpha_L = \dots = \alpha_1 =: \alpha_0$. Similarly, we would get $\beta_R = \dots = \beta_1 =: \beta_0$. Back to (8) we arrive at

$$0 = \alpha_0 V_k(a, b, -) + \beta_0 V_k(a, b, +) = (\alpha_0 - \beta_0) V_k(a, b, -).$$

Finally remark that $V_k(a, b, -) \neq 0$, otherwise the function $k \mapsto P_k\{a, b, -\}$ would be constant on $[a, b]$ and it is equal to 1 in a and to 0 in b . Therefore $\alpha_0 = \beta_0$ and the result follows. \square

3. Invariant cones

The following study reveals that the comprehension of the model essentially relies on the analysis of the behaviour of $V_k(a, b, +)$, especially in direction, as a and b tend to infinity.

In order to study the central vector $V_k(a, b, +)$, we focus on the external powers of \mathbb{R}^d of order R and L . These spaces are respectively denoted by $\wedge_R \mathbb{R}^d$ and $\wedge_L \mathbb{R}^d$ and are equipped with their usual Euclidian structure inherited from \mathbb{R}^d . Our aim is study the following R -decomposable and L -decomposable vectors

$$V_k(a, b, b + R - 1) \wedge \dots \wedge V_k(a, b, b) \in \wedge_R \mathbb{R}^d \tag{9}$$

and

$$V_k(a, b, a) \wedge \dots \wedge V_k(a, b, a - L + 1) \in \wedge_L \mathbb{R}^d, \tag{10}$$

obtained with the $V_k(a, b, \zeta)$ when ζ first varies among the possible right exit points of $[a, b]$ and then among the left exit points of that interval. Recall that Lemma 2.6 implies that the intersection of the subspaces of \mathbb{R}^d corresponding to (9) and to (10) is spanned by $V_k(a, b, +)$.

Considering for instance the right exit points, the idea is that the harmonic functions $k \mapsto P_k\{a, b, \zeta\}$ and the corresponding vectors $V_k(a, b, \zeta)$, for $\zeta \in \{b + r \mid 0 \leq r \leq R - 1\}$, have a universal algebraic structure.

This way we prove the existence of invariant cones in $\wedge_R \mathbb{R}^d$ for matrices having the same form as $(-1)^{R-1} \wedge_R M$. Notice that the signature $(-1)^{R-1}$ of a cycle of length R naturally appears.

3.1. A few definitions

We first recall some definitions about cones. One may consult Berman and Plemmons [2].

Definition 3.1. A cone \mathcal{C} in \mathbb{R}^n , $n \geq 1$, is a subset stable by non-negative linear combinations. A cone is:

- “polyhedral” if it is generated by finitely many vectors.
- “solid” if it has non-empty interior.

The set of matrices of dimensions $n \times n$ preserving a cone $\mathcal{C} \subset \mathbb{R}^n$ is written $\Pi(\mathcal{C})$. If $A \in \Pi(\mathcal{C})$ then A is “ \mathcal{C} -positive” if $A(\mathcal{C} - \{0\})$ is contained in the interior of \mathcal{C} .

We now introduce a cone in $\wedge_R \mathbb{R}^d$ that will be a central tool in our study.

Definition 3.2. Let $(e_i)_{1 \leq i \leq d}$ be the canonical basis of \mathbb{R}^d . For $i \leq j$, we set $\Sigma_i^j = e_i + \dots + e_j$. Let $\Phi = \{\varphi\}$ be the set of vectors in $\wedge_R \mathbb{R}^d$ of the form

$$\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{1+k_2} \wedge \dots \wedge \Sigma_R^{R+k_R},$$

with $0 \leq k_j \leq L - 1$ for $1 \leq j \leq R$ and $i + k_i \neq j + k_j$ if $i \neq j$.

We will see later that the previous conditions in the above definition ensure that the elements of Φ are distinct and even non proportional.

Definition 3.3. Let \mathcal{C} be the polyhedral cone in $\wedge_R \mathbb{R}^d$ generated by the elements of Φ in $\wedge_R \mathbb{R}^d$:

$$\mathcal{C} = \left\{ \sum_{\varphi \in \Phi} \alpha_\varphi \varphi \mid \alpha_\varphi \geq 0 \right\}.$$

The dual cone \mathcal{C}^* of \mathcal{C} is

$$\mathcal{C}^* = \{x \in \wedge_R \mathbb{R}^d \mid \langle x, \varphi \rangle \geq 0, \forall \varphi \in \Phi\}.$$

Standard arguments imply that \mathcal{C}^* is also polyhedral and that $\mathcal{C}^{**} = \mathcal{C}$. We now introduce the set \mathcal{M} containing the matrices having the same form as the matrix M given in (3).

Definition 3.4. Let $\mathcal{M} = \{M = M(\varepsilon_1, \dots, \varepsilon_{R-1}, \delta_1, \dots, \delta_L) \mid \varepsilon_i \geq 0, \delta_i \geq 0\}$, where

$$M = \begin{pmatrix} -a_1 & \dots & -a_{R-1} & b_L & \dots & b_1 \\ 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \end{pmatrix}, \quad \text{with } a_i = 1 + \varepsilon_1 + \dots + \varepsilon_i \text{ and } b_i = \delta_1 + \dots + \delta_i.$$

In the next paragraph we relate \mathcal{M} to the cone \mathcal{C} .

3.2. Properties of \mathcal{C} , \mathcal{C}^* and \mathcal{M}

Proposition 3.5. Let $M \in \mathcal{M}$. Then $(-1)^{R-1} \wedge_R {}^t M \in \Pi(\mathcal{C})$. Therefore $(-1)^{R-1} \wedge_R M \in \Pi(\mathcal{C}^*)$.

Proof. We set $u_i = -\Sigma_i^{R-1}$ for $1 \leq i \leq R-1$ and $v_j = \Sigma_R^{R+L-j}$ for $1 \leq j \leq L$. Let $M \in \mathcal{M}$. The first column vector of $({}^t M)$ can be decomposed in the following way:

$${}^t(-a_1, \dots, -a_{R-1}, b_L, \dots, b_1) = u_1 + \sum_{i=1}^{R-1} \varepsilon_i u_i + \sum_{j=1}^L \delta_j v_j. \tag{11}$$

Let then $\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \dots \wedge \Sigma_R^{R+k_R}$ belong to Φ , where $0 \leq k_j \leq L-1$ for $1 \leq j \leq R$ and $i+k_i \neq j+k_j$ if $i \neq j$. We get:

$$\begin{aligned} \wedge_R {}^t M \varphi &= \left(u_1 + \sum_{i=1}^{R-1} \varepsilon_i u_i + \sum_{j=1}^L \delta_j v_j + \Sigma_1^{k_1} \right) \wedge \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} \\ &= (-1)^{R-1} \sum_{j=1}^L \delta_j \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{R+L-j} \\ &\quad + \sum_{i=1}^{R-1} \varepsilon_i [-\Sigma_i^{R-1}] \wedge \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} + K, \end{aligned} \tag{12}$$

with

$$K = \begin{cases} -\Sigma_{1+k_1}^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R}, & \text{if } k_1 + 1 \leq R - 1, \\ (-1)^{R-1} \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{k_1}, & \text{if } k_1 + 1 > R - 1. \end{cases}$$

To prove the proposition, we now show that a term $S := -\Sigma_i^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R}$, with $i \leq R - 1$, belongs to $(-1)^{R-1} \Phi$. We consider two cases:

– If $i > 1$, then $S = (-1)^i \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{i-1}^{i-1+k_i} \wedge A$, where

$$A = \Sigma_i^{R-1} \wedge \Sigma_i^{i+k_{i+1}} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} = (-1)^{R-i-1} \Sigma_i^{i+k'_i} \wedge \dots \wedge \Sigma_R^{R+k'_R},$$

proceeding inductively on the number of terms. We therefore obtain the result.

– If $i = 1$. In S and if $R - 1 \geq 1 + k_2$, we add the second vector to the first one and we get $S = -\Sigma_{k_2+2}^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R}$. We are then back to the previous case. If $R - 1 < 1 + k_2$, then S is equal to

$$S = \Sigma_1^{R-1} \wedge (-\Sigma_R^{1+k_2}) \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} = (-1)^{R-1} \Sigma_1^{R-1} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{1+k_2},$$

which finishes the proof. \square

We now detail some geometrical properties of the cones \mathcal{C} and \mathcal{C}^* and also further links with the class of matrices \mathcal{M} .

Definition 3.6. We write $e = e_1 \wedge \dots \wedge e_R$ for the first vector of the canonical basis of $\wedge_R \mathbb{R}^d$.

We have the following result.

Proposition 3.7.

- (i) *The cone \mathcal{C} is solid. The cone \mathcal{C}^* is also solid, contains a neighborhood of e and for all $\varphi \in \Phi$ we have $\langle \varphi, e \rangle = 1$.*
- (ii) *The set of extremal vectors of \mathcal{C} is Φ and two elements of Φ are not proportional.*
- (iii) *The cone \mathcal{C} is minimal (excluding the degenerated cone $\{0\}$) with respect to the stability by the class $(-1)^{R-1} \wedge_R^t \mathcal{M}$. Moreover, any solid cone stable by $(-1)^{R-1} \wedge_R^t \mathcal{M}$ contains either \mathcal{C} or $-\mathcal{C}$.*
- (iv) *Let M_1, \dots, M_R be in the interior of \mathcal{M} . Then $A := (-1)^{(R-1)R} \wedge_R^t (M_1 \dots M_R) = \wedge_R^t (M_1 \dots M_R)$ is a \mathcal{C} -positive matrix.*

Proof. (i) Consider integers $1 \leq i_1 < \dots < i_R \leq d$. If $u \in \wedge_R \mathbb{R}^d$ is orthogonal to all the elements of Φ , it is in particular orthogonal to:

$$\Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \dots \wedge \Sigma_R^{i_R} \quad \text{and to} \quad \Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \dots \wedge \Sigma_R^{i_R-1}.$$

Subtracting, it is orthogonal to $\Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \dots \wedge e_{i_R}$. Then inductively, u is orthogonal to $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_R}$ and therefore $u = 0$. The statement concerning \mathcal{C}^* is direct.

(ii) We first show that two vectors of Φ are not proportional. From the fact that for all $\varphi \in \Phi$, $\langle \varphi, e \rangle = 1$, if two vectors φ_1 and φ_2 in Φ are proportional, then $\varphi_1 = \varphi_2$. This provides an equality of the form

$$\Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \dots \wedge \Sigma_R^{R+k_R} = \Sigma_1^{1+k'_1} \wedge \Sigma_2^{2+k'_2} \wedge \dots \wedge \Sigma_R^{R+k'_R}, \tag{13}$$

with $(i + k_i)_{1 \leq i \leq R} \neq (i + k'_i)_{1 \leq i \leq R}$. Let then $t \geq 1$ be the greatest index such that $k_t \neq k'_t$. If $t > 1$, we make the matrix $(-1)^{R-1} \wedge_R^t M(0, \dots, 0, \delta_1, 0, \dots, 0)$ act on the two members of (13). Using equality (12) and letting δ_1 tend to $+\infty$, we obtain that the limit in direction is

$$\Sigma_1^{1+k_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^d = \Sigma_1^{1+k'_2} \wedge \dots \wedge \Sigma_{R-1}^{R-1+k'_R} \wedge \Sigma_R^d.$$

Repeating this operation, there exists $(r_i)_{2 \leq i \leq R}$ with $0 \leq r_i \leq L - 1$ such that we have the following equality:

$$\Sigma_1^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_R^{R+r_R} = \Sigma_1^{1+k'} \wedge \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_R^{R+r_R},$$

with $k \neq k'$. Let us suppose that $k > k'$. Then:

$$0 = \Sigma_{k'+2}^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_R^{R+r_R}$$

and also:

$$0 = \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_{k'+2}^{k'+2+r_{k'+2}} \wedge \Sigma_{k'+2}^{1+k} \wedge \dots \wedge \Sigma_R^{R+r_R}.$$

We consider now the two central terms. Comparing $k' + 2 + r_{k'+2}$ to $1 + k$, we subtract the “shortest” term to the “longest” one and then shift the result to the right. One never gets 0 when subtracting, since all the endings are distinct. Finally, we arrive to a relation of the form:

$$0 = \Sigma_2^{2+r'_2} \wedge \dots \wedge \Sigma_R^{R+r'_R} \wedge \Sigma_s^{s+t_s},$$

with $s > R$ and $t_s \geq 0$, which is impossible.

Suppose now that there exists $\varphi \in \Phi$ such that $\varphi = \sum_{\psi \in \Psi} \lambda_\psi \psi$, with $\lambda_\psi > 0$, for a certain subset $\Psi \subset \Phi$. Taking the scalar product with e , we obtain $\sum_{\psi \in \Psi} \lambda_\psi = 1$. Therefore we can suppose that $\varphi \notin \Psi$. Take then the largest t such that the vector with number t is not the same for all the decomposable R -vectors φ and $\psi \in \Psi$.

Applying sufficiently many times $(-1)^{R-1} \wedge_R^t M(0, \dots, 0, \delta_1, 0, \dots, 0)$ and letting δ_1 tend to $+\infty$ as above, we are reduced to the following situation, with some $(r_j)_{2 \leq j \leq R}$ with $0 \leq r_j \leq L - 1$:

$$\Sigma_1^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_R^{R+r_R} = \sum_{\psi \in \Psi} \lambda_\psi \Sigma_1^{1+k_\psi} \wedge \Sigma_2^{2+r_2} \wedge \dots \wedge \Sigma_R^{R+r_R}, \tag{14}$$

where $\text{card}\{k, k_\psi \mid \psi \in \Psi\} \geq 2$. Using again the fact that $\sum \lambda_\psi = 1$, we consider the situation where k is distinct from all the k_ψ . Write now

$$\Sigma_1^{1+k} - \sum_{\psi \in \Psi} \lambda_\psi \Sigma_1^{1+k_\psi} = \sum_{j=1}^J \lambda_j B_j, \quad J \geq 1, \quad B_j = \Sigma_{t_{j-1}+1}^{t_j} \text{ and } \lambda_j \neq 0, \quad \forall 1 \leq j \leq J.$$

Note then that any t_k belongs to the set of $\{1 + k, 1 + k_\psi \mid \psi \in \Psi\}$ and is thus distinct from all the $(j + r_j)_{2 \leq j \leq R}$. Subtracting the right member of (14) to the left one, we use the previous equality. Repeating the above procedure of successive subtractions and shifts to the right, we get a R -vector with some last vector which is not zero and beginning with some e_s , $s > R$. This gives the result.

(iii) If a subcone $\mathcal{C}_1 \subset \mathcal{C}$ is $(-1)^{R-1} \wedge_R^t \mathcal{M}$ -stable, we show that $\Phi \subset \mathcal{C}_1$. Fix

$$\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \dots \wedge \Sigma_R^{R+k_R} \in \Phi.$$

Take $x \in \mathcal{C}_1$ and write it in the form $x = \sum_{\psi \in \Psi} \delta_\psi \psi$ with $\delta_\psi > 0$ for $\psi \in \Psi \subset \Phi$. We then apply successively to this equality the matrices

$$(-1)^{R-1} \wedge_R^t M(0, \dots, 0, \dots, 0, \delta_{L-k_j}, 0, \dots, 0), \quad \text{for } j = 1, \dots, j = R.$$

Letting then each δ_{L-k_j} tend to $+\infty$, the limit direction is φ . Similarly, consider now a solid cone \mathcal{C}_1 stable by $(-1)^{R-1} \wedge_R^t \mathcal{M}$. Let x be a point interior to \mathcal{C}_1 . As Φ generates the whole space, one can write, in a non-necessarily

unique way, $x = \sum_{\varphi \in \Phi} \alpha_\varphi \varphi$. We suppose that $c := \sum_{\varphi \in \Phi} \alpha_\varphi \neq 0$, up to perturbing by adding an element of \mathcal{C} . The previous argument works and we obtain that $c\varphi$ belongs to \mathcal{C}_1 , for all $\varphi \in \Phi$.

(iv) From (12), the image by A of any non-zero vector of \mathcal{C} is of the form $\sum_{\varphi \in \Phi} \alpha_\varphi \varphi$, with $\alpha_\varphi > 0$ for all φ . We conclude by using the fact that Φ generates the whole space. \square

Remark 1. Let us consider the question of the cardinality of Φ . When $L \geq R$, elementary calculations furnish the following formula:

$$\text{card } \Phi = \sum_{t=0}^R \sum_{\substack{1 \leq i_1 < \dots < i_t \leq R \\ i_0=0}} \left[\prod_{j=1}^t (R - i_j - (t - j))(L - R + j)^{i_j - i_{j-1} - 1} \right] (L - R + t + 1)^{R-t}.$$

Recall now from [5] that for $L = R = 2$ the matrix M is deterministically conjugated to a non-positive matrix. In the general case, if R is fixed and $L \rightarrow +\infty$, the previous formula gives $\text{card } \Phi \sim L^R$ whereas $\dim(\wedge_R \mathbb{R}^d) \sim L^R/R!$.

Consequently if one uses only the cone \mathcal{C} , there is no deterministic change of basis such that the class of the matrices $(-1)^{R-1} \wedge_R^t M$ becomes a class of matrices with non-negative coefficients. If $R > L$ there is a heavy formula for $\text{card}(\Phi)$, involving the Euclidian division of R by L .

Remark 2. If $L \leq 2$, then $\mathcal{C} \subset \mathcal{C}^*$. Indeed if $L = 1$, then $\mathcal{C} = \mathcal{C}^* = \{e\}$. If $L = 2$, then two elements φ_1 and φ_2 in Φ can be written in the form:

$$\begin{cases} \varphi_1 = e_1 \wedge \dots \wedge e_t \wedge (e_{t+1} + e_{t+2}) \wedge \dots \wedge (e_R + e_{R+1}), \\ \varphi_2 = e_1 \wedge \dots \wedge e_s \wedge (e_{s+1} + e_{s+2}) \wedge \dots \wedge (e_R + e_{R+1}). \end{cases}$$

One then checks that $\langle \varphi_1, \varphi_2 \rangle = R - (s \wedge t)$. The previous inclusion $\mathcal{C} \subset \mathcal{C}^*$ is not true for larger values of L . Indeed, if $L = 4$ and $R = 2$, we get $\langle \varphi_1, \varphi_2 \rangle = -1$ when taking:

$$\varphi_1 = (e_1 + e_2) \wedge (e_2 + e_3 + e_4) \quad \text{and} \quad \varphi_2 = (e_1 + e_2 + e_3 + e_4) \wedge (e_2).$$

4. Lyapunov spectrum and simplicity

We now introduce the Lyapunov exponents $\gamma_1(M, T) \geq \dots \geq \gamma_d(M, T)$ of the matrix M with respect to the dynamical system $(\Omega, \mathcal{F}, T, \mu)$. For a detailed presentation of Lyapunov exponents, one may consult Ledrappier [12] or Raugi [16].

4.1. Definitions and preliminary study

Definition 4.1. The Lyapunov exponents $\gamma_1(M, T) \geq \dots \geq \gamma_d(M, T)$ of M with respect to $(\Omega, \mathcal{F}, T, \mu)$ are recursively defined by the equalities:

$$\gamma_1(M, T) + \dots + \gamma_i(M, T) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\wedge_i(T^{n-1}M \dots TMM)\|, \quad \mu\text{-ae, for all } 1 \leq i \leq d.$$

The existence of Lyapunov exponents follows from the sub-additive Ergodic Theorem of Kingman. In our context these exponents are finite since the maps $\log \|M\|$ and $\log \|M^{-1}\|$ are bounded.

We also need some Oseledet’s vectors, namely some kind of eigenvectors associated to the Lyapunov spectrum.

Definition 4.2. Let $(V_i)_{1 \leq i \leq d}$ be a measurable family of vectors such that $\|V_i\| = 1$ and satisfying

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|(T^{n-1} M \cdots T M M) V_i\| = \gamma_i(M, T),$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|(T^{-n} M^{-1} \cdots T^{-2} M^{-1} T^{-1} M^{-1}) V_i\| = -\gamma_i(M, T).$$

For the existence of such vectors we refer for example to Ledrappier [12]. It is essentially a corollary of Oseledet’s Theorem [15]. Let us also recall that if an exponent $\gamma_i(M, T)$ is simple, that is if $\gamma_{i-1}(M, T) > \gamma_i(M, T) > \gamma_{i+1}(M, T)$, then the corresponding Oseledet’s vector V_i is uniquely determined in direction.

We will see in the sequel that the asymptotic properties of the model depend on the sign of the central exponent $\gamma_R(M, T)$. For the moment the next proposition shows that the other exponents of M with respect to T have fixed signs. The proof relies on ideas of Key [10] and [6].

Proposition 4.3. *The following inequalities hold:*

$$\gamma_{R-1}(M, T) > 0 > \gamma_{R+1}(M, T).$$

Proof. Taking $\zeta \in \{-1, \dots, -L\}$, we first remark that for all $k \geq 0$

$$V_k(-1, +\infty, \zeta) = T^k M \cdots M V_{-1}(-1, +\infty, \zeta).$$

We then notice that the $V_{-1}(-1, +\infty, \zeta)$ with $\zeta \in \{-1, \dots, -L\}$ span a subspace of \mathbb{R}^d of dimension at least $L - 1$. As the $V_k(-1, +\infty, \zeta)$ are bounded, the exponent of $V_{-1}(-1, +\infty, \zeta)$ with respect to M and T is ≤ 0 . Thus $\gamma_{R+1}(M, T) \leq 0$ and similarly $\gamma_{R-1}(M, T) \geq 0$.

To prove that inequalities are strict, introduce $A(r) = \text{diag}(1, r, \dots, r^{d-1})$, for some real r . Writing $M = M(a_1, \dots, a_{R-1}, b_L, \dots, b_1)$, we observe that:

$$A(r) M A(r)^{-1} = r M(a'_1, \dots, a'_{R-1}, b'_L, \dots, b'_1), \quad \text{with } a'_i = \frac{a_i}{r^i}, \quad b'_i = \frac{b_i}{r^{R+L-i}}.$$

Due to the minoration condition (1) on the transition probabilities, for r close to 1 the matrix $M(r) := M(a'_1, \dots, a'_{R-1}, b'_L, \dots, b'_1)$ is also the matrix associated to a random walk. Then $\gamma_{R+1}(M(r), T) \leq 0$ and $\gamma_{R-1}(M(r), T) \geq 0$. As the exponents of M are then deduced by translation of $\log r$ from those of $M(r)$, this concludes the proof of the proposition. \square

4.2. Simplicity of the central exponent

Theorem 4.4. *The exponent $\gamma_R(M, T)$ is simple.*

Corollary 4.5. *There exists a random vector V_R , with $\|V_R\|_1 = 1$, uniquely determined in direction, and a random scalar λ_R such that $\log |\lambda_R|$ is bounded, verifying:*

$$M V_R = \lambda_R T V_R \quad \text{and} \quad \int \log |\lambda_R| d\mu = \gamma_R(M, T).$$

The proof of Theorem 4.4 is a consequence of the following result on the simplicity of the dominant Lyapunov exponent for random matrices which are positive with respect to a solid and polyhedral cone.

Theorem 4.6. *In \mathbb{R}^n , $n \geq 1$, let \mathcal{C} be a solid and polyhedral cone such that the dual cone \mathcal{C}^* is also solid and polyhedral. Let $A \in GL_n(\mathbb{R})$ be a random matrix with respect to $(\Omega, \mathcal{F}, \mu, T)$ such that $A \in \Pi(\mathcal{C})$ and the random variables $\log \|A\|$ and $\log \|A^{-1}\|$ are bounded. Assume that*

$$\mu\{\exists n \geq 0, (T^{n-1} A \cdots T A A) \text{ is } \mathcal{C}\text{-positive}\} > 0.$$

Then:

- (i) *The dominant exponent $\gamma_1(A, T)$ is simple.*
- (ii) *There exists a vector $V \in \mathcal{C}$ satisfying $AV = \lambda_V TV$ for some random scalar λ_V such that $\log |\lambda_V|$ is bounded and $\int \log |\lambda_V| d\mu = \gamma_1(A, T)$. If there exists $W \in \mathcal{C}$ and a random scalar λ_W such that $AW = \lambda_W TW$, then V and W have the same direction.*
- (iii) *All non-zero vectors in \mathcal{C} have maximal exponent with respect to A and T .*

Proof of Theorem 4.4. Introduce the matrix $A := (-1)^{R-1} \wedge_R M$ on \mathbb{R}^d . From Proposition (3.7), $A \in \Pi(\mathcal{C})$, where \mathcal{C} is the polyhedral and solid cone defined in Definition 3.3. Recall that \mathcal{C}^* has the same properties. Also $T^{-R+1}({}^t A) \cdots T^{-1}({}^t A)({}^t A)$ is \mathcal{C} -positive. As

$$\gamma_1({}^t A, T^{-1}) = \gamma_1(A, T) = \gamma_1(\wedge_R M, T),$$

we deduce from Theorem 4.6, that $\gamma_1(\wedge_R M, T)$ is simple. Since

$$\gamma_1(\wedge_R M, T) = \sum_{i=1}^R \gamma_i(M, T) \quad \text{and} \quad \gamma_2(\wedge_R M, T) = \sum_{i=1}^{R-1} \gamma_i(M, T) + \gamma_{R+1}(M, T),$$

we obtain $\gamma_R(M, T) > \gamma_{R+1}(M, T)$. A symmetric study involving $(-1)^{L-1} \wedge_L M^{-1}$ with respect to T^{-1} gives $\gamma_{R-1}(M, T) > \gamma_R(M, T)$. This concludes the proof of the theorem. \square

4.3. Proof of Theorem 4.6

We adapt a proof of Hennion [9] on the simplicity of the dominant Lyapunov exponent for non-negative random matrices having positive iterates. For the sake of completeness, we include the details.

In a preliminary study we introduce a norm on \mathbb{R}^n and a distance on the corresponding unit ball intersected with the cone \mathcal{C} and we detail their properties.

Definition 4.7. Let Φ be the set of extremal vectors of \mathcal{C} and Ψ be the set of extremal vectors of \mathcal{C}^* . Assume that all these vectors have a norm equal to 1 with respect to the usual Euclidian norm on \mathbb{R}^n .

Definition 4.8. Denote by $\|\cdot\|_\Psi$ the following norm on \mathbb{R}^n :

$$\|x\|_\Psi = \sum_{\psi \in \Psi} |\langle x, \psi \rangle|.$$

Definition 4.9. Set $\bar{B} = \mathcal{C} \cap \{x \mid \|x\|_\Psi = 1\}$. If x and y are in \bar{B} , we define

$$m(x, y) = \sup\{s \geq 0 \mid s \langle y, \psi \rangle \leq \langle x, \psi \rangle, \forall \psi \in \Psi\} = \min_{\psi \in \Psi} \left\{ \frac{\langle x, \psi \rangle}{\langle y, \psi \rangle} \mid \langle y, \psi \rangle > 0 \right\}.$$

As $\|x\|_\Psi = \|y\|_\Psi = 1$ notice that $0 \leq m(x, y) \leq 1$. We finally set:

$$d(x, y) = \theta(m(x, y)m(y, x)), \quad \text{with } \theta(s) = \frac{1-s}{1+s}, \quad s \in [0, 1].$$

The next lemma indicates that d is a distance on \bar{B} for which a matrix in $\Pi(\mathcal{C})$ acts as a contraction. This distance is related to Hilbert's distance d_H on a cone by $d = \tanh(d_H/2)$, but it is bounded.

Lemma 4.10.

- (i) *If x, y and z are in \bar{B} , then $m(x, z)m(z, y) \leq m(x, y)$.*

If x and y are in \bar{B} , then $m(x, y)m(y, x) = 1$ if and only if $x = y$.

For x and y in \bar{B} , we have $m(x, y) = 0$ if and only if $\exists \psi \in \Psi$ such that $\langle x, \psi \rangle = 0$ and $\langle y, \psi \rangle > 0$.

(ii) The map d is a distance on \bar{B} .

Proof. Point (i) follows from the previous definitions. Concerning (ii), we first notice that $\theta'(s) = -2/(1 + s)^2$ and then θ is non-increasing on $[0, 1]$. The map $F(s) = \theta(s) + \theta(t) - \theta(st)$ verifies $F(1) = 0$ and

$$F'(s) = -\frac{2(1-t)}{(1+s)^2(1+st)^2}(1-s^2t).$$

Thus for s, t in $[0, 1]$ we have $\theta(st) \leq \theta(s) + \theta(t)$. We then use point (i). \square

The next lemma furnishes a convenient expression for $d(x, y)$, where x and y are distinct points in \bar{B} , in terms of the extremal points of the maximal segment in \bar{B} passing at x and y .

Lemma 4.11. Taking x and y in \bar{B} with $x \neq y$, set:

$$a = (1 - \lambda_1)x + \lambda_1y, \quad \lambda_1 = \inf\{\lambda \mid (1 - \lambda)x + \lambda y \in \bar{B}\},$$

$$b = (1 - \lambda_2)x + \lambda_2y, \quad \lambda_2 = \sup\{\lambda \mid (1 - \lambda)x + \lambda y \in \bar{B}\}.$$

Writing $x = u_1a + u_2b$ and $y = v_1a + v_2b$, we then have:

$$d(x, y) = \frac{|u_1v_2 - u_2v_1|}{u_1v_2 + u_2v_1}. \tag{15}$$

Proof. Observe first that λ_1 and λ_2 are finite quantities since there exists extremal vectors ψ_1 and ψ_2 such that $\langle x, \psi_1 \rangle > \langle y, \psi_1 \rangle$ and $\langle x, \psi_2 \rangle < \langle y, \psi_2 \rangle$.

Introduce $I = \{\psi \mid \langle a, \psi \rangle > 0\}$ and $J = \{\psi \mid \langle b, \psi \rangle > 0\}$. There is no inclusion between I and J and in particular none is equal to Ψ , otherwise for example there would exist $\varepsilon > 0$ such that $(1 + \varepsilon)a - \varepsilon b \in \bar{B}$, contradicting the definition of a .

Let us check the result if $x = a$. We then have $m(x, y) = 0$ and $d(x, y) = 1$ since $a \neq y$ and this corresponds to the announced formula with $u_1 = 1, u_2 = 0$ and $v_2 \neq 0$. The case $y = b$ is symmetric. Suppose then that $x \neq a, y \neq b$ and set $r = \min\{u_i/v_i \mid i = 1, 2\}$. We have:

$$r\langle y, \psi \rangle = rv_1\langle a, \psi \rangle + rv_2\langle b, \psi \rangle \leq u_1\langle a, \psi \rangle + u_2\langle b, \psi \rangle \leq \langle x, \psi \rangle.$$

Thus $m(x, y) \geq r$. For $\psi \in \Psi$, we also have the inequality:

$$m(x, y)(v_1\langle a, \psi \rangle + v_2\langle b, \psi \rangle) \leq u_1\langle a, \psi \rangle + u_2\langle b, \psi \rangle.$$

Taking $\psi_1 \in J/I$ and $\psi_2 \in I/J$, we obtain $m(x, y) \leq u_1/v_1$ and $\leq u_2/v_2$. Finally $m(x, y) = r$. As $x \neq y$, we have $m(x, y)m(y, x) < 1$ and then

$$m(x, y)m(y, x) = \min\left\{\frac{u_1v_2}{v_1u_2}, \frac{u_2v_1}{v_2u_1}\right\},$$

which provides the formula. \square

We now compare d to the distance induced by the norm $\|\cdot\|_\psi$.

Lemma 4.12.

(i) For x and y in \bar{B} : $d(x, y) \geq \frac{1}{2}\|x - y\|_\psi$.

(ii) Denote by d_1 the distance induced by $\|\cdot\|_\psi$. Then the spaces $(\text{int}(B), d)$ and $(\text{int}(B), d_1)$ are homeomorphic.

Proof. (i) Using Lemma 4.11 and the fact that $u_1 + u_2 = v_1 + v_2 = 1$, we have:

$$|u_1v_2 - u_2v_1| = |u_1(1 - v_1) - (1 - u_1)v_1| = |u_1 - v_1|.$$

However:

$$\|x - y\|_\psi = \sum_{\psi \in \Psi} |\langle x - y, \psi \rangle| \leq |u_1 - v_1| \sum |\langle a, \psi \rangle| + |u_2 - v_2| \sum |\langle b, \psi \rangle| \leq 2|u_1 - v_1|.$$

The result follows from (15) and the bound

$$0 < u_1v_2 + u_2v_1 \leq (u_1^2 + u_2^2)^{1/2} (v_1^2 + v_2^2)^{1/2} \leq (u_1 + u_2)^{1/2} (v_1 + v_2)^{1/2} = 1.$$

(ii) If $d_1(x_n, x) \rightarrow 0$, as $n \rightarrow +\infty$, then

$$m(x, x_n) = \min_{\psi \in \Psi} \left\{ \frac{\langle x, \psi \rangle}{\langle x_n, \psi \rangle} \mid \langle x_n, \psi \rangle > 0 \right\} = \min_{\psi \in \Psi} \left\{ \frac{\langle x, \psi \rangle}{\langle x_n, \psi \rangle} \mid \langle x, \psi \rangle > 0 \right\},$$

for n large enough. Thus $d(x_n, x) \rightarrow 0$. \square

As mentioned in Hennion [9], the second point of Lemma 4.12 is false for (\bar{B}, d) and (\bar{B}, d_1) since (\bar{B}, d) is not connected. We now introduce cone-preserving matrices.

Lemma 4.13. Let $g \in GL_n(\mathbb{R})$ be in $\Pi(\mathcal{C})$. Define:

$$c(g) = \sup \{d(g.x, g.y) \mid x, y \in \bar{B}\},$$

where $g.x = gx/\|gx\|_\psi$. Then:

- (i) For all $x, y \in \bar{B}$: $d(g.x, g.y) \leq c(g)d(x, y)$.
- (ii) For g and g' in $GL_n(\mathbb{R}) \cap \Pi(\mathcal{C})$: $c(gg') \leq c(g)c(g')$.
- (iii) We have $c(g) \leq 1$. Also $c(g) < 1$ if and only if g is \mathcal{C} -positive.

Proof. Take points $x \neq y$ in \bar{B} . Then $g(x) \neq g(y)$. We denote by a, b and a_1, b_1 the extremal points in \bar{B} obtained as the intersections with the line passing by (x, y) and $(g.x, g.y)$. With respect to these bases, g has a matrix of the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{with } \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0.$$

Remark that $\alpha\delta + \beta\gamma > 0$ otherwise the above matrix would be a line matrix (whereas the images are distinct) or a column matrix (but then one of the images would be zero). Since $x = u_1a + u_2b$ and $y = v_1a + v_2b$, we have:

$$\begin{aligned} d(g.x, g.y) &= \frac{|(\alpha u_1 + \beta u_2)(\gamma v_1 + \delta v_2) - (\gamma u_1 + \delta u_2)(\alpha v_1 + \beta v_2)|}{(\alpha u_1 + \beta u_2)(\gamma v_1 + \delta v_2) + (\gamma u_1 + \delta u_2)(\alpha v_1 + \beta v_2)} \\ &= \frac{|\alpha\delta - \beta\gamma||u_1v_2 - u_2v_1|}{2\alpha\gamma u_1v_1 + (\alpha\delta + \beta\gamma)(u_1v_2 + u_2v_1) + 2\beta\delta u_2v_2} \\ &\leq \frac{|\alpha\delta - \beta\gamma| |u_1v_2 - u_2v_1|}{\alpha\delta + \beta\gamma |u_1v_2 + u_2v_1|} \\ &\leq d(g.a, g.b)d(x, y) \leq c(g)d(x, y). \end{aligned}$$

This proves (i). The second point is direct. Let us show (iii). If g is \mathcal{C} -positive, we have $g.\bar{B} \subset \text{int}(B)$, thus $g.\bar{B}$ is a compact of $(\text{int}(B), d_1)$ and then of $(\text{int}(B), d)$ by Lemma 4.12. Therefore, there exist x_0 and y_0 such that $c(g) = d(g.x_0, g.y_0) < 1$. The other sense follows from Lemma 4.10 since g is an open map. \square

Lemma 4.14. *Let $g \in GL_n(\mathbb{R})$ be \mathcal{C} -positive. Then:*

$$c(g) = \max_{(\varphi, \varphi') \in \Phi} d(g.\varphi, g.\varphi') = \max_{(\psi, \psi') \in \Psi, (\varphi, \varphi') \in \Phi} \frac{|\langle g\varphi', \psi \rangle \langle g\varphi, \psi' \rangle - \langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle|}{\langle g\varphi', \psi \rangle \langle g\varphi, \psi' \rangle + \langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle}. \tag{16}$$

Therefore $c(g) = c({}^t g)$.

Proof. Let x and y be in $g.\bar{B} \subset \text{int}(B)$. We write $x = \sum_{\varphi \in \Phi} \alpha_\varphi g.\varphi$, $y = \sum_{\varphi \in \Phi} \beta_\varphi g.\varphi$ with $\alpha_\varphi \geq 0$, $\beta_\varphi \geq 0$ and $\|\sum \alpha_\varphi \varphi\|_\Psi = \|\sum \beta_\varphi \varphi\|_\Psi = 1$. We have:

$$m(x, y)m(y, x) = \min_{\psi, \psi'} \frac{\langle x, \psi \rangle \langle y, \psi' \rangle}{\langle y, \psi \rangle \langle x, \psi' \rangle}.$$

However:

$$\frac{\langle x, \psi \rangle \langle y, \psi' \rangle}{\langle x, \psi' \rangle \langle y, \psi \rangle} \geq \min_{\varphi \in \Phi} \frac{\langle g\varphi, \psi \rangle}{\langle g\varphi, \psi' \rangle} \min_{\varphi' \in \Phi} \frac{\langle g\varphi', \psi' \rangle}{\langle g\varphi', \psi \rangle} = \min_{\varphi, \varphi'} \frac{\langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle}{\langle g\varphi, \psi' \rangle \langle g\varphi', \psi \rangle}.$$

We also have:

$$\min_{\psi, \psi'} \frac{\langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle}{\langle g\varphi, \psi' \rangle \langle g\varphi', \psi \rangle} = m(g.\varphi, g.\varphi')m(g.\varphi', g.\varphi).$$

This implies the first equality as θ is non-increasing. We therefore obtain:

$$c(g) = \max_{(\psi, \psi') \in \Psi, (\varphi, \varphi') \in \Phi} \frac{\langle g\varphi', \psi \rangle \langle g\varphi, \psi' \rangle - \langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle}{\langle g\varphi', \psi \rangle \langle g\varphi, \psi' \rangle + \langle g\varphi, \psi \rangle \langle g\varphi', \psi' \rangle},$$

which gives the announced formula. \square

We now consider $A \in \Pi(\mathcal{C})$ satisfying the hypotheses of Theorem 4.6. We introduce $\tau(\omega) = \inf\{n \geq 1 \mid (T^{n-1}A \cdots TAA)(\omega) \text{ is } \mathcal{C}\text{-positive}\}$. The assumption is that $\mu\{\tau < \infty\} > 0$.

Proposition 4.15.

- (i) We have $\mu\{\tau < \infty\} = 1$, $\int \tau d\mu < \infty$ and for $n \geq \tau(\omega)$, $(T^{n-1}A \cdots A)(\omega)$ is \mathcal{C} -positive.
- (ii) We call “contraction coefficient” the following number $0 \leq \kappa < 1$:

$$\log \kappa = \lim_{n \rightarrow +\infty} \frac{1}{n} \int \log c(T^{-1}A \cdots T^{-n}A) d\mu = \inf_{n \geq 1} \frac{1}{n} \int \log c(T^{-1}A \cdots T^{-n}A) d\mu < 0.$$

Moreover:

$$\lim_{n \rightarrow +\infty} c(T^{-1}A \cdots T^{-n}A)^{1/n} = \kappa, \mu\text{-ae}, \quad \text{and} \quad \lim_{n \rightarrow +\infty} c(T^{n-1}A \cdots TAA)^{1/n} = \kappa, \mu\text{-ae}.$$

- (iii) There exists a unique measurable $V \in \bar{B}$ such that $A.V = TV$ and V belongs to $\text{int}(B)$.
- (iv) The vector V has maximal exponent with respect to A and T .

Proof. (i) There exists $N \geq 0$ such that $\mu\{(T^{N-1}A \cdots TAA) \text{ is } \mathcal{C}\text{-positive}\} > 0$. Set then:

$$\tau'(\omega) = \inf\{n \geq 1 \mid (T^{N-1}A \cdots TAA)(T^n(\omega)) \text{ is } \mathcal{C}\text{-positive}\}.$$

We deduce from Kac’s lemma that $\mu\{\tau' < \infty\} = 1$ and $\int \tau' d\mu < \infty$. Now, if two invertible matrices g and g' are in $\Pi(\mathcal{C})$ and if g' is \mathcal{C} -positive, then $g'g$ and gg' are also \mathcal{C} -positive. Thus $\tau \leq \tau' + N$ and the result follows.

(ii) We have the following inequality for $n \geq 0$ and $m \geq 0$:

$$\log c(T^{-1}A \cdots T^{-n-m}A) \leq \log c(T^{-1}A \cdots T^{-m}A) + \log c(T^{-1}A \cdots T^{-n}A) \circ T^{-m}.$$

Each term in the above inequality is ≤ 0 . One then uses Kingman’s sub-additive Ergodic Theorem. We remark that $\kappa < 1$ since there exists $n \geq 1$ such that $\int \log c(T^{-1}A \cdots T^{-n}A) < 0$. The sequence $(\log c(T^{n-1}A \cdots TAA))_{n \geq 1}$ is also sub-additive and the last point follows from the invariance of the measure μ .

(iii) We have $c(T^{-1}A \cdots T^{-n}A) \rightarrow 0$, μ -ae. Since:

$$\text{diam}((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B})) \leq c((T^{-1}A \cdots T^{-n}A)(\omega)),$$

the sequence of compact sets $((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B}))_{n \geq 1}$ decreases to a unique element, μ -ae. We set $\{V(\omega)\} = \bigcap_{n \geq 1} (T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B})$. Then $A.V = TV$. Unicity of V also follows from the fact that $((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B}))_{n \geq 1}$ decreases to $V(\omega)$.

(iv) In order to prove that V has maximal exponent, it is enough to show that all vectors of $\text{int}(\mathcal{C})$ have the same exponent with respect to A and T , since $\text{int}(\mathcal{C})$ generates the whole space \mathbb{R}^n .

For $x \in \mathbb{R}^n$, set $\|x\|' = \langle x, x \rangle^{1/2}$. A remark for what follows is that if $x \in \text{int}(\mathcal{C})$, then $\langle x, \psi \rangle > 0$ for all $\psi \in \Psi$. For every $k \geq 0$, one has:

$$\|T^{k-1}A \cdots TAAx\|_\Psi \leq \|x\|' \sum_{\psi \in \Psi} \|({}^tA)({}^tTA) \cdots (T^{k-1}{}^tA)\psi\|'. \tag{17}$$

For $y \in \mathcal{C}^*$ and $\|y\|_\Psi = 1$, one can write $y = \sum_{\psi \in \Psi} \alpha_\psi \psi$ with $\alpha_\psi \geq 0$. Thus:

$$\langle x, y \rangle \geq \left(\sum_{\psi \in \Psi} \alpha_\psi \right) \min_{\psi \in \Psi} \{\langle x, \psi \rangle\} \quad \text{and} \quad 1 \leq \left(\sum_{\psi \in \Psi} \alpha_\psi \right) \max_{\psi \in \Psi} \{\|\psi\|_\Psi\}.$$

Consequently $\langle x, y \rangle \geq \min_{\psi \in \Psi} \{\langle x, \psi \rangle\} (\max_{\psi \in \Psi} \{\|\psi\|_\Psi\})^{-1}$ if $\|y\|_\Psi = 1$. Therefore, for $k \geq 0$:

$$\begin{aligned} \|T^{k-1}A \cdots TAAx\|_\Psi &= \sum_{\psi \in \Psi} \frac{|\langle x, ({}^tA)({}^tTA) \cdots (T^{k-1}{}^tA)\psi \rangle|}{\|({}^tA)({}^tTA) \cdots (T^{k-1}{}^tA)\psi\|_\Psi} \times \|({}^tA)({}^tTA) \cdots (T^{k-1}{}^tA)\psi\|_\Psi \\ &\geq \left(\frac{\min_{\psi \in \Psi} \{\langle x, \psi \rangle\}}{\max_{\psi \in \Psi} \{\|\psi\|_\Psi\}} \right) \sum_{\psi \in \Psi} \|({}^tA)({}^tTA) \cdots (T^{k-1}{}^tA)\psi\|_\Psi. \end{aligned} \tag{18}$$

The conclusion follows from inequalities (17) and (18) and the equivalence of norms on \mathbb{R}^n . \square

We now prove that $\gamma_1(A, T)$ is simple thanks to a control of the distance to $\gamma_2(A, T)$, using κ .

Proposition 4.16. *Let $0 \leq \kappa < 1$ be the contraction coefficient introduced in Proposition 4.15 and let $\gamma_2(A, T)$ be the second Lyapunov exponent of A with respect to T . Then*

$$\gamma_2(A, T) \leq \gamma_1(A, T) + \log \kappa.$$

Proof. Introduce $\wedge_2 \mathbb{R}^n$ equipped with the Euclidian structure inherited from \mathbb{R}^n . For $x \in \mathbb{R}^n$, set $\|x\|' = \langle x, x \rangle^{1/2}$. Let d_a be the angular distance on \mathbb{R}^n defined by

$$d_a(x, y) = \frac{\|x \wedge y\|'}{\|x\|' \|y\|'}.$$

Writing $A_n = T^{n-1}A \cdots TAA$ for $n \geq 0$, we first have

$$\gamma_1(A, T) + \gamma_2(A, T) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \| \wedge_2 A_n \|, \mu\text{-ae.}$$

For any $y \in \bar{B}$

$$1 = \sum_{\psi \in \Psi} |\langle y, \psi \rangle| \leq \left(\sum_{\psi \in \Psi} \|\psi\|' \right) \|y\|' = \text{card}(\Psi) \|y\|',$$

as $\|\psi\|' = 1$ for $\psi \in \Psi$. Then for x and y in \bar{B}

$$\|x \wedge y\|' = \|x \wedge (y - x)\|' \leq \|x\|' \|y - x\|' \leq \|x\|' \|y - x\|' \text{card}(\Psi) \|y\|'.$$

From the equivalence of norms on \mathbb{R}^n and Lemma 4.12, there is a constant $C > 0$ such that

$$d_a(x, y) \leq \text{card}(\Psi) \|y - x\|' \leq C \|y - x\|_\Psi \leq 2C d(x, y).$$

For x and y in \bar{B} , we get:

$$d_a(A_n \cdot x, A_n \cdot y) \leq 2C d(A_n \cdot x, A_n \cdot y) \leq 2C c(A_n).$$

Consequently:

$$\log \|A_n x \wedge A_n y\|' - \log \|A_n x\|' - \log \|A_n y\|' \leq \log(2C) + \log c(A_n).$$

From Proposition 4.15, we deduce that for all x all y in \bar{B} :

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|A_n x \wedge A_n y\|' \leq 2\gamma_1(A, T) + \log \kappa.$$

As the cone \mathcal{C} is solid, this inequality is true for all x and y in \mathbb{R}^n , that is

$$\gamma_1(A, T) + \gamma_2(A, T) \leq 2\gamma_1(A, T) + \log \kappa,$$

giving the announced formula. \square

5. Matrices of the random walks of left and right records

In Section 2 we have defined the exit probabilities of the random walk of a given interval and this was leading to the matrix M . We now present another way of estimating these exit probabilities by considering the successive records of the random walk on a given side. This brings two other matrices G and D .

Definition 5.1. Let $a < k < b$ where $-\infty \leq a$. We define a matrix of size $R \times R$:

$$D_k(a, b) = \begin{pmatrix} P_{k+R-1}\{a, b, b+R-1\} & \dots & P_{k+R-1}\{a, b, b\} \\ \vdots & & \vdots \\ P_k\{a, b, b+R-1\} & \dots & P_k\{a, b, b\} \end{pmatrix}.$$

We also introduce $D = D_0(-\infty, 1)$:

$$D = \begin{pmatrix} 0 & 1 & \dots \\ \dots & \dots & 1 \\ P_0\{-\infty, 1, R\} & \dots & P_0\{-\infty, 1, 1\} \end{pmatrix}.$$

Remark. For any $a < k$, $\det(D_k(a, k+1)) = (-1)^{R-1} P_k\{a, k+1, k+R\}$. The following lemma follows from the Markov property.

Lemma 5.2. Let $a < k < \beta \leq b$ and $\zeta \in \{b, \dots, b + R - 1, +\}$.

(i) Making vary the departure point, we have the following matricial relations:

$$\begin{pmatrix} P_{k+R-1}\{a, b, \zeta\} \\ \vdots \\ P_k\{a, b, \zeta\} \end{pmatrix} = D_k(a, \beta) \begin{pmatrix} P_{\beta+R-1}\{a, b, \zeta\} \\ \vdots \\ P_\beta\{a, b, \zeta\} \end{pmatrix} \\ = D_k(a, k + 1) \cdots D_{b-1}(a, b)e_{R-r}, \quad \text{if } \zeta = b + r.$$

(ii) Making vary the exit points on the right side, we obtain:

$$\begin{pmatrix} P_k\{a, b, b + R - 1\} \\ \vdots \\ P_k\{a, b, b\} \end{pmatrix} = {}^t D_\beta(a, b) \begin{pmatrix} P_k\{a, \beta, \beta + R - 1\} \\ \vdots \\ P_k\{a, \beta, \beta\} \end{pmatrix} \\ = {}^t D_{b-1}(a, b) \cdots {}^t D_{k+1}(a, k + 2) \begin{pmatrix} P_k\{a, k + 1, k + R\} \\ \vdots \\ P_k\{a, k + 1, k + 1\} \end{pmatrix}.$$

(iii) We have the equality:

$$(V_k(a, b, b + R - 1) \cdots V_k(a, b, b)) = (V_k(a, \beta, \beta + R - 1) \cdots V_k(a, \beta, \beta))D_\beta(a, b).$$

Consequently:

$$V_k(a, b, b + R - 1) \wedge \cdots \wedge V_k(a, b, b) = \det(D_\beta(a, b)) \times V_k(a, \beta, \beta + R - 1) \wedge \cdots \wedge V_k(a, \beta, \beta).$$

The same study holds for the exit points on the left side of $[a, b]$.

Definition 5.3. Let $a < k < b$, where $b \leq +\infty$. We define a matrix of size $L \times L$:

$$G_k(a, b) = \begin{pmatrix} P_k\{a, b, a\} & \cdots & P_k\{a, b, a - L + 1\} \\ \vdots & & \vdots \\ P_{k-L+1}\{a, b, a\} & \cdots & P_{k-L+1}\{a, b, a - L + 1\} \end{pmatrix}.$$

We also introduce $G = G_0(-1, +\infty)$:

$$G = \begin{pmatrix} P_0\{-1, +\infty, -1\} & \cdots & P_0\{-1, +\infty, -L\} \\ 1 & \cdots & \cdots \\ \cdots & 1 & 0 \end{pmatrix}.$$

Remark. For any $k < b$,

$$\det(G_k(k - 1, b)) = (-1)^{L-1} P_k\{k - 1, b, k - L\}.$$

The following lemma also follows from the Markov property.

Lemma 5.4. Let $a \leq \alpha < k < b$ and $\zeta \in \{a, \dots, a - L + 1, -\}$.

(i) Making vary the departure point, we have the following matricial relations:

$$\begin{pmatrix} P_k\{a, b, \zeta\} \\ \vdots \\ P_{k-L+1}\{a, b, \zeta\} \end{pmatrix} = G_k(\alpha, b) \begin{pmatrix} P_\alpha\{a, b, \zeta\} \\ \vdots \\ P_{\alpha-L+1}\{a, b, \zeta\} \end{pmatrix} \\ = G_k(k - 1, b) \cdots G_{\alpha+1}(a, b)e_{1+l}, \quad \text{if } \zeta = a - l. \tag{19}$$

(ii) Making vary the exit points on the left side, we obtain:

$$\begin{aligned} \begin{pmatrix} P_k\{a, b, a\} \\ \vdots \\ P_k\{a, b, a - L + 1\} \end{pmatrix} &= {}^t G_\alpha(a, b) \begin{pmatrix} P_k\{\alpha, b, \alpha\} \\ \vdots \\ P_k\{\alpha, b, \alpha - L + 1\} \end{pmatrix} \\ &= {}^t G_{a+1}(a, b) \cdots {}^t G_{k-1}(k-2, b) \begin{pmatrix} P_k\{k-1, b, k-1\} \\ \vdots \\ P_k\{k-1, b, k-L\} \end{pmatrix}. \end{aligned}$$

(iii) We have the equality:

$$V_k(a, b, a) \cdots (V_k(a, b, a - L + 1)) = (V_k(\alpha, b, \alpha) \cdots V_k(\alpha, b, \alpha - L + 1)) G_\alpha(a, b).$$

Consequently:

$$V_k(a, b, a) \wedge \cdots \wedge V_k(a, b, a - L + 1) = \det(G_\alpha(a, b)) \times V_k(\alpha, b, \alpha) \wedge \cdots \wedge V_k(\alpha, b, \alpha - L + 1).$$

Remark 1. Notice that the sums on the rows of the matrix G and of the matrix D are ≤ 1 . Therefore these matrices don't increase the norm $\| \cdot \|_\infty$ and then the maximal Lyapunov exponents of (G, T) and (D, T^{-1}) verify $\gamma_{\max}(G, T) \leq 0$ and $\gamma_{\max}(D, T^{-1}) \leq 0$.

Remark 2. The matrices G and D can be easily interpreted. Consider D for example. Let then $(\xi_n^+(\omega))_{n \geq 0}$ be the random walk on \mathbb{Z} deduced from $(\xi_n(\omega))_{n \geq 0}$, when the transitions to the left are suppressed, the transitions to the R right neighbors at the point k are changed into $(T^k P_0\{-\infty, 1, r\}(\omega))_{1 \leq r \leq R}$ and the probability to stay definitely in k is $T^k P_0\{-\infty, 1, -\}(\omega)$.

This Markov chain is the random walk of the successive records on the right side for the initial random walk and it “evolves” with D . Similarly, the matrix G is related to the sequence of records on the left side. We will see a little further that the central exponent $\gamma_R(M, T)$ compares the influences of these two random walks.

6. Main exponents of G and D and central exponent $\gamma_R(M, T)$

We relate the Lyapunov exponents of the matrices G and D to the exponents of M . A reason for this is that G and D describe rather easily and explicitly the properties of the random walk (see Lemma 7.1) but their definition involve implicit quantities. On the contrary M is directly built with the transition probabilities.

In a first step, we build the main eigenvector or Oseledet's main vector of $(-1)^{R-1} \wedge_R M$. Using a symmetric construction for $(-1)^{L-1} \wedge_L M^{-1}$, we deduce a formula for the central exponent $\gamma_R(M, T)$ in terms of the maximal exponents of G and D .

Definition 6.1. Let $a < k < b$. With the gradient-vectors (see Definition 2.3) associated to the exit points of $[a, b]$ on the right side and on the left side, we define:

$$\begin{cases} \mathcal{R}_k(a, b) = V_k(a, b, b + R - 1) \wedge \cdots \wedge V_k(a, b, b) \in \wedge_R \mathbb{R}^d, \\ \mathcal{R}_k^*(a, b) = V_k(a, b, b + R - 1) \wedge \cdots \wedge V_k(a, b, b + 1) \in \wedge_{R-1} \mathbb{R}^d, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_k(a, b) = V_k(a, b, a) \wedge \cdots \wedge V_k(a, b, a - L + 1) \in \wedge_L \mathbb{R}^d, \\ \mathcal{L}_k^*(a, b) = V_k(a, b, a - 1) \wedge \cdots \wedge V_k(a, b, a - L + 1) \in \wedge_{L-1} \mathbb{R}^d. \end{cases}$$

We then have the following result.

Proposition 6.2.

- (i) For any $a < k < b$, $\mathcal{R}_k(a, k + 1) \in (-1)^R \text{int}(\mathcal{C}^*)$.
- (ii) The following convergence holds:

$$\frac{\mathcal{R}_{-1}(-n, 1)}{P_{-1}\{-n, 1, -\}} \rightarrow V, \quad \text{as } n \rightarrow +\infty, \mu\text{-}ae,$$

where $V \neq 0$ is such that $\log \|V\|$ is bounded.

- (iii) Moreover V satisfies:

$$\left[(-1)^{R-1} \wedge_R M\right]V = \lambda TV,$$

where, $\mu\text{-}ae$:

$$\lambda = \frac{1}{P_0\{-\infty, 1, R\}} \lim_{n \rightarrow +\infty} \frac{P_0\{-n, 1, -\}}{P_{-1}\{-n, 0, -\}} \tag{20}$$

and V has maximal exponent with respect to $(-1)^{R-1} \wedge_R M$ and T .

Proof. (i) We first notice that $(-1)^R \mathcal{R}_k(a, k + 1) = [-V_k(a, k + 1, k + R)] \wedge \dots \wedge [-V_k(a, k + 1, k + 1)]$. For simplicity we denote by X this vector.

Let now $\varphi = \sum_1^{1+k_1} \wedge \dots \wedge \sum_R^{1+k_R}$ be an extremal vector of \mathcal{C} . Setting $f_r^s = P_{k+R-r-k_r}\{a, k + 1, k + s\}$ for $1 \leq r \leq R$ and $1 \leq s \leq R$, we obtain:

$$\langle X, \varphi \rangle = \begin{vmatrix} 1 - f_1^R & -f_2^R & \dots & -f_R^R \\ -f_1^{R-1} & 1 - f_2^{R-1} & \dots & -f_R^{R-1} \\ \vdots & \vdots & \dots & \vdots \\ -f_1^1 & \dots & \dots & 1 - f_R^1 \end{vmatrix}. \tag{21}$$

Remark that for all $1 \leq r \leq R$, we have:

$$\sum_{1 \leq s \leq R, s \neq R-r+1} f_r^s = (1 - f_r^{R-r+1}) - P_{k+R-r-k_r}\{a, k + 1, -\}.$$

Therefore the matrix appearing in (21) has a strictly dominating diagonal. Introducing parameters outside the diagonal and letting them decrease from 1 to 0, the determinant does not pass to zero. For a constant $C > 0$, we obtain:

$$\langle X, \varphi \rangle = C \prod_{r=1}^R (1 - f_r^{R-r+1}) > 0.$$

Consequently $\langle X, \varphi \rangle > 0$ for all $\varphi \in \Phi$ and then $(-1)^R \mathcal{R}_k(a, k + 1) \in \text{int}(\mathcal{C}^*)$.

- (ii) and (iii) We first have for all $n \geq 1$ and Lemma 5.2:

$$\mathcal{R}_0(-n, 1) = \wedge_R M \mathcal{R}_{-1}(-n, 1) = \left[(-1)^{R-1} \wedge_R M\right] \mathcal{R}_{-1}(-n, 0) P_0\{-n, 1, R\}.$$

Therefore:

$$(-1)^{R-1} \wedge_R M \left[\frac{\mathcal{R}_{-1}(-n, 0)}{P_{-1}\{-n, 0, -\}} \right] = \frac{P_0\{-n, 1, -\}}{P_{-1}\{-n, 0, -\} P_0\{-n, 1, R\}} \left[\frac{\mathcal{R}_0(-n, 1)}{P_0\{-n, 1, -\}} \right]. \tag{22}$$

To study the above vector $\mathcal{R}_0(-n, 1)/P_0\{-n, 1, -\}$, we set for $0 \leq l \leq L - 1$ and $0 \leq r \leq R - 1$:

$$f_{-l}^{R-r}(-n) := \sum_{s=0}^r (P_{-l}\{-n, 1, R - s\} - P_{-l+1}\{-n, 1, R - s\}).$$

We then observe that:

$$\mathcal{R}_0(-n, 1) = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ f_0^R(-n) \\ \vdots \\ f_{-L+1}^R(-n) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -1 \\ \vdots \\ 0 \\ f_0^{R-1}(-n) \\ \vdots \\ f_{-L+1}^{R-1}(-n) \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ f_0^1(-n) \\ \vdots \\ f_{-L+1}^1(-n) \end{pmatrix}.$$

Recall that $f_{-l}^1(-n) = -P_{-l}\{-n, 1, -\} + P_{-l}\{-n, 1, -\}$ and $f_0^1(-n) = -P_0\{-n, 1, -\}$. The first $R - 1$ vectors in the above decomposable R -vector converge as $n \rightarrow +\infty$ to linearly independent vectors and to prove (ii) it is enough to show that the last vector divided by $P_0\{-n, 1, -\}$ converges to a bounded vector.

Let then U be the matrix of dimensions $L \times L$ with a diagonal of (-1) and a sub-diagonal of ones. We have:

$$\begin{pmatrix} f_0^1(-n) \\ \vdots \\ f_{-L+1}^1(-n) \end{pmatrix} = U \begin{pmatrix} P_0\{-n, 1, -\} \\ \vdots \\ P_{-L+1}\{-n, 1, -\} \end{pmatrix} = U G_0(-1, 1) \cdots G_{-n+1}(-n, 1) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Notice then that the product of any L matrices of the form $G_r(s, t)$ is a positive matrix. Also from the minoration condition (1) on the transition probabilities and the value of the directional contraction coefficient given in Lemma 6.3 applied to matrices of the form $G_k(k - 1, b)$ in the cone \mathbb{R}^L , we obtain that $G_0(-1, 1) \cdots G_{-n+1}(-n, 1)^t(1, \dots, 1)$ has a limit direction which is strictly within the positive cone of \mathbb{R}^L uniformly in ω .

Dividing by $P_0\{-n, 1, -\}$, this vector converges and the logarithm of its norm is bounded. This proves (ii). The first part of (iii) follows then from (22). To see that V is the main eigenvector with respect to $(-1)^{R-1} \wedge_R M$ and T , notice that from (i) the vector $(-1)^R V$ is in \mathcal{C}^* . Then Theorem 4.6 indicates that the main eigenvector is the unique eigenvector in \mathcal{C}^* . \square

Remark. If $R = 1$, Proposition 6.2(i) reduces to the fact that the map $k \mapsto P_k\{a, b, +\}$ is non-decreasing. In the general case, some more complex condition of geometrical type holds.

We now give an expression of the central exponent $\gamma_R(M, T)$ of M with respect to T in terms of the maximal exponents $\gamma_{\max}(G, T)$ and $\gamma_{\max}(D, T^{-1})$ of (G, T) and (D, T^{-1}) .

Theorem 6.3. *We have the equality:*

$$\gamma_R(M, T) = \gamma_{\max}(G, T) - \gamma_{\max}(D, T^{-1}).$$

The proof of this theorem follows from the next proposition.

Proposition 6.4. *The following equalities hold:*

(i)

$$\begin{cases} \gamma_1(M, T) + \cdots + \gamma_R(M, T) = \gamma_{\max}(G, T) - \int \log P_0\{-\infty, 1, R\} d\mu, \\ \gamma_R(M, T) + \cdots + \gamma_d(M, T) = -\gamma_{\max}(D, T^{-1}) + \int \log P_0\{-1, +\infty, -L\} d\mu. \end{cases}$$

(ii)

$$\int \log \left(\frac{p_{-L}}{p_R} \frac{P_0\{-\infty, 1, R\}}{P_0\{-1, +\infty, -L\}} \right) d\mu = 0.$$

(iii)

$$\gamma_1(M, T) + \dots + \gamma_d(M, T) = \int \log(p_{-L}/p_R) d\mu. \tag{23}$$

Proof. Observe first that (iii) is an application of a classical and general result since $|\det(M)| = p_{-L}/p_R$. We refer for example to Ledrappier [12]. We now prove the first equality in (i). Using (20) and the T -invariance of the measure μ , we obtain:

$$(\gamma_1 + \dots + \gamma_R)(M, T) = \lim_{n \rightarrow +\infty} \int \log \frac{P_0\{-n, 1, -\}}{P_0\{-n+1, 1, -\}} d\mu - \int \log P_0\{-\infty, 1, R\} d\mu. \tag{24}$$

Considering the first term of the right member of (24), we have:

$$P_0\{-n, 1, -\} = \langle e_1, G_0(-1, 1) \cdots G_{-n+1}(-n, 1)u \rangle, \quad \text{where } u = {}^t(1, \dots, 1).$$

Set now $v_n = {}^tG_0(-1, n-1) \cdots {}^tG_{n-2}(n-3, n-1)e_1$. We then get:

$$\frac{P_0\{-n, 1, -\}}{P_0\{-n+1, 1, -\}} = T^{-n+2} \left(\frac{\langle G_{-1}(-2, n-1)u, v_n \rangle}{\langle u, v_n \rangle} \right),$$

Next we observe that v_n converges in direction to the positive vector W with $\|W\| = 1$ and verifying ${}^tG_{-1}(-2, +\infty)W = \rho T^{-1}W$ with $\int \log \rho d\mu = \gamma_{\max}({}^tG, T^{-1}) = \gamma_{\max}(G, T)$. Note that $\log \rho$ is bounded.

Moreover for $n \geq L$, v_n is strictly within the positive cone of \mathbb{R}^L , uniformly in ω since the product of any L matrices of the form $G_k(k-1, b)$ has positive entries all minored by a positive constant. We therefore obtain:

$$\lim_{n \rightarrow +\infty} \int \log \frac{P_0\{-n, 1, -\}}{P_0\{-n+1, 1, -\}} d\mu = \int \log \frac{\langle G_{-1}(-2, +\infty)u, W \rangle}{\langle u, W \rangle} d\mu = \int \log \rho d\mu = \gamma_{\max}(G, T).$$

This proves the first formula of (i). The method for the second one is symmetric.

(ii) Let $a < k < b$. A first remark is that $|\mathcal{L}_k(a, b) \wedge \mathcal{R}_k^*(a, b)| = |\mathcal{L}_k^*(a, b) \wedge \mathcal{R}_k(a, b)|$. Using the relation (4) and the determinant of the matrix M , we have the equality:

$$|\mathcal{L}_{a+1}(a, b) \wedge \mathcal{R}_{a+1}^*(a, b)| \times \prod_{k=a+2}^{b-1} \frac{p_{-L}(k)}{p_R(k)} = |\mathcal{L}_{b-1}(a, b) \wedge \mathcal{R}_{b-1}^*(a, b)|. \tag{25}$$

Noticing that for all $0 \leq l \leq L-1$:

$$V_k(a, b, a-l) = V_k(a, b-1, a-l) + V_k(a, b-1, b-1)P_{b-1}\{a, b, a-l\},$$

we get, using Lemma 5.2:

$$\begin{aligned} |\mathcal{L}_{a+1}^*(a, b) \wedge \mathcal{R}_{a+1}(a, b)| &= |\mathcal{L}_{a+1}^*(a, b) \wedge \mathcal{R}_{a+1}(a, b-1)| P_{b-1}\{a, b, b+R-1\} \\ &= |\mathcal{L}_{a+1}^*(a, b-1) \wedge \mathcal{R}_{b-1}(a, b-1)| P_{b-1}\{a, b, b+R-1\}. \end{aligned}$$

We thus deduce:

$$|\mathcal{L}_{a+1}^*(a, b) \wedge \mathcal{R}_{a+1}(a, b)| = |\mathcal{L}_{a+1}^*(a, a+2) \wedge \mathcal{R}_{a+1}(a, a+2)| \prod_{k=a+2}^{b-1} P_k\{a, k+1, k+R\}. \tag{26}$$

Similarly we have:

$$|\mathcal{L}_{b-1}(a, b) \wedge \mathcal{R}_{b-1}^*(a, b)| = |\mathcal{L}_{b-1}(b-2, b) \wedge \mathcal{R}_{b-1}^*(b-2, b)| \prod_{k=a+1}^{b-2} P_k\{k-1, b, k-L\}. \tag{27}$$

Now for any k we remark that $|\mathcal{L}_k(k-1, k+1) \wedge \mathcal{R}_k^*(k-1, k+1)| = p_{-L}(k)$ which is not 0. Using relations (25)–(27) with $a = -n - 1$ and $b = n + 1$ we finally obtain:

$$2n \int \log\left(\frac{p_{-L}}{p_R}\right) d\mu + \sum_{k=-2n-1}^{-2} \int \log P_0\{k, 1, R\} d\mu = \sum_{k=2}^{2n+1} \int \log P_0\{-1, k, -L\} d\mu.$$

Dividing by $2n$ the two members of the previous equality and letting $n \rightarrow +\infty$, the monotone convergence theorem gives the announced formula. \square

7. Asymptotic behaviour of the model

7.1. A recurrence criterion

Let us remark that the set $\{P_0\{-\infty, 1, +\} < 1\}$ is T -invariant. Therefore the ergodicity of μ with respect to T implies that $P_0\{-\infty, 1, +\} < 1, \mu$ -ae, or $P_0\{-\infty, 1, +\} = 1, \mu$ -ae.

Lemma 7.1. *The following statements are equivalent:*

- (i) $\gamma_{\max}(D, T^{-1}) < 0$.
- (ii) $P_0\{-\infty, 1, +\} < 1, \mu$ -ae.
- (iii) $\sup_{n \geq 0} \xi_n(\omega) < +\infty, \mathcal{P}_0^\omega$ -ae, μ -ae, meaning that $\xi_n(\omega) \rightarrow -\infty, \mathcal{P}_0^\omega$ -ae, μ -ae.

Proof. (i) \Rightarrow (iii) We have $P_0\{-\infty, n, +\} = \langle e_R, DT D \cdots T^{n-1} D u \rangle$, with $u := \sum_{i=1}^R e_i$. Thus:

$$\sum_{n=0}^{+\infty} P_0\{-\infty, n, +\} = \sum_{n=0}^{+\infty} \langle T^{n-1}({}^t D) \cdots ({}^t D) e_R, u \rangle < +\infty, \mu$$
-ae,

as $\gamma_{\max}(D, T^{-1}) = \gamma_{\max}({}^t D, T)$. The conclusion follows the lemma of Borel–Cantelli.

(iii) \Rightarrow (ii) We have that μ -ae there exists $N \geq 1$ such that $P_0\{-\infty, N, -\} > 0$. Considering the first position after the last visit in $\{1, \dots, N\}$, there exists $-L + 1 \leq x \leq 0$ with $P_x\{-\infty, 1, -\} > 0$. This gives $P_0\{-\infty, 1, -\} > 0$.

(ii) \Rightarrow (i) We have:

$$DT D \cdots T^{R-1} D = \begin{pmatrix} P_{R-1}\{-\infty, R, R+R-1\} & \cdots & P_{R-1}\{-\infty, R, R\} \\ \vdots & \vdots & \vdots \\ P_0\{-\infty, R, R+R-1\} & \cdots & P_0\{-\infty, R, R\} \end{pmatrix}.$$

Denote by $\| \cdot \|_\infty$ the norm subordinated to the infinite norm. Then:

$$\|DT D \cdots T^{R-1} D\|_\infty \leq \max_{0 \leq l \leq R-1} P_l\{-\infty, R, +\} =: \eta < 1, \mu$$
-ae.

Take $N > 1$ such that $U := \{\eta < 1 - 1/N\}$ verifies $\mu(U) > 0$. Denote by $(\tau_n(\omega))_{n \geq 1}$ the passage times in U . Kac’s Lemma then implies that $\tau_n/n \rightarrow 1/\mu(U), \mu$ -ae. For any $n \geq 1$, we choose $p = p(n)$ such that $R - 1 + R\tau_p < n \leq R - 1 + R\tau_{p+1}$. Since $\|D\|_\infty \leq 1$, we get:

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|D \cdots T^{n-1} D\|_\infty \leq \limsup_{p \rightarrow +\infty} \frac{1}{R\tau_p} \log(1 - 1/N)^p \leq \frac{\mu(U)}{R} \log(1 - 1/N) < 0.$$

This concludes the proof of the lemma. \square

From the whole previous study we deduce the following recurrence criterion for the random walk in random environment $(\xi_n(\omega))_{n \geq 0}$.

Theorem 7.2.

(i) If $\gamma_R(M, T) > 0$, then

$$\xi_n(\omega) \rightarrow -\infty, \quad \mathcal{P}_0^\omega\text{-}ae, \quad \mu\text{-}ae.$$

(ii) If $\gamma_R(M, T) = 0$, then

$$-\infty = \liminf \xi_n(\omega) < \limsup \xi_n(\omega) = +\infty, \quad \mathcal{P}_0^\omega\text{-}ae, \quad \mu\text{-}ae.$$

(iii) If $\gamma_R(M, T) < 0$, then

$$\xi_n(\omega) \rightarrow +\infty, \quad \mathcal{P}_0^\omega\text{-}ae, \quad \mu\text{-}ae.$$

Proof. Recall that $\gamma_{\max}(D, T^{-1}) \leq 0$ and $\gamma_{\max}(G, T) \leq 0$. Suppose now that $\gamma_R(M, T) > 0$. From Theorem 6.3 we get $\gamma_{\max}(D, T^{-1}) < 0$. Then Lemma 7.1 implies that the random walk is transient to $-\infty$. This gives (i). The proof for (iii) is similar.

Let us consider (ii), that is $\gamma_R(M, T) = 0$. If $\gamma_{\max}(D, T^{-1}) < 0$ and $\gamma_{\max}(G, T) < 0$ then Lemma 7.1 implies that the random walk is transient to $-\infty$ and to $+\infty$, which is impossible. Therefore $\gamma_{\max}(D, T^{-1}) = \gamma_{\max}(G, T) = 0$ and the random walk visits $-\infty$ and $+\infty$, that is recurrent. \square

Remark. It is proved in Letchikov [13] and in [7] that the above theorem is equivalent to Key's Theorem [10]. We mention that one can establish in a different way the above recurrence criterion by calculating explicitly the exit probabilities of an interval, using the same method as in [5].

7.2. Computation of $\gamma_R(M, T)$

We now turn to the computation of the central Lyapunov exponent $\gamma_R(M, T)$. Using the previous study on stable cones, we show that there is an exponential algorithm giving access to the value of this exponent. Therefore, conditionally to a numerical knowledge of the dynamical system $(\Omega, \mathcal{F}, \mu, T)$, the recurrence criterion (7.2) can be handled easily.

Denoting as before $g.x = gx/\|gx\|$ for an invertible matrix g and a non zero vector x , an algorithm is the following.

Step 1. For $\omega \in \Omega$, one evaluates the following decomposable R -vector in $\wedge_R \mathbb{R}^d$:

$$V_N(\omega) := [(-1)^{R-1} \wedge_R M(T^{-1}\omega)] \cdots [(-1)^{R-1} \wedge_R M(T^{-N}\omega)] \cdot (e_1 \wedge \cdots \wedge e_R).$$

From the condition of minoration (1) on the transition probabilities of the random walk, the convergence is uniformly exponential in ω with a rate given by the explicit expression (16) for the directional contraction constant of the matrix $(-1)^{R-1} \wedge_R M$ in the cone \mathcal{C} . When N is taken large enough in terms of the previous quantity, V_N is an approximation of the vector of maximal exponent of $((-1)^{R-1} \wedge_R M, T)$. Therefore:

$$\gamma_1(M, T) + \cdots + \gamma_R(M, T) \simeq \int \log \| \wedge_R M V_N \| d\mu. \quad (28)$$

Step 2. One repeats this procedure for the L last exponents by considering the decomposable L -vector in $\wedge_L \mathbb{R}^d$:

$$W_N(\omega) := [(-1)^{L-1} \wedge_L M^{-1}(T\omega)] \cdots [(-1)^{L-1} \wedge_L M^{-1}(T^N\omega)] \cdot (e_R \wedge \cdots \wedge e_d).$$

For the same reason as above, when N is large enough, W_N is an approximation of the vector of maximal exponent of $((-1)^{L-1} \wedge_L M^{-1}, T^{-1})$. This gives:

$$\gamma_R(M, T) + \dots + \gamma_d(M, T) \simeq - \int \log \| \wedge_L M^{-1} W_N \| d\mu. \tag{29}$$

Step 3. Finally one deduces an approximation of $\gamma_R(M, T)$ by considering (28) + (29) – (23). This then gives a concrete idea of the asymptotic behaviour of the random walk.

8. On the form of the central vector V_R

In a next step we consider the central vector V_R , with $\|V_R\| = 1$, defined in Corollary 4.5. This vector is uniquely determined in direction and verifies $MV_R = \lambda_R T V_R$, where $\int \log |\lambda_R| d\mu = \gamma_R(M, T)$. We have the following proposition:

Proposition 8.1. *The vector $V_{-1}(-m, n, +)$ converges in direction to V_R , as m and n tend to $+\infty$.*

Proof. Using Proposition 6.2, $\mathcal{R}_{-1}(-n, 0)$ converges in direction to the main eigenvector of the matrix $(-1)^{R-1} \wedge_R M$ with respect to T . From the definition of the Oseledet’s vectors, the previous vector is $V_1 \wedge \dots \wedge V_R$. Similarly, one would show that $\mathcal{L}_{-1}(-1, n)$ converges in direction to $V_R \wedge \dots \wedge V_d$.

However, for any $m \geq 1$ and $n \geq 0$, one observes that $\mathcal{R}_{-1}(-m, 0)$ and $\mathcal{R}_{-1}(-m, n)$ on the one side and $\mathcal{L}_{-1}(-1, n)$ and $\mathcal{L}_{-1}(-m, n)$ on the other side have the same direction.

As the subspaces corresponding to $\mathcal{R}_{-1}(-m, n)$ and $\mathcal{L}_{-1}(-m, n)$ intersect in the direction of $V_{-1}(-m, n, +)$ and since $\text{Vect}\{V_j \mid 1 \leq j \leq R\} \cap \text{Vect}\{V_j \mid R \leq j \leq d\}$ is a one-dimensional subspace, we deduce that $V_{-1}(-m, n, +)$ converges in direction to V_R . \square

We now give an expression of V_R . We restrict our study to the transient cases, up to using the matrix $M(r) := (1/r)A(r)MA(r)^{-1}$ with $A(r) = \text{diag}(1, r, \dots, r^{d-1})$ and r close to 1 if $\gamma_R(M, T) = 0$, as in the proof of Proposition 4.3.

We introduce the following non-invertible matrix, close to D of Definition 5.1.

Definition 8.2. Let $a \leq k$ and set $f_{k-L+1}^r(a) = P_{k-L+1}\{a, k-L+2, k-L+1+r\}$, for $1 \leq r \leq R$. We introduce a random matrix $\mathcal{D}_k(a)$ of dimensions $d \times d$:

$$\mathcal{D}_k(a) = \begin{pmatrix} 0 & 1 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ 0 & \dots & 0 & f_{k-L+1}^R(a) & \dots & f_{k-L+1}^1(a) \end{pmatrix}.$$

We now define the main eigenvector and the corresponding eigenvalue of the matrix D for the Lyapunov spectrum.

Definition 8.3. Let W be the unique positive vector in \mathbb{R}^R with $\langle W, e_R \rangle = 1$ and ρ the unique positive scalar ρ such that $DTW = \rho W$. Remark that the map ρ is bounded.

Considering the case when the random walk is transient to the left, we have the following proposition.

Proposition 8.4. *Assume that $\gamma_{\max}(D, T^{-1}) < 0$. Then there exists \tilde{V}_R and $\tilde{\lambda}_R$ such that $M\tilde{V}_R = \tilde{\lambda}_R T \tilde{V}_R$ and \tilde{V}_R has the direction of V_R where:*

$$\tilde{V}_R = T^{-L+1} \begin{pmatrix} (1/T\rho \cdots T^{d-2}\rho)(1 - 1/T^{d-1}\rho) \\ \vdots \\ (1 - 1/T\rho) \\ (\rho - 1) \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}_R = 1/T^{-L+2}\rho.$$

As a corollary, there is a random variable $\eta(\omega) > 0$ such that $\log \eta$ is bounded satisfying

$$\lambda_R = \frac{\eta}{T\eta} \tilde{\lambda}_R.$$

Proof. We first remark that for all $m \geq 1$ and all $n \geq 1$, we have

$$V_{-1}(-m, n, +) = (\mathcal{D}_{-1}(-m) - I)\mathcal{D}_0(-m) \cdots \mathcal{D}_{n+L-2}(-m) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then Proposition 8.1 gives that $V_{-1}(-m, n, +)$ converges in direction to V_R . Notice that if $\gamma_{\max}(D, T^{-1}) < 0$, the matrix $(\mathcal{D}_{-1}(-\infty) - I)$ is invertible.

Since the vectors $\mathcal{D}_0(-m) \cdots \mathcal{D}_{n+L-2}(-m)^t(1, \dots, 1)$ converge in direction as m and n tend to $+\infty$ to $T^{-L+1}(1/(\rho \cdots T^{d-2}\rho), \dots, 1/\rho, 1)$, we obtain the result. \square

Remark. If $\gamma_{\max}(G, T) < 0$, one obtains a similar expression for V_R . It is asserted in Lëitchikov [13] that V_R has positive components, but the proof is incorrect. Indeed, when $L = R = 2$ one can build an *iid* medium satisfying the minoration condition (1) such that V_2 lies in a neighbourhood of ${}^t(1, -1, 1)$ with positive probability. Therefore the statement of [13] is not true in general. However restrictions on the support of the law of $(p_z)_{z \in \Lambda}$ may ensure that it is verified.

9. Law of Large Numbers

We mention in this section that the Law of Large Numbers is always valid for the random walk. This is a direct corollary of an argument of [6], relying on the study of the random walks of the left and right records and the formalism introduced by Kozlov [11] for the absolutely continuous invariant measure for the random walk of “the environments seen from the particle”.

For integers $a < k < b$, denote by $E_k\{a, b\}$ the expectation under the measure P_k of the time to reach $(-\infty, a] \cup [b, +\infty)$. We then have the following result.

Proposition 9.1.

(1) *If $\int E_0\{-\infty, 1\} d\mu = +\infty$, then $\limsup \xi_n(\omega)/n \leq 0$, \mathcal{P}_0^{ω} -ae, μ -ae. If on the contrary $\int E_0\{-\infty, 1\} d\mu < +\infty$, then:*

$$\frac{1}{n} \xi_n(\omega) \rightarrow c = \frac{\int (\sum_{r=1}^R r P_0\{-\infty, 1, r\}) \pi_1 d\mu}{\int E_0\{-\infty, 1\} \pi_1 d\mu} > 0, \quad \mathcal{P}_0^{\omega}$$
-ae, μ -ae (30)

where $\pi_1 = \langle \Pi_1, e_1 \rangle$ and Π_1 is the positive eigenvector such that $T\Pi_1 = {}^t D\Pi_1$ and $\|\Pi_1\|_1 = 1$.

(2) If $\int E_0\{-1, +\infty\} d\mu = +\infty$, then $\liminf \xi_n(\omega)/n \geq 0$, \mathcal{P}_0^ω -ae, μ -ae. If on the contrary $\int E_0\{-1, +\infty\} d\mu < +\infty$, then:

$$\frac{1}{n} \xi_n(\omega) \rightarrow c = -\frac{\int (\sum_{l=1}^L l P_0\{-1, +\infty, -l\}) \pi_2 d\mu}{\int E_0\{-1, +\infty\} \pi_2 d\mu} < 0, \quad \mathcal{P}_0^\omega\text{-ae, } \mu\text{-ae} \tag{31}$$

where $\pi_2 = \langle \Pi_2, e_1 \rangle$ and Π_2 is the positive eigenvector such that $T^{-1} \Pi_2 = {}^t G \Pi_2$ and $\|\Pi_2\|_1 = 1$.

We thus deduce from the previous proposition the validity of the Law of Large Numbers.

Theorem 9.2. *There exists a constant c such that*

$$\frac{1}{n} \xi_n(\omega) \rightarrow c, \quad \mathcal{P}_0^\omega\text{-ae, } \mu\text{-ae.}$$

In the recurrent case this implies that the average deviations of the random walk are sub-linear.

Corollary 9.3. *Assume that $\gamma_R(M, T) = 0$. Then*

$$\frac{1}{n} \xi_n(\omega) \rightarrow 0, \quad \mathcal{P}_0^\omega\text{-ae, } \mu\text{-ae.}$$

10. Concluding remarks

The present study of the model stays at the level of Lyapunov exponents. As suggested by [6] when $R = 1$, the precise behaviour of the random walk should be related to the properties of λ_R with respect to the dynamical system $(\Omega, \mathcal{F}, \mu, T)$.

In this direction, it seems important to clarify the geometry of the spaces of eigenvectors associated to the Lyapunov spectrum of M . This would then give a better access to λ_R but it may involve a complete study of the space of harmonic functions on an interval that are barycenter at each point of their L left neighbours and their R right neighbours.

We also mention that one can compute explicitly the extremal vectors of the cone C^* . However in this study other completely determined cones appear to be stable by the class of matrices $(-1)^{R-1} R M$. Their interpretation in terms of random walks or harmonic functions is to be precised.

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