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On the rate of convergence in the central limit theorem for martingale difference sequences

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Abstract

We established the rate of convergence in the central limit theorem for stopped sums of a class of martingale difference sequences.

Résumé

On établit la vitesse de convergence dans le théorème limite central pour les sommes arrêtées issues d'une classe de suites de différences de martingale.

MSC: 60G42; 60F05

Keywords: Central limit theorem; Martingale difference sequence; Rate of convergence

1. Introduction

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall say that $(X_k)_{k\in\mathbb{N}}$ is a martingale difference sequence if, for any $k \geqslant 0$

1. $\mathbb{E}\{|X_k|\} < +\infty$.
2. $\mathbb{E}\{X_{k+1} \mid \mathcal{F}_k\} = 0$, where $\mathcal{F}_k$ is the σ-algebra generated by X_i, $i \leqslant k$.

For each integer $n \geqslant 1$ and x real number, we denote

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$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t, \qquad \sigma_{n-1}^2 = \mathbb{E}\{X_n^2 \mid \mathcal{F}_{n-1}\},$$

$$\nu(n) = \inf\left\{k \in \mathbb{N}^*\colon \sum_{i=0}^{k} \sigma_i^2 \geqslant n\right\}, \qquad S_{\nu(n)}^2 = \sum_{k=1}^{+\infty} S_k^2 I_{\nu(n)=k}, \qquad \sigma_{\nu(n)}^2 = \sum_{k=1}^{+\infty} \sigma_k^2 I_{\nu(n)=k},$$

$$F_n(x) = \mathbb{P}(S_{\nu(n)} \leqslant x\sqrt{n}\,), \qquad S'_{\nu(n)} = S_{\nu(n)} + \sqrt{\gamma(n)}X_{\nu(n)+1}, \qquad H_n(x) = \mathbb{P}(S'_{\nu(n)} \leqslant x\sqrt{n}\,),$$

and $\gamma(n)$ is a random variable such that

$$\sum_{i=0}^{\nu(n)-1} \sigma_i^2 + \gamma(n)\sigma_{\nu(n)}^2 = n \quad \text{a.s.} \tag{1}$$

If the random variables X_i are independent and identically distributed with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$, we have by the central limit theorem (CLT)

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{R}} \left|\mathbb{P}(S_n \leqslant x\sqrt{n}\,) - \phi(x)\right| = 0.$$

By the theorem of Berry ([1], 1941) and Esseen ([5], 1942), if moreover, $\mathbb{E}|X_i^3| < +\infty$, the rate of convergence in the limit is of order $n^{-1/2}$. If $(X_i)_{i\in\mathbb{N}}$ is an ergodic martingale difference sequence with $\mathbb{E}X_i^2 = 1$, by the theorem of Billingsley ([2], 1968) and Ibragimov (([11], 1963), see also ([10], 1980)) we have the CLT. The rate of convergence can, however, be arbitrarily slow even if X_i are bounded and α-mixing (cf. [4]). There are several results showing that with certain assumption on the conditional variance $\mathbb{E}(X_i^2 \mid \mathcal{F}_{i-1})$, the rate of convergence becomes polynomial (Kato ([12], 1979), Grams ([7], 1972), Nakata ([9], 1976), Bolthausen ([3], 1982), Haeusler ([8], 1988), ...).

In 1963, Ibragimov [11] has shown that for X_i uniformly bounded, if instead of usual sums S_n, the stopped sums $S_{\nu(n)}$ or $S'_{\nu(n)}$ are considered, one gets the rate of convergence of order $n^{-1/4}$; the only assumption beside boundedness is that $\sum_{i=0}^{+\infty} \sigma_i^2$ diverge to infinity a.s.

In the present paper we give a rate of convergence for a larger class of martingale difference sequences, the Ibragimov's case will be a particular one.

2. Main result

We consider a sequence $(X_i)_{i\in\mathbb{N}}$ of square integrable martingale differences.

Theorem 1. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and if there exists a nondecreasing sequence $(Y_i)_{i\in\mathbb{N}}$ adapted to the filtration $(\mathcal{F}_i, i \in \mathbb{N})$ such that, for all $i \in \mathbb{N}^*$*

$$\mathbb{E}(Y_i^4) < +\infty, \qquad 1 \leqslant Y_i \quad \text{and} \quad \mathbb{E}\big(|X_i|^3 \mid \mathcal{F}_{i-1}\big) \leqslant Y_{i-1}\sigma_{i-1}^2 \quad \text{a.s.}$$

then for all n sufficiently large

$$\sup_{x\in\mathbb{R}} \left|F_n(x) - \phi(x)\right| \leqslant \frac{{a_n}^{1/2}}{\pi n^{1/4}} \left(11 + \frac{3}{4n^{1/4}} + \frac{2}{9n^{1/2}} + \frac{1}{8n^{3/4}}\right), \tag{2}$$

$$\sup_{x\in\mathbb{R}} \left|H_n(x) - \phi(x)\right| \leqslant \frac{{a_n}^{1/2}}{\pi n^{1/4}} \left(11 + \frac{9}{4n^{1/4}} + \frac{2}{9n^{1/2}} + \frac{1}{8n^{3/4}}\right), \tag{3}$$

where $a_n = (\mathbb{E}Y^4_{\nu(n)})^{1/2}$.

If we put $Y_i = M$ a.s. where $M > 0$ is a constant, one obtains the following corollaries:

Corollary 1. *If the series* $\sum_{i=0}^{+\infty} \sigma_i^2$ *diverges a.s. and there exists* $M > 0$ *such that, for all* $i \in \mathbb{N}^*$, $\mathbb{E}(|X_i|^3 \mid \mathcal{F}_{i-1}) \leqslant M\mathbb{E}(X_i^2 \mid \mathcal{F}_{i-1})$ *a.s. then there is a constant* $0 < c_M < +\infty$

$$\sup_{x\in\mathbb{R}}\bigl|F_n(x) - \phi(x)\bigr| \leqslant \frac{c_M}{n^{1/4}}, \tag{4}$$

$$\sup_{x\in\mathbb{R}}\bigl|H_n(x) - \phi(x)\bigr| \leqslant \frac{c_M}{n^{1/4}}. \tag{5}$$

Corollary 2. *If there exists* $0 < \alpha \leqslant M < +\infty$ *satisfying* $\sigma_{i-1}^2 \geqslant \alpha$ *and* $\mathbb{E}(|X_i|^3 \mid \mathcal{F}_{i-1}) \leqslant M$ *a.s. for all* $i \in \mathbb{N}^*$, *then there is a constant* $0 < c_{(\alpha,M)} < +\infty$ *such that* (4) *and* (5) *hold.*

Moreover, if we suppose that $(X_i)_{i\in\mathbb{N}}$ is uniformly bounded, we obtain the result of Ibragimov [11].

Corollary 3. *If the series* $\sum_{i=0}^{+\infty} \sigma_i^2$ *diverges a.s. and* $|X_i| \leqslant M < +\infty$ *a.s. for all* $i \geqslant 0$, *then* (4) *and* (5) *hold.*

Example. Let $A = (A_k)_{k\in\mathbb{N}}$ be a sequence of real valued random variables such that $\sup_{k\in\mathbb{N}} \mathbb{E}(A_k^4)^{1/4} = \beta < \infty$ and consider an arbitrary sequence of variables $\zeta = (\zeta_k)_{k\in\mathbb{N}^*}$ with zero means, unit variances, bounded third moments and which are also independent of A. We define $X = (A_{k-1}\zeta_k)_{k\in\mathbb{N}^*}$ and $\mathcal{F}_k$ the σ-algebra generated by $A_0, A_1, \ldots, A_k$.

Clearly $(X_k, \mathcal{F}_k, k \in \mathbb{N}^*)$ is a martingale difference sequence, and for all $k \in \mathbb{N}^*$,

$$\mathbb{E}(A_{k-1}^2\zeta_k^2 \mid \mathcal{F}_{k-1}) = A_{k-1}^2 \quad \text{a.s.},$$
$$\mathbb{E}\bigl(|A_{k-1}\zeta_k|^3 \mid \mathcal{F}_{k-1}\bigr) \leqslant |A_{k-1}| \sup_{i\in\mathbb{N}^*} \mathbb{E}\bigl(|\zeta_i|^3\bigr) A_{k-1}^2 \quad \text{a.s.}$$

If $(|A_k|)_{k\in\mathbb{N}}$ is nondecreasing, then using Theorem 1, one obtains

$$\sup_{x\in\mathbb{R}}\bigl|F_n(x) - \phi(x)\bigr| \leqslant c\beta \frac{\sup_{k\in\mathbb{N}^*} \mathbb{E}(|\zeta_k|^3)^{1/4}}{n^{1/4}},$$

where c is a positive constant.

3. Proof of theorem

According to Esseen's theorem (see, e.g., ([6], 1954) p. 210 and ([13], 1955) p. 285), for all $y > 0$,

$$\sup_{x\in\mathbb{R}}\bigl|F_n(x) - \phi(x)\bigr| \leqslant \frac{1}{\pi}\int_{-y}^{y} \left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right)\right\} - \exp\left(-\frac{t^2}{2}\right)\right| \frac{\mathrm{d}t}{|t|} + \frac{24}{\pi\sqrt{2\pi}\,y}. \tag{6}$$

Below we shall prove the following inequalities

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\} - 1\right| \leqslant a_n \mathrm{e}^{\frac{t^2}{2}}\left(\frac{|t|}{3\sqrt{n}} + \frac{t^2}{4n} + \frac{a_n|t|^3}{3n^{3/2}} + \frac{a_n t^4}{4n^2}\right), \tag{7}$$

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2}\right)\right\}\right|\leqslant\frac{a_nt^2}{2n}\exp\left(\frac{t^2}{2}\right),\tag{8}$$

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS'_{\nu(n)}}{\sqrt{n}}\right)\right\}\right|\leqslant\frac{3a_nt^2}{2n},\tag{9}$$

where $a_n=(\mathbb{E}Y_{\nu(n)}^4)^{1/2}$.

3.1. *Proof of the inequality (7)*

We have

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\\
&\quad=\sum_{k=1}^{+\infty}\mathbb{E}\left\{\left(\exp\left(\frac{itS_k}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{k-1}\sigma_p^2\right)-1\right)I_{\nu(n)=k}\right\}\\
&\quad=\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{\exp\left(\frac{itS_{j-1}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(\mathrm{e}^{\frac{itX_j}{\sqrt{n}}}-\mathrm{e}^{-\frac{t^2\sigma_{j-1}^2}{2n}}\right)I_{\nu(n)=k}\right\}.
\end{aligned}$$

For real x, put

$$\mathrm{e}^{ix}=1+ix+\frac{(ix)^2}{2}+u(x),\qquad \mathrm{e}^{-x}=1-x+\beta(x)\frac{x^2}{2}.\tag{$*$}$$

It is easily seen that, for all $x\in\mathbb{R}$

$$\left|u(x)\right|\leqslant\frac{|x|^3}{6},\qquad\left|u(x)\right|\leqslant\frac{x^2}{2},\quad\text{and}\quad\left|\beta\left(|x|\right)\right|\leqslant1.$$

Observing that the random variable $W_{j-1}^n=\exp(\frac{itS_{j-1}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2)$ is measurable with respect to the σ-algebra $\mathcal{F}_{j-1}$ and using the identities ($*$), we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\\
&\quad=\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{W_{j-1}^n\mathbb{E}\left\{\left(\frac{itX_j}{\sqrt{n}}-\frac{t^2X_j^2}{2n}+u\left(\frac{tX_j}{\sqrt{n}}\right)+\frac{t^2\sigma_{j-1}^2}{2n}+\beta\left(\frac{t^2\sigma_{j-1}^2}{2n}\right)\frac{t^4\sigma_{j-1}^4}{8n^2}\right)I_{\nu(n)=k}\,\middle|\,\mathcal{F}_{j-1}\right\}\right\}.
\end{aligned}\tag{10}$$

Since $\{\nu(n)=k\}$ is measurable with respect to the σ-algebra $\mathcal{F}_k$, for all $j\geqslant2$, we have

$$\sum_{k=1}^{j-1}\mathbb{E}\{X_jI_{\nu(n)=k}\mid\mathcal{F}_{j-1}\}=\sum_{k=1}^{j-1}\mathbb{E}\left\{(X_j^2-\sigma_{j-1}^2)I_{\nu(n)=k}\mid\mathcal{F}_{j-1}\right\}=0.$$

On the other hand, for all $j\geqslant1$ we have

$$\sum_{k=1}^{+\infty}\mathbb{E}\{X_jI_{\nu(n)=k}\mid\mathcal{F}_{j-1}\}=\sum_{k=1}^{+\infty}\mathbb{E}\left\{(X_j^2-\sigma_{j-1}^2)I_{\nu(n)=k}\mid\mathcal{F}_{j-1}\right\}=0.$$

It follows that, for all $j \geqslant 1$

$$\sum_{k\geqslant j}\mathbb{E}\{X_j I_{\nu(n)=k} \mid \mathcal{F}_{j-1}\} = \sum_{k\geqslant j}\mathbb{E}\big\{(X_j^2-\sigma_{j-1}^2)I_{\nu(n)=k} \mid \mathcal{F}_{j-1}\big\} = 0.$$

So, from (10) we derive

$$\begin{aligned}
&\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right| \\
&\quad= \left|\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{W_{j-1}^n\mathbb{E}\left\{\left(u\left(\frac{tX_j}{\sqrt{n}}\right)+\beta\left(\frac{t^2\sigma_{j-1}^2}{2n}\right)\frac{t^4\sigma_{j-1}^4}{8n^2}\right)I_{\nu(n)=k}\,\middle|\,\mathcal{F}_{j-1}\right\}\right\}\right| \\
&\quad\leqslant \sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\left\{\left(\frac{|t|^3|X_j|^3}{6n^{3/2}}+\frac{t^4\sigma_{j-1}^4}{8n^2}\right)I_{\nu(n)=k}\,\middle|\,\mathcal{F}_{j-1}\right\}\right\}.
\end{aligned} \tag{11}$$

For any $j \geqslant 2$ and any real function ψ such that $\mathbb{E}(\psi(X_k)) < \infty$ for any positive k, we have

$$\begin{aligned}
&\sum_{k=1}^{j-1}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\big\{\psi(X_j)I_{\nu(n)=k} \mid \mathcal{F}_{j-1}\big\}\right\} \\
&\quad= \sum_{k=1}^{j-1}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\big\{\psi(X_j) \mid \mathcal{F}_{j-1}\big\}I_{\nu(n)=k}\right\}.
\end{aligned} \tag{12}$$

On the other hand, for all $j \geqslant 1$, we have

$$\begin{aligned}
&\sum_{k=1}^{+\infty}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\big\{\psi(X_j)I_{\nu(n)=k} \mid \mathcal{F}_{j-1}\big\}\right\} \\
&\quad= \mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\psi(X_j)\right\} \\
&\quad= \sum_{k=1}^{+\infty}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\big\{\psi(X_j) \mid \mathcal{F}_{j-1}\big\}I_{\nu(n)=k}\right\}.
\end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}
&\sum_{j=1}^{+\infty}\sum_{k\geqslant j}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\left\{\big\{\psi(X_j)I_{\nu(n)=k} \mid \mathcal{F}_{j-1}\big\}\right\}\right. \\
&\quad= \sum_{j=1}^{+\infty}\sum_{k\geqslant j}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\mathbb{E}\big\{\psi(X_j) \mid \mathcal{F}_{j-1}\big\}I_{\nu(n)=k}\right\}.
\end{aligned} \tag{14}$$

Applying (11) and (14) for $\psi(x) = |x|^3$ we deduce that

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right|$$

$$\leqslant \sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(\mathbb{E}\left\{\frac{|t|^3|X_j|^3}{6n^{3/2}}\,\middle|\,\mathcal{F}_{j-1}\right\}I_{\nu(n)=k}+\mathbb{E}\left\{\frac{t^4\sigma_{j-1}^4}{8n^2}I_{\nu(n)=k}\,\middle|\,\mathcal{F}_{j-1}\right\}\right)\right\}$$

$$\leqslant \sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(\frac{|t|^3Y_{j-1}\sigma_{j-1}^2}{6n^{3/2}}I_{\nu(n)=k}+\frac{t^4\sigma_{j-1}^4}{8n^2}I_{\nu(n)=k}\right)\right\}. \tag{15}$$

By the Hölder inequality, for all $j \in \mathbb{N}^*$

$$\sigma_{j-1}^2 = \mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1}) \leqslant \mathbb{E}\big(|X_j|^3 \mid \mathcal{F}_{j-1}\big)^{2/3} \leqslant Y_{j-1}^{2/3}\sigma_{j-1}^{4/3} \quad \text{a.s.},$$

whence

$$\sigma_{j-1}^2 \leqslant Y_{j-1}^2 \quad \text{a.s.} \tag{16}$$

From (15), (16) and using the fact that $Y_k \geqslant Y_{j-1} \geqslant 1$ for all $j \leqslant k$, we deduce that

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right|$$

$$\leqslant \left(\frac{|t|^3}{6n^{3/2}}+\frac{t^4}{8n^2}\right)\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{Y_{j-1}^2\sigma_{j-1}^2\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)I_{\nu(n)=k}\right\}$$

$$\leqslant \left(\frac{|t|^3}{6n^{3/2}}+\frac{t^4}{8n^2}\right)\sum_{k=1}^{+\infty}\mathbb{E}\left\{Y_k^2\sum_{j=1}^{k}\sigma_{j-1}^2\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)I_{\nu(n)=k}\right\}. \tag{17}$$

To bound up the terms appearing in (17), we will use the following elementary lemma.

Lemma 1. *Let $k \geqslant 1$, then on the event $\{\nu(n)=k\}$ we have*

$$\sum_{j=1}^{k}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}\sigma_{j-1}^2 \leqslant \exp\left(\frac{t^2}{2}\right)\left(1+\frac{Y_k^2t^2}{n}\right).$$

Proof. On the event $\{\nu(n)=k\}$, we have

$$\exp\left(\frac{t^2}{2}\right) \geqslant \exp\left(\frac{t^2}{2n}\sum_{p=0}^{k-1}\sigma_p^2\right)-\exp\left(\frac{t^2}{2n}\sigma_0^2\right)$$

$$\geqslant \sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(\exp\left(\frac{t^2\sigma_j^2}{2n}\right)-1\right).$$

Using the inequality, $\exp(x)-1 \geqslant x$ for all $x \geqslant 0$, one obtains

$$\exp\left(\frac{t^2}{2}\right) \geqslant \sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}\sigma_j^2.$$

Therefore

$$\sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}\sigma_{j-1}^2$$

$$
\begin{aligned}
&\leqslant \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}(\sigma_{j-1}^2-\sigma_j^2)+\exp\left(\frac{t^2}{2}\right)\\
&= \sum_{j=1}^{k-2}\left(\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j}\sigma_p^2\right)-\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\right)\frac{t^2}{2n}\sigma_j^2-\frac{t^2}{2n}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{k-2}\sigma_p^2\right)\sigma_{k-1}^2\\
&\quad+\frac{t^2}{2n}\exp\left(\frac{t^2}{2n}\sigma_0^2\right)\sigma_0^2+\exp\left(\frac{t^2}{2}\right)\\
&\leqslant \frac{t^2}{2n}Y_k^2\sum_{j=1}^{k-2}\left(\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j}\sigma_p^2\right)-\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\right)+\frac{t^2}{2n}Y_k^2\exp\left(\frac{t^2}{2n}\sigma_0^2\right)+\exp\left(\frac{t^2}{2}\right)\\
&\leqslant \left(1+\frac{t^2}{2n}Y_k^2\right)\exp\left(\frac{t^2}{2}\right).
\end{aligned}
$$

We conclude the proof of the lemma by noting that $\sigma_{k-1}^2 \leqslant Y_k^2$ and $\sum_{p=0}^{k-1}\sigma_p^2 \leqslant n$ a.s. □

Finally, according to Lemma 1 and the (17) we get

$$
\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right| \leqslant a_n\exp\left(\frac{t^2}{2}\right)\left(\frac{|t|}{3\sqrt{n}}+\frac{t^2}{4n}+\frac{a_n|t|^3}{3n^{3/2}}+\frac{a_nt^4}{4n^2}\right),
$$

where $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{1/2}$.

3.2. *Proof of the inequality (8)*

Using (1) and the inequality $|1-\exp(-x)| \leqslant x$, for all $x \geqslant 0$ we see that

$$
\begin{aligned}
&\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2}\right)\right\}\right|\\
&\quad=\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2}\right)\left(\exp\left(-\frac{t^2}{2n}\gamma(n)\sigma_{\nu(n)}^2\right)-1\right)\right\}\right|\\
&\quad\leqslant \mathbb{E}\left\{\left|1-\exp\left(-\frac{t^2}{2n}\gamma(n)\sigma_{\nu(n)}^2\right)\right|\right\}\exp\left(\frac{t^2}{2}\right)\\
&\quad\leqslant \mathbb{E}\left\{\frac{t^2}{2n}\left|\gamma(n)\right|\sigma_{\nu(n)}^2\right\}\exp\left(\frac{t^2}{2}\right)\\
&\quad\leqslant (\mathbb{E}Y_{\nu(n)}^4)^{1/2}\frac{t^2}{2n}\exp\left(\frac{t^2}{2}\right).
\end{aligned}
$$

Therefore (8) holds true.

From (7) and (8) we conclude that

$$
\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right)\right\}-\exp\left(-\frac{t^2}{2}\right)\right| \leqslant a_n\left(\frac{|t|}{3\sqrt{n}}+\frac{3t^2}{4n}+\frac{|t|^3}{3n^{3/2}}a_n+\frac{t^4}{4n^2}a_n\right).
$$

Using Esseen's theorem, we derive

$$
\sup_{x\in\mathbb{R}}\left|F_n(x)-\phi(x)\right| \leqslant \frac{a_n}{\pi}\int_{-y}^{y}\left(\frac{1}{3\sqrt{n}}+\frac{3|t|}{4n}+\frac{t^2}{3n^{3/2}}a_n+\frac{|t|^3}{4n^2}a_n\right)\mathrm{d}t+\frac{24}{\pi\sqrt{2\pi}\,y}.
$$

Hence

$$\sup_{x\in\mathbb{R}}\bigl|F_n(x)-\phi(x)\bigr|\leqslant\frac{a_n}{\pi}\left(\frac{2y}{3\sqrt{n}}+\frac{3y^2}{4n}+\frac{2y^3}{9n^{3/2}}a_n+\frac{y^4}{8n^2}a_n\right)+\frac{24}{\pi\sqrt{2\pi}\,y}.$$

Choosing y in such a way that $y/\sqrt{n}=1/(ya_n)$, i.e. $y=(n/a_n^2)^{1/4}$, we infer that

$$\sup_{x\in\mathbb{R}}\bigl|F_n(x)-\phi(x)\bigr|\leqslant\frac{{a_n}^{1/2}}{\pi n^{1/4}}\left(11+\frac{3}{4n^{1/4}}+\frac{2}{9n^{1/2}}+\frac{1}{8n^{3/4}}\right).$$

The proof of the inequality (2) in theorem is complete.

3.3. Proof of the inequality (9)

Observing that the random events $\{\gamma(n)\leqslant x\}\cap\{\nu(n)=k\}$ and consequently the random variables $\sqrt{\gamma(n)}I_{\nu(n)=k}$ are measurable with respect to $\mathcal{F}_k$, we find that

$$\begin{aligned}
&\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS'_{\nu(n)}}{\sqrt{n}}\right)\right\}\right|\\
&\quad=\left|\sum_{k=0}^{+\infty}\mathbb{E}\left\{\left(\exp\left(\frac{itS_k}{\sqrt{n}}\right)-\exp\left(\frac{itS_k}{\sqrt{n}}+\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{\nu(n)+1}\right)\right)I_{\nu(n)=k}\right\}\right|\\
&\quad\leqslant\sum_{k=0}^{+\infty}\left|\mathbb{E}\left\{\exp\left(\frac{itS_k}{\sqrt{n}}\right)\left(1-\exp\left(\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{\nu(n)+1}\right)\right)I_{\nu(n)=k}\right\}\right|\\
&\quad=\sum_{k=0}^{+\infty}\left|\mathbb{E}\left\{\exp\left(\frac{itS_k}{\sqrt{n}}\right)\left(-\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1}+\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u\left(\frac{t\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1}\right)\right)I_{\nu(n)=k}\right\}\right|\\
&\quad=\sum_{k=0}^{+\infty}\left|\mathbb{E}\left\{\exp\left(\frac{itS_k}{\sqrt{n}}\right)\mathbb{E}\left\{-\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1}+\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u\left(\frac{t\sqrt{\gamma(n)}X_{k+1}}{\sqrt{n}}\right)\,\middle|\,\mathcal{F}_k\right\}I_{\nu(n)=k}\right\}\right|\\
&\quad=\sum_{k=0}^{+\infty}\left|\mathbb{E}\left\{\exp\left(\frac{itS_k}{\sqrt{n}}\right)\left(\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u\left(\frac{t}{\sqrt{n}}\sqrt{\gamma(n)}X_{k+1}\right)\right)I_{\nu(n)=k}\right\}\right|\\
&\quad\leqslant\sum_{k=0}^{+\infty}\mathbb{E}\left\{I_{\nu(n)=k}\frac{3t^2}{2n}\gamma(n)X_{k+1}^2\right\}\\
&\quad\leqslant\frac{3t^2}{2n}\sum_{k=0}^{+\infty}\mathbb{E}\bigl\{I_{\nu(n)=k}\mathbb{E}\{X_{k+1}^2\mid\mathcal{F}_k\}\bigr\}\\
&\quad\leqslant\frac{3t^2}{2n}\mathbb{E}\bigl(Y_{\nu(n)}^4\bigr)^{1/2}.
\end{aligned}$$

The proof of the inequality (9) is complete.

3.4. Proof of the inequality (3)

According to Esseen's theorem where $y=(n/a_n^2)^{1/4}$ and the inequality (9), one obtains

$$\sup_{x\in\mathbb{R}}\left|H_n(x)-\phi(x)\right| \leqslant \frac{1}{\pi}\int_{-y}^{y}\left|\mathbb{E}\left\{\exp\left(\frac{itS'_{\nu(n)}}{\sqrt{n}}\right)\right\}-\exp\left(-\frac{t^2}{2}\right)\right|\frac{\mathrm{d}t}{|t|}+\frac{24}{\pi\sqrt{2\pi}\,y}$$
$$\leqslant \frac{{a_n}^{1/2}}{\pi n^{1/4}}\left(11+\frac{3}{4n^{1/4}}+\frac{2}{9n^{1/2}}+\frac{1}{8n^{3/4}}\right)+\frac{1}{\pi}\int_{-y}^{y}\frac{3|t|}{2n}E(Y^4_{\nu(n)})^{1/2}\,\mathrm{d}t$$
$$\leqslant \frac{{a_n}^{1/2}}{\pi n^{1/4}}\left(11+\frac{3}{4n^{1/4}}+\frac{2}{9n^{1/2}}+\frac{1}{8n^{3/4}}\right)+\frac{3}{2\pi\sqrt{n}}$$
$$\leqslant \frac{{a_n}^{1/2}}{\pi n^{1/4}}\left(11+\frac{9}{4n^{1/4}}+\frac{2}{9n^{1/2}}+\frac{1}{8n^{3/4}}\right).$$

The proof of theorem is complete. □

Proofs of Corollaries 1, 2 and 3 are easy so, it is left to the reader.

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