

A new class of Ornstein transformations with singular spectrum

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Abstract

It is shown that for any family of probability measures in Ornstein type constructions, the corresponding transformation has almost surely a singular spectrum. This is a new generalization of Bourgain's theorem [J. Bourgain, On the spectral type of Ornstein class one transformations, *Israel J. Math.* 84 (1993) 53–63], same result is proved for Rudolph's construction [D. Rudolph, An example of a measure-preserving map with minimal self-joining and applications, *J. Anal. Math.* 35 (1979) 97–122].

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Résumé

On montre que pour toute famille de mesures de probabilités dans la construction d'Ornstein, les transformations résultantes ont un spectre presque sûrement singulier. On obtient ainsi une nouvelle généralisation d'un théorème dû à Bourgain [J. Bourgain, On the spectral type of Ornstein class one transformations, *Israel J. Math.* 84 (1993) 53–63]. Un résultat similaire est obtenu pour les transformations de Rudolph [D. Rudolph, An example of a measure-preserving map with minimal self-joining and applications, *J. Anal. Math.* 35 (1979) 97–122].

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1. Introduction

In this note we investigate the spectral analysis of a generalized class of Ornstein transformations. There are several generalizations of Ornstein transformations. Here we are concerned with arbitrary product probability space associated to random constructions of rank one transformations. Namely, in the Ornstein's construction, the probability space is equipped with the infinite product of uniform probability measures on some finite subsets of \mathbb{Z} . Here, the probability space is equipped with the infinite product of probability measures $(\xi_m)_{m \in \mathbb{N}}$ on a family $(X_m)_{m \in \mathbb{N}}$ of finite subsets

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of \mathbb{Z} . We establish that for any choice of the family $(\xi_m)_{m \in \mathbb{N}}$ the associated Ornstein transformations has almost surely singular spectrum.

Let us recall that Ornstein introduced these transformations in 1967 in [15] and proved that the mixing property occurs almost surely. Until 1991, these transformations which have simple spectrum appeared as a candidate for an affirmative answer to Banach's well-known problem whether a dynamical system (X, \mathcal{B}, μ) may have simple Lebesgue spectrum. But, in 1991, J. Bourgain in [4], using Riesz products techniques, proved that Ornstein transformations have almost surely singular spectrum. Subsequently, I. Klemes [12], I. Klemes and K. Reinhold [13] obtain that the spectrum of the mixing subclass of staircase transformations of T. Adams [1] and T. Adams and N. Friedman [2] have singular spectrum. They conjectured that rank one transformations always have singular spectrum.

In this paper, using the techniques of J. Bourgain generalized in [8], we extend Bourgain's theorem to the generalized Ornstein transformations associated to a large family of random constructions.

Firstly, we shall recall some basic facts from spectral theory. A nice account can be found in the appendix of [16]. We shall assume that the reader is familiar with the method of cutting and stacking for constructing rank one transformations.

Given $T : (X, \mathcal{B}, \mu) \mapsto (X, \mathcal{B}, \mu)$ a measure preserving invertible transformation and denoting by $U_T f$ the operator $U_T f(x) = f(T^{-1}x)$ on $L^2(X, \mathcal{B}, \mu)$, recall that to any $f \in L^2(X)$ there corresponds a positive measure σ_f on \mathbb{T} , the unit circle, defined by $\hat{\sigma}_f(n) = \langle U_T^n f, f \rangle$.

Definition 1.1. The maximal spectral type of T is the equivalence class of Borel measures σ on \mathbb{T} (under the equivalence relation $\mu_1 \sim \mu_2$ if and only if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$), such that $\sigma_f \ll \sigma$ for all $f \in L^2(X)$ and if ν is another measure for which $\sigma_f \ll \nu$ for all $f \in L^2(X)$ then $\sigma \ll \nu$.

There exists a Borel measure $\sigma = \sigma_f$ for some $f \in L^2(X)$, such that σ is in the equivalence class defining the maximal spectral type of T . By abuse of notation, we will call this measure the maximal spectral type measure. The reduced maximal type σ_0 is the maximal spectral type of U_T on $L^2_0(X) \stackrel{\text{def}}{=} \{f \in L^2(X) : \int f d\mu = 0\}$. The spectrum of T is said to be discrete (resp. continuous, resp. singular, resp. absolutely continuous, resp. Lebesgue) if σ_0 is discrete (resp. continuous, resp. singular, resp. absolutely continuous with respect to the Lebesgue measure or equivalent to the Lebesgue measure). We write

$$Z(h) \stackrel{\text{def}}{=} \overline{\text{span}\{U_T^n h, n \in \mathbb{Z}\}}.$$

T is said to have simple spectrum, if there exists $h \in L^2(X)$ such that

$$Z(h) = L^2(X).$$

2. Rank one transformation by construction

Using the cutting and stacking method described in [10,11], one defines inductively a family of measure preserving transformations, called rank one transformations, as follows.

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k th stage we divide the stack obtained at the $(k-1)$ th stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k . If during the k th stage of our construction the number of spacers put above the j th column of the $(k-1)$ th stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq p_k$, then we have (see Fig. 1)

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}, \quad \forall k \geq 1,$$

$$h_0 = 1.$$

Proceeding thus we get a rank one transformation T on a certain measure space (X, \mathcal{B}, ν) which may be finite or σ -finite depending on the number of spacers added.

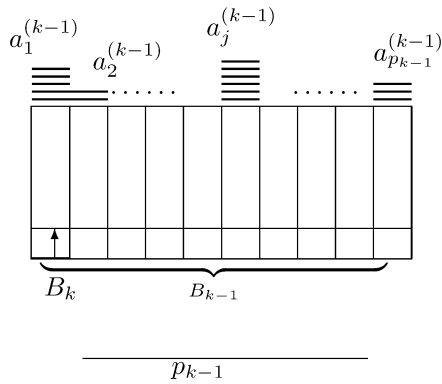


Fig. 1. k th-tower.

The construction of any rank one transformation thus needs two parameters $(p_k)_{k=0}^\infty$ (cutting parameter) and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^\infty$ (spacers parameter). We put

$$T \stackrel{\text{def}}{=} T_{(p_k, (a_j^{(k)})_{j=1}^{p_k})_{k=0}^\infty}.$$

In [4,5] and [13] it is proved that up to some discrete measure, the spectral type of this transformation is given by

$$d\sigma = W^* \lim \prod_{k=1}^n |P_k|^2 d\lambda, \tag{2.1}$$

where

$$P_k(z) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k-1} z^{-(jh_k + \sum_{i=1}^j a_i^{(k)})} \right).$$

λ denotes the normalized Lebesgue measure on torus \mathbb{T} and W^* lim denotes weak* limit in the space of bounded Borel measures on \mathbb{T} . The polynomials P_k appear naturally from the induction relation between the bases B_k . Indeed

$$B_k = \bigcup_{j=0}^{p_k-1} T^{h_k + s_k(j)} B_{k+1},$$

$$\nu(B_k) = p_k \nu(B_{k+1}),$$

where $s_k(n) = a_1^{(k)} + \dots + a_n^{(k)}$ and $s_k(0) = 0$.

Put

$$f_k = \frac{1}{\sqrt{\nu(B_k)}} \chi_{B_k},$$

that is the indicator function of the k th-base normalized in the L^2 -norm. So

$$f_k = P_k(U_T) f_{k+1},$$

It follows that

$$d\sigma_k = |P_k|^2 d\sigma_{k+1} = \dots = \prod_{j=0}^{m-1} |P_{k+j}|^2 d\sigma_{k+m},$$

where σ_p is the spectral measure of f_p , $p \geq 0$.

3. Generalized Ornstein’s class of transformations

In Ornstein’s construction, the p_k ’s are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, are chosen randomly. This may be organized in different ways as pointed by J. Bourgain in [4]. Here, We suppose given (t_k) , (p_k) a sequence of positive integers and (ξ_k) a sequence of probability measure such that the support of each ξ_k is a subset of $X_k = \{-\frac{t_k}{2}, \dots, \frac{t_k}{2}\}$. We choose now independently, according to ξ_k the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in \mathbb{N} . We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = t_k + x_{k,i} - x_{k,i-1}, \quad \text{with } x_{k,0} = 0.$$

It follows

$$h_{k+1} = p_k(h_k + t_k) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^\infty$, $(t_k)_{k=0}^\infty$ and $(x_{k,p_k})_{k=0}^\infty$ determine completely the sequence of heights $(h_k)_{k=0}^\infty$. The total measure of the resulting measure space is finite if

$$\sum_{k=0}^\infty \frac{t_k}{h_k} + \sum_{k=0}^\infty \frac{x_{k,p_k}}{p_k h_k} < \infty. \tag{3.1}$$

We will assume that this requirement is satisfied.

We thus have a probability space of Ornstein transformations $\Omega = \prod_{l=0}^\infty X_l^{p_l-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{l=1}^\infty P_l$, where $P_l \stackrel{\text{def}}{=} \otimes_{j=1}^{p_l-1} \xi_l$; ξ_l is the probability measure on X_l . We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leq i \leq p_k - 1$, is the projection from Ω onto the i th co-ordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \leq i \leq p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=0}^\infty$ in Ω defines the spacers and therefore a rank one transformation $T_{\omega,x}$, where $x = (x_{k,p_k})$.

The definition above gives a more general definition of random construction due to Ornstein. In the particular case of Ornstein’s transformations constructed in [15], $t_k = h_{k-1}$, ξ_k is the uniform distribution on X_k and $p_k \gg h_{k-1}$.

We recall that Ornstein in [15] proved that there exist a sequence $(p_k, x_{k,p_k})_{k \in \mathbb{N}}$ such that, $T_{\omega,x}$ is almost surely mixing. Later in [17], Prikhod’ko obtain the same result for some special choice of the sequence of the distribution (ξ_m) and recently, using the idea of D. Creutz and C.E. Silva [6] one can extend this result to a large family of the probability measure associated to Ornstein construction. In our general construction, according to (2.1) the spectral type of each T_ω , up to a discrete measure, is given by

$$\sigma_{T_\omega} = \sigma_{\chi_{B_0}(\omega)} = \sigma^{(\omega)} = W^* \lim \prod_{l=1}^N \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+t_l)+x_{l,p}} \right|^2 d\lambda.$$

In the case of Rudolph construction, we choose independently, using the uniform distribution on the set $\{1, \dots, t_k\}$ the numbers $(x_{k,i})_{i=1}^{p_k}$. We put, for $1 \leq i \leq p_k - 1$,

$$a_i^{(k)} = t_k + 2 + x_{k,i+1} - x_{k,i},$$

$$a_{p_k}^{(k)} = t_k + 1 - x_{k,p_k}.$$

We define $a_{-1}^{(k)}$ as de number of spacers we stack below the first column. We put

$$a_{-1}^{(k)} = x_{k,1} + 1.$$

One sees that

$$h_{k+1} = p_k(h_k + t_k + 2).$$

For Rudolph construction, according to (2.1) the spectral type of each T_ω , up to a discrete measure, is given by

$$\sigma_{T_\omega} = \sigma_{\chi_{B_0}(\omega)} = \sigma^{(\omega)} = W^* \lim \prod_{l=1}^N \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+t_l+2)+x_{l,p+1}-x_{l,1}} \right|^2 d\lambda.$$

With the above notation, we state our main result.

Theorem 3.1. For every choice of $(p_k), (t_k), (x_{k,p_k})$ and for any family of probability measures ξ_k on $X_k = \{-t_k, \dots, t_k\}$ of \mathbb{Z} , $k \in \mathbb{N}^*$, the associated generalized Ornstein transformations has almost surely singular spectrum, i.e.

$$\mathbb{P}\{\omega: \sigma^{(\omega)} \perp \lambda\} = 1,$$

where $\mathbb{P} \stackrel{\text{def}}{=} \bigotimes_{l=0}^{\infty} \bigotimes_{j=1}^{p_l-1} \xi_l$ is the probability measure on $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$, X_l is finite subset of \mathbb{Z} .

Before we proceed to the proof, we remark that it is an easy exercise to see that the spectrum of Ornstein’s transformation is always singular if the cutting parameter p_k is bounded. In fact, Klemes and Reinhold proved that if $\sum_{k=0}^{\infty} \frac{1}{p_k^2} = \infty$ then the associated rank one transformation is singular. Henceforth, we assume that the series $\sum_{k=0}^{\infty} \frac{1}{p_k^2}$ converges.

We shall adapt Bourgain’s proof. For that, we need a local version of the singularity criterion used by Bourgain. Let F be a Borel set then with the above notation, we will state local singularity criterion in the following form.

Theorem 3.2 ((Local Singularity Criterion)). *The following are equivalent*

- (i) $\sigma_F \perp \lambda$, where $\sigma_F = \chi_F d\sigma$, χ_F is a indicator function of F .
- (ii) $\int_F \prod_{l=1}^n |P_l(z)| d\lambda \xrightarrow{n \rightarrow \infty} 0$.
- (iii) $\inf\{\int_F \prod_{l=1}^k |P_{n_l}(z)| d\lambda, k \in \mathbb{N}, n_1 < n_2 < \dots < n_k\} = 0$.

One can adapt the proof of Theorem 4.3 in [13], or in [8,14].

Now, using Lebesgue’s dominated convergence theorem and the local singularity criterion, we obtain

Proposition 3.3. *The following are equivalent*

- (i) $\sigma_F^{(\omega)} \perp \lambda$ \mathbb{P} a.s.
- (ii) $\int_F \prod_{l=1}^n |P_l(z)| d\lambda d\mathbb{P} \xrightarrow{n \rightarrow \infty} 0$.
- (iii) $\inf\{\int_F \prod_{l=1}^k |P_{n_l}(z)| d\lambda d\mathbb{P}, k \in \mathbb{N}, n_1 < n_2 < \dots < n_k\} = 0$.

Fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \dots < n_k\}, k \in \mathbb{N}, m > n_k$ and put

$$Q(z) = \prod_{i=1}^k |P_{n_i}(z)|.$$

Following [4] (see also [12] or [9]), we have, for $m > n_k$.

Lemma 3.4.

$$\int_F Q|P_m| d\lambda \leq \frac{1}{2} \left(\int_F Q d\lambda + \int_F Q|P_m|^2 d\lambda \right) - \frac{1}{8} \left(\int_F Q||P_m|^2 - 1| d\lambda \right)^2.$$

Now, we assume that F is a closed set, it follows

Lemma 3.5.

$$\limsup_{m \rightarrow \infty} \int_F Q|P_m(z)|^2 d\lambda(z) \leq \int_F Q d\lambda(z).$$

Proof. The sequence of probability measures $|P_m(z)|^2 d\lambda(z)$ converges weakly to the Lebesgue measure. Then the lemma follows from the classical portmanteau theorem.¹ \square

From Lemmas 3.4 and 3.5 we get the following

Lemma 3.6.

$$\liminf \iint_F Q |P_m| d\lambda d\mathbb{P} \leq \iint_F Q d\lambda d\mathbb{P} - \frac{1}{8} \left(\limsup \iint_F Q |P_m|^2 - 1 d\lambda d\mathbb{P} \right)^2.$$

Clearly, we need to estimate the quantity

$$\iint_F Q \left| |P_m(z)|^2 - 1 \right| d\lambda(z) d\mathbb{P}. \tag{3.2}$$

For that, following Bourgain we shall prove the following

Proposition 3.7. *There exists an absolute constant $K > 0$ such that*

$$\limsup \iint_F Q \left| |P_m|^2 - 1 \right| d\lambda d\mathbb{P} \geq K \left(\iint_F Q d\lambda d\mathbb{P} - \liminf \iint_F Q(z) \phi_m(z) d\lambda d\mathbb{P} \right)^2,$$

where $\phi_m(z) = \left| \sum_{p=-t_m/2}^{p=t_m/2} \xi_m(p) z^p \right|^2, z \in \mathbb{T}$.

We shall give the proof of Proposition 3.7 in the following section.

4. Khintchine–Bonami inequality

Fix $z \in \mathbb{T}$ and $m \in \mathbb{N}^*$. Define τ and $(\tau_p)_{p=1}^{p_m-1}$ by

$$\begin{aligned} \tau : \mathbb{Z} &\longrightarrow \mathbb{T} \\ s &\longmapsto z^s. \end{aligned}$$

τ_p is given by $\tau_p = \tau \circ x_{m,p}, x_{m,p}$ is the p th projection on $\Omega_m = X_m^{p_m-1}$. So

$$\left| P_m(z) \right|^2 - 1 = \sum_{p \neq q, p \neq 0, q \neq 0} a_{pq} \tau_p(\omega) \overline{\tau_q(\omega)} + 2 \operatorname{Re} \left\{ \frac{1}{p_m} \sum_{p \neq 0} z^p \tau_p(\omega) \right\}. \tag{4.1}$$

Where

$$a_{pq} = \frac{z^{(p-q)(h_m+t_m)}}{p_m}.$$

The random variables $(\tau_p)_{p=1}^{p_m-1}$ are independent. Put

$$\tau_p^\circ = \tau_p - \int \tau_p d\mathbb{P}, \quad p = 1, \dots, p_m - 1 \tag{4.2}$$

and write

$$\begin{aligned} &\sum_{p \neq q, p \neq 0, q \neq 0} a_{pq} \tau_p \overline{\tau_q} \\ &= \left(\sum a_{pq} \right) \left| \int \tau_1 d\mathbb{P} \right|^2 + \sum a_{pq} \left(\left(\int \overline{\tau_1} d\mathbb{P} \right) \tau_p^\circ + \left(\int \tau_1 d\mathbb{P} \right) \overline{\tau_q^\circ} \right) + \sum a_{pq} \tau_p^\circ \overline{\tau_q^\circ}. \end{aligned} \tag{4.3}$$

¹ See for example [7].

Now, using the same arguments as J. Bourgain, let us consider a random sign $\varepsilon = \{\varepsilon_1, \dots, \varepsilon_{p_m-1}\} \in \{-1, 1\}^{p_m-1}$, and the probability space

$$Z_m = \Omega_m \times \{-1, 1\}^{p_m-1}, \quad \text{where } \Omega_m = \left\{ -\frac{t_m}{2}, \dots, \frac{t_m}{2} \right\}^{p_m-1}.$$

Taking the conditional expectation of the following quantity

$$\sum_{p \neq q, p \neq 0, q \neq 0} a_{pq} \left(\int \bar{\tau}_1 d\mathbb{P} \tau_p^\circ + \int \tau_1 d\mathbb{P} \bar{\tau}_q^\circ \right) + \sum_{p \neq q, p \neq 0, q \neq 0} a_{pq} \tau_p^\circ \bar{\tau}_q^\circ$$

with respect to the σ -algebra \mathcal{B}_ε given by the cylinders sets $A(I, x)$ where $I \subset \{1, \dots, p_m - 1\}$, $x \in \Omega_m$ and

$$A(I, x) = \prod_{i \in I} \{x_i\} \times \left\{ -\frac{t_m}{2}, \dots, \frac{t_m}{2} \right\}^{|I^c|} \times \{1\}^{|I|} \times \{-1\}^{|I^c|}$$

(I corresponds to $\varepsilon_i = 1, \forall i \in I$ and $\varepsilon_i = -1, \forall i \notin I$), in other words, taking conditional expectation with respect to the random variables τ_p for which $\varepsilon_p = 1$, one finds the following polynomial expression in ε of degree 2

$$\sum a_{pq} \left(\frac{1 + \varepsilon_p}{2} \int \bar{\tau}_1 d\mathbb{P} \tau_p^\circ + \frac{1 + \varepsilon_q}{2} \int \tau_1 d\mathbb{P} \bar{\tau}_q^\circ \right) + \sum a_{pq} \frac{1 + \varepsilon_p}{2} \frac{1 + \varepsilon_q}{2} \tau_p^\circ \bar{\tau}_q^\circ. \tag{4.4}$$

So

$$\int ||P_m(z)|^2 - 1| d\mathbb{P} = \iint \mathbb{E}(|P_m(z)|^2 - 1|_{\mathcal{B}_\varepsilon}) d\mathbb{P} d\varepsilon \geq \iint |\mathbb{E}(|P_m(z)|^2 - 1|_{\mathcal{B}_\varepsilon})| d\mathbb{P} d\varepsilon. \tag{4.5}$$

It follows, by the Khintchine–Bonami inequality,² [3], that there exists a positive constant K such that

$$\begin{aligned} \iint |\mathbb{E}(|P_m(z)|^2 - 1|_{\mathcal{B}_\varepsilon})| d\mathbb{P} d\varepsilon &\geq K \int \left(\int |\mathbb{E}(|P_m(z)|^2 - 1|_{\mathcal{B}_\varepsilon})|^2 d\varepsilon \right)^{1/2} d\mathbb{P} \\ &= K \int \left(\sum_{p \neq q} |a_{pq}(z) \tau_p^\circ(z) \bar{\tau}_q^\circ(z)|^2 \right)^{1/2} d\mathbb{P}. \end{aligned}$$

But all these random variables are bounded by 2. Hence

$$\begin{aligned} \int ||P_m(z)|^2 - 1| d\mathbb{P} &\geq K' \int \frac{1}{p_m^2} \sum |\tau_p^\circ(z) \bar{\tau}_q^\circ(z)|^2 d\mathbb{P} \\ &= K' \frac{1}{p_m^2} \sum \left(\int |\tau_p^\circ(z)|^2 d\mathbb{P} \right)^2 \\ &= K' \frac{(p_m - 1)(p_m - 2)}{p_m^2} \left(\int |\tau_1^\circ(z)|^2 d\mathbb{P} \right)^2. \end{aligned} \tag{4.6}$$

Since

$$\int |\tau_1^\circ(z)|^2 d\mathbb{P} = \text{var}(\tau_1(z)) = 1 - \left| \sum_{s=-t_m/2}^{t_m/2} \xi_m(s) z^s \right|^2 = 1 - \phi_m(z). \tag{4.7}$$

Now, combined (4.6) with (4.7) to obtain

$$\int ||P_m(z)|^2 - 1| d\mathbb{P} \geq K' \frac{(p_m - 1)(p_m - 2)}{p_m^2} \left(1 - \left| \sum_{s=-t_m/2}^{t_m/2} \xi_m(s) z^s \right|^2 \right)^2. \tag{4.8}$$

² One can extend easily this inequality to bounded sequences of independent real random variables, with vanishing expectation.

Finally, multiply (4.8) by

$$\int \prod_{j \in \mathcal{N}} |P_j(z)| \, d\mathbb{P}. \tag{4.9}$$

Using the independence of (4.9) and $|1 - |P_m(z)||^2$. Integrating over F with respect to the Lebesgue measure to get

$$\iint_{\Omega F} Q ||P_m|^2 - 1| \, d\mathbb{P} \, d\lambda \geq K' \frac{(P_m - 1)(P_m - 2)}{P_m^2} \iint_{\Omega F} (1 - \phi_m(t)) \, d\mathbb{P} \, d\lambda. \tag{4.10}$$

Apply Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \iint_{\Omega F} Q(1 - \phi_m(z)) \, d\lambda \, d\mathbb{P} &\leq \left(\iint_{\Omega F} Q \, d\lambda \, d\mathbb{P} \right)^{1/2} \left(\iint_{\Omega F} Q(1 - \phi_m(z))^2 \, d\lambda \, d\mathbb{P} \right)^{1/2} \\ &\leq \left(\iint_{\Omega F} Q(1 - \phi_m(z))^2 \, d\lambda \, d\mathbb{P} \right)^{1/2}. \end{aligned} \tag{4.11}$$

Combine (4.10) and (4.11) and take \liminf to finish the proof of Proposition 3.7.

Now, passing to a subsequence we may assume that ϕ_m converge weakly in $L^2(\lambda)$ to some function ϕ in $L^2(\lambda)$. Then,

$$\hat{\phi}(n) = \lim_{m \rightarrow \infty} \widehat{\phi}_m(n) \geq 0 \quad \text{for any } n \in \mathbb{Z},$$

and

$$\sum_n \hat{\phi}(n) \leq 1.$$

Hence, the Fourier series of ϕ converges absolutely. We still denoted by

$$\phi(z) = \sum_n \hat{\phi}(n) z^n.$$

In particular ϕ is a continuous function. We deduce that the set $\{\phi(z) = 1\}$ is either the torus or a finite subgroup of the torus.

Remark 4.1. It is any easy exercise to see that if the set $\{\phi = 1\}$ is not a null set with respect to Lebesgue measure then, for any $z \in \mathbb{T}$, $\phi(z) = 1$. In fact,

$$\phi(z) = 1 \text{ everywhere} \quad \text{iff} \quad \widehat{\phi}_m(0) = \sum \xi_m(s)^2 \xrightarrow{m \rightarrow \infty} 1 \text{ hence} \quad \text{iff} \quad \max_{s \in X_m} \xi_m(s) \xrightarrow{m \rightarrow \infty} 1.$$

5. On the Ornstein probability space for which $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$

In this section, we assume that $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$. So, we may choose a subsequence of ϕ_m which converge in weak* to ϕ . Let $\varepsilon > 0$,

$$F_\varepsilon = \{z \in \mathbb{T}: 1 - \phi(z) \geq \varepsilon\}.$$

F_ε is a closed set and we have also the following proposition

Proposition 5.1. *There exists an absolute constant $K > 0$ such that*

$$\underline{\lim} \iint_{F_\varepsilon} Q ||P_m(z)|^2 - 1| \, d\lambda \, d\mathbb{P} \geq K \varepsilon^2 \left(\iint_{F_\varepsilon} Q \, d\lambda \, d\mathbb{P} \right)^2.$$

Proof. Apply Proposition 3.7 to get that there exists an constant $K > 0$ for which

$$\begin{aligned} \liminf \int_{F_\varepsilon} Q |P_m(z)|^2 - 1 | d\lambda \mathbb{P} &\geq K \left(\int_F Q d\lambda d\mathbb{P} - \overline{\lim} \int_F Q(z) \left| \sum_{p=-t_m/2}^{p=t_m/2} \xi_m(p) z^p \right|^2 \lambda d\mathbb{P} \right)^2 \\ &\geq K \left(\int_{F_\varepsilon} Q(1 - \phi(z)) \lambda d\mathbb{P} \right)^2 \geq K \varepsilon^2 \left(\int_{F_\varepsilon} Q \lambda d\mathbb{P} \right)^2. \end{aligned}$$

The proof of the proposition is complete. \square

Proof of Theorem 3.1 in the case of $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$. Firstly, for fixed $\varepsilon > 0$, let us choose the subsequence \mathcal{N} . From Propositions 3.6. and 5.1. one can write

$$\overline{\lim} \int_{F_\varepsilon} Q |P_m(z)| d\lambda(z) d\mathbb{P} \leq \int_{F_\varepsilon} Q - \frac{1}{8} K^2 \varepsilon^4 \left(\int_{F_\varepsilon} Q d\lambda d\mathbb{P} \right)^4,$$

and from this last inequality we shall construct \mathcal{N} . In fact, suppose we have chosen the k first elements of the subsequence \mathcal{N} . We wish to define the $(k + 1)$ th element. Let $m > n_k$ such that

$$\int_{F_\varepsilon} Q |P_m(z)| d\lambda(z) d\mathbb{P} \leq \int_{F_\varepsilon} Q d\lambda d\mathbb{P} - \frac{1}{8} K^2 \varepsilon^4 \left(\int_{F_\varepsilon} Q d\lambda d\mathbb{P} \right)^4,$$

and put $n_{k+1} = m$. It follows that the elements of the subsequence \mathcal{N} verify

$$\int_{F_\varepsilon} \prod_{i=1}^{k+1} |P_{n_i}(z)| d\lambda d\mathbb{P} \leq \int_{F_\varepsilon} \prod_{i=1}^k |P_{n_i}(z)| d\lambda d\mathbb{P} - \frac{1}{8} K^2 \varepsilon^4 \left(\int_{F_\varepsilon} \prod_{i=1}^k |P_{n_i}(z)| d\lambda d\mathbb{P} \right)^4.$$

We deduce that the sequence $(\int_{F_\varepsilon} \prod_{i=1}^k |P_{n_i}(z)| d\lambda \mathbb{P})_{k \geq 1}$ is decreasing and converges to the limit l_ε which verifies

$$l_\varepsilon \leq l_\varepsilon - \frac{1}{8} K^2 \varepsilon^4 l_\varepsilon^4,$$

and this implies that $l_\varepsilon = 0$. Hence, $\sigma_{F_\varepsilon}^{(\omega)}$ is singular. But, $\bigsqcup_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} F_\varepsilon = \{1 - \phi \neq 0\} = \{z: \phi(z) \neq 1\}$ is a conull set with respect to the Lebesgue measure. This completes the proof of Theorem 3.1 when $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$. \square

6. On the Ornstein probability spaces for which $\lim \max_{s \in X_m} \xi_m(s) = 1$

We have the following

Lemma 6.1.

$$\limsup_{m \rightarrow \infty} \int_{F_\varepsilon} Q |P_m|^2 - 1 | d\lambda d\mathbb{P} \geq \int_{F_\varepsilon} Q d\lambda d\mathbb{P}.$$

Proof. We have

$$\int_{F_\varepsilon} Q |P_m|^2 - 1 | d\lambda d\mathbb{P} \geq \int \left| \int Q (|P_m|^2 - 1) d\mathbb{P} \right| d\lambda. \tag{6.1}$$

But, from (4.1)

$$\int (|P_m|^2 - 1) d\mathbb{P} = 2 \operatorname{Re} \left\{ (G_{p_m}(z^{h_m+t_m})) \left(\int \tau_1 d\mathbb{P} \right) \right\} + \left(F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right) \phi_m(z). \tag{6.2}$$

Where, F_p and G_p is given, for any $p \in \mathbb{N}^*$, by

$$F_p(z) = \left| \frac{1}{\sqrt{p}} \sum_{k=1}^{p-1} z^k \right|^2,$$

$$G_p(z) = \frac{1}{p} \sum_{k=1}^{p-1} z^k.$$

Combine (6.1) and (6.2) to obtain

$$\iint Q ||P_m|^2 - 1| d\lambda d\mathbb{P} \geq \iint Q \left(\left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \phi_m(z) \right| \right) d\lambda d\mathbb{P} - 2 \iint Q |G_{p_m}(z^{h_m+t_m})| \left| \int \tau_1 d\mathbb{P} \right| d\lambda d\mathbb{P}. \tag{6.3}$$

But, on one hand, we have

$$\int Q |G_{p_m}(z^{h_m+t_m})| \left| \int \tau_1 d\mathbb{P} \right| d\lambda \leq \left(\int |G_{p_m}(z^{h_m+t_m})|^2 d\lambda \right)^{1/2} \left(\int Q^2 d\lambda \right)^{1/2} \leq \left(\int |G_{p_m}(z^{h_m+t_m})|^2 d\lambda \right)^{1/2} = \frac{1}{\sqrt{p_m}} \xrightarrow{m \rightarrow \infty} 0.$$

On the other hand, since $|X_m| \leq t_m, \sum_{k \in X_m} (\xi_m\{k\})^2 \xrightarrow{m \rightarrow \infty} 1$ and for any $f \in L^1$, we have

$$\widehat{f_{(m)}}(n) = \begin{cases} 0 & \text{if } n \text{ is not divisible by } m, \\ \widehat{f}\left(\frac{n}{m}\right) & \text{otherwise.} \end{cases}$$

Where $f_{(m)}(z) = f(z^m)$, we get that

$$\left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right| \left| \sum_{k \in X_m} \xi_m(k) z^k \right|^2 d\lambda$$

converge to $K \cdot \lambda$, with $K \geq 1$. In fact

$$\begin{aligned} \int \left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right| \left| \sum_{k \in X_m} \xi_m(k) z^k \right|^2 d\lambda &= \sum_{k \in X_m} (\xi_m\{k\})^2 \int \left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right| d\lambda \\ &\geq \sum_{k \in X_m} (\xi_m\{k\})^2 \int \left(F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right) z^{h_m+t_m} d\lambda \\ &= \sum_{k \in X_m} (\xi_m\{k\})^2 \left(\frac{p_m - 2}{p_m} \right) \xrightarrow{m \rightarrow \infty} 1 \end{aligned}$$

and the lemma follows from (6.3). \square

Proof of Theorem 3.1 in the case of $\lim \max_{s \in X_m} \xi_m(s) = 1$. As in the case of $\lim \max_{s \in X_m} \xi_m(s) < 1$, we use the Lemma 6.1 to establish that

$$\lim_{n \rightarrow \infty} \int \prod_{k=1}^n |P_k(z)| d\lambda d\mathbb{P} = 0$$

and the proof of Theorem 3.1. is complete. \square

Remark 6.2. It is an easy exercise to apply our machinery to the Rudolph construction in [18].

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