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Periodic solutions of second order Hamiltonian systems bifurcating from infinity

Les solutions périodiques, émanants de l'infini des systèmes hamiltoniens autonomes de second ordre

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Abstract

The goal of this article is to study closed connected sets of periodic solutions, of autonomous second order Hamiltonian systems, emanating from infinity. The main idea is to apply the degree for SO(2)-equivariant gradient operators defined by the second author in [S. Rybicki, SO(2)-degree for orthogonal maps and its applications to bifurcation theory, Nonlinear Anal. TMA 23 (1) (1994) 83–102]. Using the results due to Rabier [P. Rabier, Symmetries, topological degree and a theorem of Z.Q. Wang, Rocky Mountain J. Math. 24 (3) (1994) 1087–1115] we show that we cannot apply the Leray–Schauder degree to prove the main results of this article. It is worth pointing out that since we study connected sets of solutions, we also cannot use the Conley index technique and the Morse theory.

Résumé

Le but de cet article est l'étude des ensembles fermés et connexes de solutions périodiques, émanant de l'infini, des systèmes hamiltoniens autonomes de second ordre. L'idée principale consiste à appliquer le degré aux opérateurs de gradient SO(2)-équivariants définis par le second auteur dans [S. Rybicki, SO(2)-degree for orthogonal maps and its applications to bifurcation theory, Nonlinear Anal. TMA 23 (1) (1994) 83–102]. Moyennant un résultat de Rabier [P. Rabier, Symmetries, topological degree and a theorem of Z.Q. Wang, Rocky Mountain J. Math. 24 (3) (1994) 1087–1115], on démontre que l'on ne peut pas appliquer le degré de Leray–Schauder pour obtenir le résultat principal de ce travail. Il est important de souligner que, vu que l'on étudie des ensembles connexes de solutions, ni la technique de l'indice de Conley, ni la théorie de Morse ne peuvent être appliquées ici.

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1. Introduction

Consider the following family of autonomous second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = -\nabla_u V\big(u(t), \lambda\big), \\ u(0) = u(2\pi), \\ \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{1.1}$$

where $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and the gradient $\nabla_x V$ (with respect to the first coordinate) is asymptotically linear at infinity, i.e. $\nabla_x V(x, \lambda) = A(\lambda)x + \mathrm{o}(\|x\|)$ as $\|x\| \to \infty$ uniformly on bounded λ-intervals and $A(\lambda)$ is a real symmetric matrix for every $\lambda \in \mathbb{R}$.

Our purpose is to prove sufficient conditions for the existence of closed connected sets of non-stationary 2π-periodic solutions of system (1.1) emanating from infinity. Moreover, we describe the possible minimal periods of solutions bifurcating from infinity and study the symmetry-breaking of solutions.

Bifurcations from infinity of solutions of second order ODEs have been studied among the others in [9,12,14,23,24]. The authors applied the idea of the Hopf bifurcation from infinity or the Leray–Schauder degree to study solutions of the Liénard, Rayleigh and Sturm–Liouville equations. The assumptions considered in those articles are of different nature than these in our article. For example in the case of the Hopf bifurcation they considered asymptotically linear equation of the form $\ddot{x}(t) = A(\lambda)x(t) + a(x, \lambda)$, where the matrix $A(\lambda)$ has a simple eigenvalue $\mathrm{i}\omega_0$ ($0 \neq \omega_0 \in \mathbb{R}$) at $\lambda = \lambda_0$ and $a(x, \lambda) \to 0$ as $x \to 0$ i.e. the matrix $A(\lambda_0)$ is not symmetric. Moreover, they do not obtain any estimation of minimal periods of bifurcating solutions and information about the symmetry-breaking phenomenon.

We treat solutions of system (1.1) as critical orbits of an SO(2)-invariant C^2-functional $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ whose gradient (with respect to the first coordinate) is an SO(2)-equivariant C^1-operator of the form compact perturbation of the identity.

The basic idea is to apply the degree for SO(2)-equivariant gradient maps defined and discussed in [19–22]. Our degree is an element of the tom Dieck ring $U(\mathrm{SO}(2))$, see Section 2 for the definition of this ring. The first degree for SO(2)-equivariant gradient maps, which is a rational number, is due to Dancer [5]. The degree for equivariant gradient maps in the presence of symmetries of any compact Lie group G, which is an element of the tom Dieck ring $U(G)$, is due to Gęba [8], see [6] for the definition of $U(G)$.

For other applications of the degree for SO(2)-equivariant gradient maps to Hamiltonian systems we refer the reader to [7,13,17,18].

It is worth in pointing out that application of classical invariants like the Conley index technique and the Morse theory does not ensure the existence of closed connected sets of critical points of variational problems, see [2,3,10,15,25] for examples and discussion.

Since the gradient of the functional Φ_V is of the form compact perturbation of the identity, it is natural to try to relate the degree for SO(2)-equivariant gradient maps to the Leray–Schauder degree. We are aware of theorems similar to Theorem 3.1 which have been proved for operators of the form compact perturbation of the identity (without gradient and equivariant structures), see for instance Theorem 2.6 of [11].

However the choice of the degree for SO(2)-equivariant gradient maps seems to be the best adapted to our theory. The advantage of using the degree for SO(2)-equivariant gradient maps lies in the fact that the index of an isolated nontrivial SO(2)-orbit can be a nonzero element of the tom Dieck ring $U(\mathrm{SO}(2))$. Whereas the index of this orbit computed by the Leray–Schauder degree equals $0 \in \mathbb{Z}$, see [16].

After this introduction our article is organized as follows.

In Section 2, for the convenience of the reader, we have summarized without proofs the relevant material on the degree for SO(2)-equivariant gradient maps, thus making our exposition self-contained.

Section 3 is devoted to the study of closed connected sets of critical SO(2)-orbits of asymptotically linear SO(2)-equivariant gradient maps of the form compact perturbation of the identity. Using the degree for SO(2)-equivariant gradient maps we define a bifurcation index $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \in U(\mathrm{SO}(2))$, see Definition 3.1. Nontriviality of the bifurcation index implies the existence of an unbounded closed connected set of critical SO(2)-orbits, see Theorem 3.1. If the set of stationary solutions of second order Hamiltonian system is bounded then the bifurcation index $\mathrm{Bif}_{\mathrm{LS}}(\infty, [\lambda_-, \lambda_+]) \in \mathbb{Z}$ computed by the Leray–Schauder degree is trivial. We discuss this situation in Remarks 3.1, 3.4 and Corollary 3.1. In Theorem 3.2 we indicate points at which an unbounded closed connected set of critical SO(2)-orbits meets infinity. In Lemmas 3.2, 3.3 we control the isotropy groups of SO(2)-orbits. The phenomenon of symmetry-breaking of SO(2)-orbits is discussed in Corollaries 3.5, 3.4.

In Section 4 the main results of this article are stated and proved. In this section we study closed connected sets of periodic solutions of autonomous second order Hamiltonian systems. Theorems 4.1, 4.3 are consequences of Theorems 3.1, 3.2, respectively. In these theorems we have formulated sufficient conditions for the existence of unbounded closed connected sets of 2π-periodic solutions of system (1.1). We emphasize that assumptions of these theorems are expressed directly in terms of the right hand sight of system (1.1) i.e. potential V. In Corollary 4.3 we have described the minimal periods of solutions of system (1.1) which are sufficiently close to infinity. In Theorem 4.4 we study periodic solutions of a special case of system (1.1) i.e. we assume that $V(x, \lambda) = \lambda^2 V(x)$. In this theorem we indicate all the points at which closed connected sets of periodic solutions of system (1.1) meet infinity. The minimal periods of solutions of system (1.1) are discussed in Corollary 4.5.

In Section 5 we consider three real second order Hamiltonian systems in order to illustrate the main results of this paper.

2. Preliminaries

In this section, for the convenience of the reader, we remind the main properties of the degree for SO(2)-equivariant gradient maps defined in [19]. This degree will be denoted briefly by $\nabla_{\mathrm{SO}(2)}$-deg to underline that it is a special degree theory for SO(2)-equivariant gradient maps.

Put $U(\mathrm{SO}(2)) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}$ and define the actions

$$+, \star : U(\mathrm{SO}(2)) \times U(\mathrm{SO}(2)) \to U(\mathrm{SO}(2)),$$
$$\cdot : \mathbb{Z} \times U(\mathrm{SO}(2)) \to U(\mathrm{SO}(2)),$$

as follows

$$\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k, \ldots), \tag{2.1}$$
$$\alpha \star \beta = (\alpha_0\beta_0, \alpha_0\beta_1 + \beta_0\alpha_1, \ldots, \alpha_0\beta_k + \beta_0\alpha_k, \ldots), \tag{2.2}$$
$$\gamma \cdot \alpha = (\gamma\alpha_0, \gamma\alpha_1, \ldots, \gamma\alpha_k, \ldots), \tag{2.3}$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_k, \ldots) \in U(\mathrm{SO}(2))$ and $\gamma \in \mathbb{Z}$. It is easy to check that $(U(\mathrm{SO}(2)), +, \star)$ is a commutative ring with the trivial element $\Theta = (0, 0, \ldots) \in U(\mathrm{SO}(2))$ and the unit $\mathbb{I} = (1, 0, \ldots) \in U(\mathrm{SO}(2))$. The ring $(U(\mathrm{SO}(2)), +, \star)$ is called the tom Dieck ring of the group SO(2). For the definition of the tom Dieck ring $U(G)$, where G is any compact Lie group, we refer the reader to [6].

If $\delta_1, \ldots, \delta_q \in U(\mathrm{SO}(2))$, then we write $\prod_{j=1}^{q} \delta_j$ for $\delta_1 \star \cdots \star \delta_q$. Moreover, it is understood that $\prod_{j\in\emptyset} \delta_j = \mathbb{I} \in U(\mathrm{SO}(2))$.

A representation of the group SO(2) (an SO(2)-representation) is a pair $\mathbb{V} = (\mathbb{V}_0, \rho)$, where $\mathbb{V}_0$ is a real, linear space and $\rho : \mathrm{SO}(2) \to \mathrm{GL}(\mathbb{V}_0)$ is a continuous homomorphism into the group of all linear automorphisms of $\mathbb{V}_0$. Notice that if $\mathbb{V} = (\mathbb{V}_0, \rho)$ is an SO(2)-representation, then letting $gv = \rho(g)(v)$ we obtain a linear SO(2)-action on $\mathbb{V}_0$. For simplicity of notation, we do not distinguish between $\mathbb{V}$ and $\mathbb{V}_0$ using the same letter $\mathbb{V}$ for a representation and the corresponding linear space $\mathbb{V}_0$.

Let $\mathbb{V}$ be a real, finite-dimensional and orthogonal SO(2)-representation. If $v \in \mathbb{V}$ then the subgroup $\mathrm{SO}(2)_v = \{g \in \mathrm{SO}(2)\colon\ gv = v\}$ is said to be the isotropy group of $v \in \mathbb{V}$. Moreover, the set $\mathrm{SO}(2)v = \{gv\colon\ g \in \mathrm{SO}(2)\}$ is called the SO(2)-orbit of $v \in \mathbb{V}$.

Let $\Omega \subset \mathbb{V}$ be an open, bounded and an SO(2)-invariant subset and let $H \subset \mathrm{SO}(2)$ be a closed subgroup. Then we define

- $\Omega^H = \{v \in \Omega\colon\ H \subset \mathrm{SO}(2)_v\} = \{v \in \Omega\colon\ gv = v\ \forall g \in H\}$,
- $\Omega_H = \{v \in \Omega\colon\ H = \mathrm{SO}(2)_v\}$.

Fix $k \in \mathbb{N}$ and set $C^k_{SO(2)}(\mathbb{V}, \mathbb{R}) = \{f \in C^k(\mathbb{V}, \mathbb{R}) : f \text{ is SO(2)-invariant}\}$.

Let $f_0 \in C^1_{SO(2)}(\mathbb{V}, \mathbb{R})$. Since $\mathbb{V}$ is an orthogonal SO(2)-representation, the gradient $\nabla f_0 : \mathbb{V} \to \mathbb{V}$ is an SO(2)-equivariant C^0-map. If $H \subset SO(2)$ is a closed subgroup then $\mathbb{V}^H$ is a finite-dimensional SO(2)-representation. If $(\nabla f_0)^H = \nabla f_{0|\mathbb{V}^H}$ and $f_0^H = f_{0|\mathbb{V}^H}$ then it is easy to verify that $(\nabla f_0)^H = \nabla(f_0^H) : \mathbb{V}^H \to \mathbb{V}^H$ is well-defined SO(2)-equivariant gradient map.

Choose an open, bounded and SO(2)-invariant subset $\Omega \subset \mathbb{V}$ such that $(\nabla f_0)^{-1}(0) \cap \partial\Omega = \emptyset$. Under these assumptions we have defined in [19] the degree for SO(2)-equivariant gradient maps $\nabla_{SO(2)}\text{-deg}(\nabla f_0, \Omega) \in U(SO(2))$ with coordinates

$$\nabla_{SO(2)}\text{-deg}(\nabla f_0, \Omega) = \big(\nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla f_0, \Omega), \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_1}(\nabla f_0, \Omega), \ldots, \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}(\nabla f_0, \Omega), \ldots\big).$$

Remark 2.1. To define the degree for SO(2)-equivariant gradient maps of ∇f_0 we choose (in a homotopy class of the SO(2)-equivariant gradient map ∇f_0) a "sufficiently good" SO(2)-equivariant gradient map ∇f_1 and define this degree for ∇f_1. The definition does not depend on the choice of the map ∇f_1. Roughly speaking the main steps of the definition of the degree for SO(2)-equivariant gradient maps of $\nabla f_0 : (\text{cl}(\Omega), \partial\Omega) \to (\mathbb{V}, \mathbb{V} \setminus \{0\})$ are the following:

Step 1. There is a potential $f \in C^1_{SO(2)}(\mathbb{V} \times [0,1], \mathbb{R})$ such that

(a1) $(\nabla_v f)^{-1}(0) \cap (\partial\Omega \times [0,1]) = \emptyset$,
(a2) $\nabla_v f(\cdot, 0) = \nabla f_0(\cdot)$,
(a3) $\nabla_v f_1 \in C^1_{SO(2)}(\mathbb{V}, \mathbb{V})$, where we abbreviate $\nabla_v f(\cdot, 1)$ to $\nabla_v f_1$,
(a4) $(\nabla_v f_1)^{-1}(0) \cap \Omega^{SO(2)} = \{v_1, \ldots, v_p\}$ and
 (i) $\det \nabla^2_{vv} f_1(v_j) \neq 0$, for all $j = 1, \ldots, p$,
 (ii) $$\nabla^2_{vv} f_1(v_j) = \begin{bmatrix} \nabla^2_{vv}\big(f_1^{SO(2)}\big)(v_j) & 0 \\ 0 & \mathrm{Id} \end{bmatrix} : \begin{matrix} \mathbb{V}^{SO(2)} \\ \oplus \\ \big(\mathbb{V}^{SO(2)}\big)^\perp \end{matrix} \longrightarrow \begin{matrix} \mathbb{V}^{SO(2)} \\ \oplus \\ \big(\mathbb{V}^{SO(2)}\big)^\perp \end{matrix} \quad \text{for all } j = 1, \ldots, p,$$
(a5) $(\nabla_v f_1)^{-1}(0) \cap (\Omega \setminus \Omega^{SO(2)}) = \{SO(2)w_1, \ldots, SO(2)w_q\}$ and
 (i) $\dim\ker \nabla^2_{vv} f_1(w_j) = 1$, for all $j = 1, \ldots, q$,
 (ii) $$\nabla^2_{vv} f_1(w_j) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_j & 0 \\ 0 & 0 & \mathrm{Id} \end{bmatrix} :$$
$$\begin{matrix} T_{w_j}\big(SO(2)w_j\big) \\ \oplus \\ T_{w_j}\big(\mathbb{V}_{SO(2)_{w_j}}\big) \ominus T_{w_j}\big(SO(2)w_j\big) \\ \oplus \\ \big(T_{w_j}(\mathbb{V}_{SO(2)_{w_j}})\big)^\perp \end{matrix} \longrightarrow \begin{matrix} T_{w_j}\big(SO(2)w_j\big) \\ \oplus \\ T_{w_j}\big(\mathbb{V}_{SO(2)_{w_j}}\big) \ominus T_{w_j}\big(SO(2)w_j\big) \\ \oplus \\ \big(T_{w_j}(\mathbb{V}_{SO(2)_{w_j}})\big)^\perp \end{matrix}$$
 for all $j = 1, \ldots, q$.

Step 2. The first coordinate of the degree for SO(2)-equivariant gradient maps is defined by

$$\nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla f_0, \Omega) = \sum_{j=1}^{p} \operatorname{sign}\det \nabla^2_{vv}\big(f_1^{SO(2)}\big)(v_j).$$

In other words since $\nabla(f_1^{SO(2)}) = (\nabla f_1)^{SO(2)}$, we obtain

$$\nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla f_0, \Omega) = \deg_B\big((\nabla f_1)^{SO(2)}, \Omega^{SO(2)}, 0\big),$$

where $\deg_B$ denotes the Brouwer degree.

Step 3. Fix $k \in \mathbb{N}$ and define

$$\nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}(\nabla f_0, \Omega) = \sum_{\{j \in \{1,\ldots,q\}:\ SO(2)_{w_j} = \mathbb{Z}_k\}} \operatorname{sign}\det Q_j,$$

Notice that since

$$\deg_{\mathrm{B}}\big((\nabla f_1)^{\mathrm{SO}(2)}, \Omega^{\mathrm{SO}(2)}, 0\big) = \deg_{\mathrm{B}}(\nabla f_1, \Omega, 0) \quad \text{and} \quad \deg_{\mathrm{B}}(\nabla f_1, \Omega, 0) = \deg_{\mathrm{B}}(\nabla f_0, \Omega, 0)$$

(see [16]), directly by Step 2. we obtain $\nabla_{\mathrm{SO}(2)}\text{-}\deg_{\mathrm{SO}(2)}(\nabla f_0, \Omega) = \deg_{\mathrm{B}}(\nabla f_0, \Omega, 0)$. Moreover, immediately from Step 3. we obtain that if $k \in \mathbb{N}$ and $\mathrm{SO}(2)_v \neq \mathbb{Z}_k$ for every $v \in \Omega$, then $\nabla_{\mathrm{SO}(2)}\text{-}\deg_{\mathbb{Z}_k}(\nabla f_0, \Omega) = 0$.

For $\gamma > 0$ and $v_0 \in \mathbb{V}^{\mathrm{SO}(2)}$ we put $B_\gamma(\mathbb{V}, v_0) = \{v \in \mathbb{V}\colon |v - v_0| < \gamma\}$. For simplicity of notation, we write $B_\gamma(\mathbb{V})$ instead of $B_\gamma(\mathbb{V}, 0)$. In the following theorem we formulate the main properties of the degree for SO(2)-equivariant gradient maps.

Theorem 2.1. [19] *Under the above assumptions the degree for* SO(2)*-equivariant gradient maps has the following properties*

(1) *if* $\nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega) \neq \Theta$, *then* $(\nabla f)^{-1}(0) \cap \Omega \neq \emptyset$,
(2) *if* $\nabla_{\mathrm{SO}(2)}\text{-}\deg_H(\nabla f, \Omega) \neq 0$, *then* $(\nabla f)^{-1}(0) \cap \Omega^H \neq \emptyset$,
(3) *if* $\Omega = \Omega_0 \cup \Omega_1$ *and* $\Omega_0 \cap \Omega_1 = \emptyset$, *then*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega) = \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega_0) + \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega_1),$$

(4) *if* $\Omega_0 \subset \Omega$ *is an open* SO(2)*-invariant subset and* $(\nabla f)^{-1}(0) \cap \Omega \subset \Omega_0$, *then*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega) = \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega_0),$$

(5) *if* $f \in C^1_{\mathrm{SO}(2)}(\mathbb{V} \times [0,1], \mathbb{R})$ *is such that* $(\nabla_v f)^{-1}(0) \cap (\partial\Omega \times [0,1]) = \emptyset$, *then*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f_0, \Omega) = \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f_1, \Omega),$$

(6) *if* $\mathbb{W}$ *is an orthogonal* SO(2)*-representation, then*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg\big((\nabla f, \mathrm{Id}), \Omega \times B_\gamma(\mathbb{W})\big) = \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f, \Omega),$$

(7) *if* $f \in C^2_{\mathrm{SO}(2)}(\mathbb{V}, \mathbb{R})$ *is such that* $\nabla f(0) = 0$ *and* $\nabla^2 f(0)$ *is an* SO(2)*-equivariant self-adjoint isomorphism then there is* $\gamma > 0$ *such that*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg\big(\nabla f, B_\gamma(\mathbb{V})\big) = \nabla_{\mathrm{SO}(2)}\text{-}\deg\big(\nabla^2 f(0), B_\gamma(\mathbb{V})\big).$$

Below we formulate the product formula for the degree for SO(2)-equivariant gradient maps.

Theorem 2.2. [20] *Let* $\Omega_i \subset \mathbb{V}_i$ *be an open, bounded and* SO(2)*-invariant subset of a finite-dimensional, orthogonal* SO(2)*-representation* $\mathbb{V}_i$, *for* $i = 1, 2$. *Let* $f_i \in C^1_{\mathrm{SO}(2)}(\mathbb{V}_i, \mathbb{R})$ *be such that* $(\nabla f_i)^{-1}(0) \cap \partial\Omega_i = \emptyset$, *for* $i = 1, 2$. *Then*

$$\nabla_{\mathrm{SO}(2)}\text{-}\deg\big((\nabla f_1, \nabla f_2), \Omega_1 \times \Omega_2\big) = \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f_1, \Omega_1) \star \nabla_{\mathrm{SO}(2)}\text{-}\deg(\nabla f_2, \Omega_2).$$

For $k \in \mathbb{N}$ define a map $\rho^k : \mathrm{SO}(2) \to \mathrm{GL}(2, \mathbb{R})$ as follows

$$\rho^k\big(\mathrm{e}^{\mathrm{i}\theta}\big) = \begin{bmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{bmatrix}, \quad 0 \leqslant \theta < 2\pi.$$

For $j, k \in \mathbb{N}$ we denote by $\mathbb{R}[j,k]$ the direct sum of j copies of $(\mathbb{R}^2, \rho^k)$, we also denote by $\mathbb{R}[j,0]$ the trivial j-dimensional SO(2)-representation. We say that two SO(2)-representations $\mathbb{V}$ and $\mathbb{W}$ are equivalent if there exists an SO(2)-equivariant, linear isomorphism $T : \mathbb{V} \to \mathbb{W}$. The following classic result gives complete classification (up to equivalence) of finite-dimensional representations of the group SO(2) (see [1]).

Theorem 2.3. [1] *If* $\mathbb{V}$ *is a finite-dimensional* SO(2)*-representation, then there exist finite sequences* $\{j_i\}$, $\{k_i\}$ *satisfying*:

$$k_i \in \{0\} \cup \mathbb{N}, \quad j_i \in \mathbb{N}, \ 1 \leqslant i \leqslant r, \ k_1 < k_2 < \cdots < k_r, \tag{$*$}$$

such that $\mathbb{V}$ *is equivalent to* $\bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$. *Moreover, the equivalence class of* $\mathbb{V}$ ($\mathbb{V} \approx \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$) *is uniquely determined by* $\{k_i\}$, $\{j_i\}$ *satisfying* $(*)$.

We will denote by $m^-(L)$ the Morse index of a symmetric matrix L i.e. the sum of algebraic multiplicities of negative eigenvalues of L.

To apply successfully any degree theory we need computational formulas for this invariant. Below we show how to compute the degree for SO(2)-equivariant gradient maps of a linear, self-adjoint, SO(2)-equivariant isomorphism.

Lemma 2.1. [19] *If* $\mathbb{V} \approx \mathbb{R}[j_0, 0] \oplus \mathbb{R}[j_1, k_1] \oplus \cdots \oplus \mathbb{R}[j_r, k_r]$, $L : \mathbb{V} \to \mathbb{V}$ *is a self-adjoint,* SO(2)*-equivariant, linear isomorphism and* $\gamma > 0$ *then*

(1) $L = \operatorname{diag}(L_0, L_1, \dots, L_r)$,

(2) $$\nabla_{SO(2)}\text{-}\deg_H\big(L, B_\gamma(\mathbb{V})\big) = \begin{cases} (-1)^{m^-(L_0)}, & \text{for } H = SO(2), \\ (-1)^{m^-(L_0)} \cdot \dfrac{m^-(L_i)}{2}, & \text{for } H = \mathbb{Z}_{k_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{k_1}, \dots, \mathbb{Z}_{k_r}\}, \end{cases}$$

(3) *in particular, if* $L = -\mathrm{Id}$*, then*
$$\nabla_{SO(2)}\text{-}\deg_H\big(-\mathrm{Id}, B_\gamma(\mathbb{V})\big) = \begin{cases} (-1)^{j_0}, & \text{for } H = SO(2), \\ (-1)^{j_0} \cdot j_i, & \text{for } H = \mathbb{Z}_{k_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{k_1}, \dots, \mathbb{Z}_{k_r}\}. \end{cases}$$

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be an infinite-dimensional, separable Hilbert space which is an orthogonal SO(2)-representation and let $C^k_{SO(2)}(\mathbb{H}, \mathbb{R})$ denote the set of SO(2)-invariant C^k-functionals. Fix $\Phi \in C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ such that $\nabla\Phi(u) = u - \nabla\eta(u)$, where $\nabla\eta : \mathbb{H} \to \mathbb{H}$ is an SO(2)-equivariant compact operator. Let $\mathcal{U} \subset \mathbb{H}$ be an open, bounded and SO(2)-invariant set such that $(\nabla\Phi)^{-1}(0) \cap \partial\mathcal{U} = \emptyset$. In this situation $\nabla_{SO(2)}\text{-}\deg(\mathrm{Id} - \nabla\eta, \mathcal{U}) \in U(SO(2))$ is well-defined, see [19] for details and basic properties of this degree.

Remark 2.2. We would like to underline that the infinite-dimensional version of the degree for SO(2)-equivariant gradient maps has the following two important properties

(1) $\nabla_{SO(2)}\text{-}\deg_{SO(2)}(\nabla\Phi, \mathcal{U}) = \deg_{LS}(\nabla\Phi, \mathcal{U}, 0)$, where $\deg_{LS}$ denotes the Leray–Schauder degree,
(2) if $\Phi \in C^1_{SO(2)}(\mathbb{H} \times [\lambda_-, \lambda_+], \mathbb{R})$, $Q \subset \mathbb{H} \times [\lambda_-, \lambda_+]$ is an open bounded SO(2)-invariant subset and there is $\gamma > 0$ such that
  (a) $Q \cap (\mathbb{H} \times \{\lambda_-, \lambda_+\}) = B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$,
  (b) $(\nabla_u\Phi)^{-1}(0) \cap \partial Q \subset B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$, then
$$\nabla_{SO(2)}\text{-}\deg\big(\nabla_u\Phi(\cdot, \lambda_+), Q_{\lambda_+}\big) = \nabla_{SO(2)}\text{-}\deg\big(\nabla_u\Phi(\cdot, \lambda_-), Q_{\lambda_-}\big), \quad \text{where } Q_{\lambda_\pm} = \big\{(u, \lambda_\pm) \in Q\big\}.$$

The second property is a slight generalization of the homotopy invariance of the degree for SO(2)-equivariant gradient maps and is called the generalized homotopy invariance.

Let $L : \mathbb{H} \to \mathbb{H}$ be a linear, bounded, self-adjoint, SO(2)-equivariant operator with spectrum $\sigma(L) = \{\lambda_i\}$. By $\mathbb{V}_L(\lambda_i)$ we will denote the eigenspace of L corresponding to the eigenvalue λ_i and we put $\mu_L(\lambda_i) = \dim \mathbb{V}_L(\lambda_i)$. In other words $\mu_L(\lambda_i)$ is the multiplicity of the eigenvalue λ_i. Since operator L is linear, bounded, self-adjoint, and SO(2)-equivariant, $\mathbb{V}_L(\lambda_i)$ is a finite-dimensional, orthogonal SO(2)-representation. For $\gamma > 0$ and $v_0 \in \mathbb{H}^{SO(2)}$ set $B_\gamma(\mathbb{H}, u_0) = \{u \in \mathbb{H}\colon \|u - u_0\| < \gamma\}$. For abbreviation, let $B_\gamma(\mathbb{H})$ stand for $B_\gamma(\mathbb{H}, 0)$. Note that $B_\gamma(\mathbb{H}, u_0)$ is open and SO(2)-invariant for every $u_0 \in \mathbb{H}^{SO(2)}$.

Combining Theorem 4.5 in [19] with Theorem 2.2 we obtain the following theorem.

Theorem 2.4. *Under the above assumptions if* $1 \notin \sigma(L)$*, then*
$$\nabla_{SO(2)}\text{-}\deg\big(\mathrm{Id} - L, B_\gamma(\mathbb{H})\big) = \prod_{\lambda_i > 1} \nabla_{SO(2)}\text{-}\deg\big(-\mathrm{Id}, B_\gamma\big(\mathbb{V}_L(\lambda_i)\big)\big) \in U\big(SO(2)\big).$$
It is understood that if $\sigma(L) \cap [1, +\infty) = \emptyset$*, then*
$$\nabla_{SO(2)}\text{-}\deg\big(\mathrm{Id} - L, B_\gamma(\mathbb{H})\big) = \mathbb{I} \in U\big(SO(2)\big).$$

3. Abstract results

In this section we study global bifurcation from infinity of critical orbits of SO(2)-invariant functionals.

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be as in the previous section. We consider $\mathbb{H}\times\mathbb{R}$ as an SO(2)-representation with SO(2)-action given by $g(u,\lambda) = (gu,\lambda)$, where $(u,\lambda)\in\mathbb{H}\times\mathbb{R}$ and $g\in SO(2)$.

Put $C^k_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R}) = \{\Phi\in C^k(\mathbb{H}\times\mathbb{R},\mathbb{R})\colon \Phi \text{ is } SO(2)\text{-invariant}\}$. It is clear that if $\Phi\in C^k_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$, then the gradient $\nabla_u\Phi:\mathbb{H}\times\mathbb{R}\to\mathbb{H}$ is an SO(2)-equivariant C^{k-1}-operator.

Consider a potential $\Phi\in C^2_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ such that:

$$\text{(c1)}\quad \Phi(u,\lambda) = \frac{1}{2}\langle u,u\rangle_{\mathbb{H}} - g(u,\lambda),\quad \text{where } \nabla_u g:\mathbb{H}\times\mathbb{R}\to\mathbb{H} \text{ is compact.}$$

From now on we study solutions of the following system

$$\nabla_u\Phi(u,\lambda) = 0. \tag{3.1}$$

The set $(\nabla_u\Phi)^{-1}(0)\cap(\mathbb{H}^{SO(2)}\times\mathbb{R})$ is called the set of trivial solutions of Eq. (3.1). Put

$$\mathcal{N}(\nabla_u\Phi) = \big\{(u,\lambda)\in\big(\mathbb{H}\setminus\mathbb{H}^{SO(2)}\big)\times\mathbb{R}\colon\ \nabla_u\Phi(u,\lambda)=0\big\}.$$

Assume that there exist $\lambda_-,\lambda_+>0$ and $\gamma>0$ such that

$$\big(\nabla_u\Phi(\cdot,\lambda_\pm)\big)^{-1}(0)\cap\big(\big(\mathbb{H}\setminus B_\gamma(\mathbb{H})\big)\times\{\lambda_\pm\}\big)=\emptyset. \tag{3.2}$$

Definition 3.1. An element $\mathrm{Bif}(\infty,[\lambda_-,\lambda_+])\in U(SO(2))$ defined as follows

$$\mathrm{Bif}\big(\infty,[\lambda_-,\lambda_+]\big) = \nabla_{SO(2)}\text{-deg}\big(\nabla_u\Phi(\cdot,\lambda_+), B_\gamma(\mathbb{H})\big) - \nabla_{SO(2)}\text{-deg}\big(\nabla_u\Phi(\cdot,\lambda_-), B_\gamma(\mathbb{H})\big)$$

is called the bifurcation index at $(\infty,[\lambda_-,\lambda_+])$.

The following lemma will be extremely useful in the proof of the next theorem.

Lemma 3.1. [4] *Let A and B be disjoint closed subsets of a compact space K. If there is no closed connected subset of K that intersects both A and B, then there exist disjoint closed subsets K_A and K_B of K such that $A\subset K_A$, $B\subset K_B$ and $K = K_A\cup K_B$.*

The following theorem is the most general result of this section. Namely, we prove the sufficient condition for the existence of an unbounded closed connected set of critical orbits of SO(2)-invariant functionals. In the proof of this theorem we combine Lemma 3.1 with the degree for SO(2)-equivariant gradient maps.

Theorem 3.1. *Let $\Phi\in C^2_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ satisfy condition* (c1) *and let $\lambda_\pm\in\mathbb{R}$, $\gamma>0$ be such that* (3.2) *holds. If* $\mathrm{Bif}(\infty,[\lambda_-,\lambda_+])\neq\Theta\in U(SO(2))$, *then there exists an unbounded closed connected component C of* $(\nabla_u\Phi)^{-1}(0)\cap(\mathbb{H}\times[\lambda_-,\lambda_+])$ *such that* $C\cap(B_\gamma(\mathbb{H})\times\{\lambda_-,\lambda_+\})\neq\emptyset$.

Proof. First of all we claim that for every $\xi\geqslant\gamma$ there exists a closed connected component C_ξ of $\nabla_u\Phi^{-1}(0)\cap(\mathbb{H}\times[\lambda_-,\lambda_+])$ such that

$$C_\xi\cap\big(B_\gamma(\mathbb{H})\times\{\lambda_-,\lambda_+\}\big)\neq\emptyset\quad\text{and}\quad C_\xi\cap\big(\partial B_\xi(\mathbb{H})\times[\lambda_-,\lambda_+]\big)\neq\emptyset.$$

Suppose, contrary to our claim, that there exists $\xi\geqslant\gamma$ such that at least one of the following conditions is fulfilled

(i) $\mathcal{C}\cap(B_\gamma(\mathbb{H})\times\{\lambda_-,\lambda_+\})=\emptyset$,
(ii) $\mathcal{C}\cap(\partial B_\xi(\mathbb{H})\times[\lambda_-,\lambda_+])=\emptyset$,

for every closed connected component $\mathcal{C}$ of $\nabla_u\Phi^{-1}(0)\cap(\mathbb{H}\times[\lambda_-,\lambda_+])$.

Put in Lemma 3.1

(i) $K = \nabla_u \Phi^{-1}(0) \cap (\mathrm{cl}(B_\xi(\mathbb{H})) \times [\lambda_-, \lambda_+])$,
(ii) $A = \nabla_u \Phi^{-1}(0) \cap (\mathrm{cl}(B_\xi(\mathbb{H})) \times \{\lambda_-, \lambda_+\})$,
(iii) $B = \nabla_u \Phi^{-1}(0) \cap (\partial B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+])$.

Since $\nabla_u \Phi$ is of the form compact perturbation of the identity and $\mathrm{cl}(B_\xi(\mathbb{H})) \times [\lambda_-, \lambda_+]$ is closed and bounded, K is compact. Recall that $\nabla_u \Phi^{-1}(0) \cap (\mathrm{cl}(B_\xi(\mathbb{H})) \times \{\lambda_-, \lambda_+\}) \subset B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$. Thus $A \cap B = \emptyset$. By assumption, there is no closed connected subset of K that intersects both A and B. Applying Lemma 3.1, we obtain compact sets K_A, K_B with desired properties.

Choose $\alpha > 0$ such that $K_A(\alpha)$, $K_B(\alpha)$ are disjoint α-neighborhoods of the sets K_A, K_B. Define

$$\begin{aligned} Q &= \mathrm{SO}(2)\big(\big(B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+]\big) \setminus \mathrm{cl}\big(K_B(\alpha)\big)\big) \\ &= \big\{(gv, \lambda):\ v \in \big(B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+]\big) \setminus \mathrm{cl}\big(K_B(\alpha)\big) \text{ and } g \in \mathrm{SO}(2)\big\}. \end{aligned}$$

We claim that Q is open, SO(2)-invariant and $(\nabla_u \Phi)^{-1}(0) \cap \partial Q \subset B_\xi(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$. Since $B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+]$ is open in $\mathbb{H} \times [\lambda_-, \lambda_+]$, it is clear that Q is open. Moreover, since Q is a sum of SO(2)-orbits, it is SO(2)-invariant. What is left is to show that $(\nabla_u \Phi)^{-1}(0) \cap \partial Q \subset B_\xi(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$. Suppose, contrary to our claim that, $(\nabla_u \Phi)^{-1}(0) \cap (\partial Q \setminus (B_\xi(\mathbb{H}) \times \{\lambda_-, \lambda_+\})) \neq \emptyset$ and fix $(u_0, \lambda_0) \in \partial Q \setminus (B_\xi(\mathbb{H}) \times \{\lambda_-, \lambda_+\})$ such that $\nabla_u \Phi(u_0, \lambda_0) = 0$. Hence there are $(\tilde{u}_0, \lambda_0) \in \partial((B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+]) \setminus \mathrm{cl}(K_B(\alpha)))$ and $g \in \mathrm{SO}(2)$ such that $(g\tilde{u}_0, \lambda_0) = (u_0, \lambda_0)$. Since $\nabla \Phi$ is SO(2)-equivariant, we obtain

$$0 = \nabla_u \Phi(u_0, \lambda_0) = \nabla_u \Phi(g\tilde{u}_0, \lambda_0) = g \nabla_u \Phi(\tilde{u}_0, \lambda_0)$$

and consequently $\nabla_u \Phi(\tilde{u}_0, \lambda_0) = 0$, which contradicts the definition of $K_B(\alpha)$.

Put $Q_\lambda = \{(u, \lambda) \in Q\}$ for every $\lambda \in [\lambda_-, \lambda_+]$.

Since $(\nabla_u \Phi)^{-1}(0) \cap \partial Q \subset B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}$, from the generalized homotopy invariance of the degree for SO(2)-equivariant gradient maps (see Remark 2.2), we obtain that:

$$\begin{aligned} \Theta &= \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_+), Q_{\lambda_+}\big) - \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_-), Q_{\lambda_-}\big) \\ &= \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_+), B_\xi(\mathbb{H})\big) - \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_-), B_\xi(\mathbb{H})\big) \\ &= \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_+), B_\gamma(\mathbb{H})\big) - \nabla_{\mathrm{SO}(2)}\text{-deg}\big(\nabla_u \Phi(\cdot, \lambda_-), B_\gamma(\mathbb{H})\big) \\ &= \mathrm{Bif}\big(\infty, [\lambda_-, \lambda_+]\big) \neq \Theta, \end{aligned}$$

a contradiction.

Suppose, contrary to our claim that, the theorem is false i.e. every closed connected component C of $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ such that $C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$ is bounded. Choose an increasing sequence $\{\gamma_n\} \subset \mathbb{N}$ such that $\gamma_n \geqslant \gamma$ for every $n \in \mathbb{N}$. From the first part of the proof it is known that for every $n \in \mathbb{N}$ there exists a bounded closed connected component C_{γ_n} of $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ such that $C_{\gamma_n} \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$ and $C_{\gamma_n} \cap (\partial B_{\gamma_n}(\mathbb{H}) \times [\lambda_-, \lambda_+]) \neq \emptyset$. Choose $(u_n, \lambda_n) \in C_{\gamma_n} \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\})$ for every $n \in \mathbb{N}$. Without loosing of generality, one can assume that $\lambda_n = \lambda_+$ for every $n \in \mathbb{N}$. Note that $cl\{(u_n, \lambda_+)\}$ is compact, as a closed subset of the compact set $(\nabla_u \Phi)^{-1}(0) \cap (\mathrm{cl}(B_\gamma(\mathbb{H})) \times \{\lambda_+\})$. Thus, there exists convergent subsequence $(u_{n_k}, \lambda_+) \to (u_0, \lambda_+)$. Denote by C a closed connected component of $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ containing (u_0, λ_+). Since C is bounded, there is $\xi \geqslant \gamma$ such that $C \subset B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+]$.

Put in Lemma 3.1

(i) $K = \nabla_u \Phi^{-1}(0) \cap (\mathrm{cl}(B_\xi(\mathbb{H})) \times [\lambda_-, \lambda_+])$,
(ii) $A = C$,
(iii) $B = \nabla_u \Phi^{-1}(0) \cap (\partial B_\xi(\mathbb{H}) \times [\lambda_-, \lambda_+])$.

Applying Lemma 3.1, we obtain compact subsets $K_A, K_B \subset K$ such that $A \subset K_A$, $B \subset K_B$, $K_A \cap K_B = \emptyset$ and $K_A \cup K_B = K$. Note that almost all $(u_{n_k}, \lambda_+) \in K_B$. Indeed, $(u_{n_k}, \lambda_+) \in C_{\gamma_{n_k}}$ and $C_{\gamma_{n_k}} \cap (\partial B_{\gamma_{n_k}}(\mathbb{H}) \times [\lambda_-, \lambda_+]) \neq \emptyset$. Hence $(u_{n_k}, \lambda_+) \in K_B$ for all $k \in \mathbb{N}$ such that $\gamma_{n_k} \geqslant \xi$. Thus (u_0, λ_+), as the limit of elements from the closed set K_B, belongs to K_B. On the other hand, $(u_0, \lambda_+) \in A \subset K_A$ and $\mathrm{dist}(K_A, K_B) > 0$, a contradiction. We have just proved that $C \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ is unbounded. □

Remark 3.1. Since $\nabla_u \Phi(\cdot, \lambda_\pm)$ is of the form compact perturbation of the identity, one can define a bifurcation index $\mathrm{Bif}_{\mathrm{LS}}(\infty, [\lambda_-, \lambda_+]) \in \mathbb{Z}$ as follows

$$\mathrm{Bif}_{\mathrm{LS}}\big(\infty, [\lambda_-, \lambda_+]\big) = \deg_{\mathrm{LS}}\big(\nabla_u \Phi(\cdot, \lambda_+), B_\gamma(\mathbb{H}), 0\big) - \deg_{\mathrm{LS}}\big(\nabla_u \Phi(\cdot, \lambda_-), B_\gamma(\mathbb{H}), 0\big).$$

We realize that theorems similar to Theorem 3.1 has been proved for operators of the form compact perturbation of the identity (without gradient and equivariant structures), see for instance Theorem 2.6 of [11].

However directly from the definition of the degree for SO(2)-equivariant gradient maps it follows that if $\mathrm{Bif}_{\mathrm{LS}}(\infty, [\lambda_-, \lambda_+]) \neq 0 \in \mathbb{Z}$ then $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \neq \Theta \in U(\mathrm{SO}(2))$. On the other hand it can happen that $\mathrm{Bif}_{\mathrm{LS}}(\infty, [\lambda_-, \lambda_+]) = 0$ and $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \neq \Theta$.

Definition 3.2. Let $C \subset \mathbb{H} \times \mathbb{R}$ be closed and connected. We say that a symmetry breaking phenomenon for C occurs if there are $(u_0, \lambda_0) \in C$ and sequence $\{(u_n, \lambda_n)\} \subset C$ converging to (u_0, λ_0) such that $\mathrm{SO}(2)_{u_n} \neq \mathrm{SO}(2)_{u_0}$ for every $n \in \mathbb{N}$.

Corollary 3.1. *Let assumptions of Theorem* 3.1 *be satisfied. Moreover, suppose that* $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+])$ *is bounded. Then, there exists an unbounded closed connected component C of* $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ *such that the symmetry breaking phenomenon for C occurs or there exists at least one nontrivial solution of Eq.* (3.1) *such that* $(u, \lambda) \in (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \cap C$.

Proof. By Theorem 3.1 we obtain an unbounded component C of $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ such that $C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$. Since $(\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]) \cap (\nabla_u \Phi)^{-1}(0)$ is bounded, without loss of generality, one can assume that

$$(\nabla_u \Phi)^{-1}(0) \cap \big(\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]\big) \subset B_\gamma(\mathbb{H}) \times [\lambda_-, \lambda_+]. \tag{3.3}$$

Therefore the isotropy group of every element $u \in C \cap ((\mathbb{H} \setminus B_\gamma(\mathbb{H})) \times [\lambda_-, \lambda_+])$ is different from SO(2). Thus, if $C \cap (\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]) \neq \emptyset$, then the symmetry breaking phenomenon for C occurs. Otherwise $C \subset \mathcal{N}(\nabla_u \Phi)$ and $C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$, which completes the proof. □

Remark 3.2. Notice that if in Corollary 3.1 we have

$$(\nabla_u \Phi)^{-1}(0) \cap \big(B_\gamma(\mathbb{H}) \times \{\lambda_\pm\}\big) \subset \mathbb{H}^{\mathrm{SO}(2)} \times \{\lambda_\pm\},$$

then the symmetry breaking phenomenon for C occurs.

Remark 3.3. Notice that if in Corollary 3.1 we have

$$(\nabla_u \Phi)^{-1}(0) \cap \big(\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]\big) = \{u_1, \dots, u_q\} \times [\lambda_-, \lambda_+]$$

and $\nabla_u^2 \Phi(u_i, \lambda)$ is an isomorphism for every $\lambda \in [\lambda_-, \lambda_+]$, $i = 1, \dots, q$, then $C \subset \mathcal{N}(\nabla_u \Phi)$.

Remark 3.4. Under the assumptions of Corollary 3.1. Since

$$(\nabla_u \Phi)^{-1}(0) \cap \big(\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]\big)$$

is bounded, there is $\gamma > 0$ such that

$$(\nabla_u \Phi)^{-1}(0) \cap \big(\mathbb{H}^{\mathrm{SO}(2)} \times [\lambda_-, \lambda_+]\big) \subset B_\gamma(\mathbb{H}) \times [\lambda_-, \lambda_+].$$

Therefore we obtain

$$\deg_{\mathrm{LS}}\big(\big(\nabla_u \Phi(\cdot, \lambda_-)\big)^{\mathrm{SO}(2)}, B_\gamma(\mathbb{H})^{\mathrm{SO}(2)}, 0\big) = \deg_{\mathrm{LS}}\big(\big(\nabla_u \Phi(\cdot, \lambda_+)\big)^{\mathrm{SO}(2)}, B_\gamma(\mathbb{H})^{\mathrm{SO}(2)}, 0\big).$$

As a direct consequence of results due to Rabier [16] we obtain

$$\deg_{\mathrm{LS}}\big(\nabla_u \Phi(\cdot, \lambda_\pm), B_\gamma(\mathbb{H}), 0\big) = \deg_{\mathrm{LS}}\big(\big(\nabla_u \Phi(\cdot, \lambda_\pm)\big)^{\mathrm{SO}(2)}, B_\gamma(\mathbb{H})^{\mathrm{SO}(2)}, 0\big). \tag{3.4}$$

Summing up, we have obtained $\mathrm{Bif}_{\mathrm{LS}}(\infty, [\lambda_-, \lambda_+]) = 0 \in \mathbb{Z}$.

The following lemma is a parameterized extension of Corollary 3.1 of [7].

Lemma 3.2. *Let $\Phi \in C^2_{\mathrm{SO}(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ satisfy assumption* (c1). *Then for every $(u_0,\lambda_0) \in (\nabla_u\Phi)^{-1}(0)\cap(\mathbb{H}^{\mathrm{SO}(2)}\times\mathbb{R})$ there exist $\gamma>0$ such that if $(u,\lambda)\in(\nabla_u\Phi)^{-1}(0)\cap(B_\gamma(\mathbb{H},u_0))\times(\lambda-\gamma,\lambda+\gamma))$, then there exists $v\in \ker\nabla^2_u\Phi(u_0,\lambda_0)$ such that $\mathrm{SO}(2)_u=\mathrm{SO}(2)_v$.*

Proof. Since $\nabla^2_u\Phi(u_0,\lambda_0):\mathbb{H}\to\mathbb{H}$ is a self-adjoint Fredholm operator of index 0, we obtain $\mathbb{H}=\ker\nabla^2_u\Phi(u_0,\lambda_0)\oplus \operatorname{im}\nabla^2_u\Phi(u_0,\lambda_0)$. Let $\pi:\mathbb{H}\to\ker\nabla^2_u\Phi(u_0,\lambda_0)$ and $\mathrm{Id}-\pi:\mathbb{H}\to\operatorname{im}\nabla^2_u\Phi(u_0,\lambda_0)$ stand for SO(2)-equivariant orthogonal projections. Obviously

$$\nabla_u\Phi(u,\lambda)=0 \quad\Leftrightarrow\quad (\pi\circ\nabla_u\Phi)(u,\lambda)=0 \text{ and } \big((\mathrm{Id}-\pi)\circ\nabla_u\Phi\big)(u,\lambda)=0.$$

By the SO(2)-equivariant version of the implicit function theorem, we obtain that solutions of

$$\big((\mathrm{Id}-\pi)\circ\nabla_u\Phi\big)(u,\lambda)=0$$

are of the form $(v,\omega(v,\lambda),\lambda)$, where $v\in B_\gamma(\ker\nabla^2_u\Phi(u_0,\lambda_0),u_0)$, $\lambda\in(\lambda_0-\gamma,\lambda_0+\gamma)$ for sufficiently small $\gamma>0$ and $(v,\lambda)\to\omega(v,\lambda)$ is an SO(2)-equivariant C^1-mapping.

Let $(u,\lambda)\in(\nabla_u\Phi)^{-1}(0)\cap(B_\gamma(\mathbb{H},u_0))\times(\lambda_0-\gamma,\lambda_0+\gamma)$. Therefore $(u,\lambda)=(v,\omega(v,\lambda),\lambda)$. Since ω is SO(2)-equivariant, $\mathrm{SO}(2)_{(v,\lambda)}\subset\mathrm{SO}(2)_{\omega(v,\lambda)}$ and consequently

$$\mathrm{SO}(2)_u=\mathrm{SO}(2)_{(u,\lambda)}=\mathrm{SO}(2)_{(v,\omega(v,\lambda),\lambda)}=\mathrm{SO}(2)_{(v,\lambda)}\cap\mathrm{SO}(2)_{\omega(v,\lambda)}=\mathrm{SO}(2)_{(v,\lambda)}=\mathrm{SO}(2)_v. \qquad \square$$

As a direct consequence of Lemma 3.2 we obtain the following corollary.

Corollary 3.2. *Let assumptions of Theorem* 3.1 *be satisfied. Additionally, suppose that* $\ker\nabla^2_u\Phi(u,\lambda)\subset\mathbb{H}^{\mathrm{SO}(2)}$ *for every $u\in\mathbb{H}^{\mathrm{SO}(2)}$, $\lambda\in[\lambda_-,\lambda_+]$. Then,*

$$\text{either } C\subset\mathbb{H}^{\mathrm{SO}(2)}\times[\lambda_-,\lambda_+] \text{ or } C\subset\mathcal{N}(\nabla_u\Phi).$$

If moreover $(\nabla_u\Phi)^{-1}(0)\cap B_\gamma(\mathbb{H})\times\{\lambda_-,\lambda_+\}\subset\mathbb{H}^{\mathrm{SO}(2)}\times\{\lambda_-,\lambda_+\}$, then the symmetry breaking phenomenon for C does not occur.

Proof. First of all notice that the set C obtained by Theorem 3.1 is closed and connected. Suppose, contrary to our claim, that $C\cap(\mathbb{H}^{\mathrm{SO}(2)}\times[\lambda_-,\lambda_+])\neq\emptyset$ and $C\cap\mathcal{N}(\nabla_u\Phi)\neq\emptyset$. Then there exists $(u_0,\lambda_0)\in C\cap(\mathbb{H}^{\mathrm{SO}(2)}\times[\lambda_-,\lambda_+])$ such that in its any neighborhood there exists an element $(u,\lambda)\in C\cap\mathcal{N}(\nabla_u\Phi)$. Taking into account that $\mathrm{SO}(2)_u\neq\mathrm{SO}(2)_{u_0}=\mathrm{SO}(2)$, the assumption and Lemma 3.2 we obtain a contradiction. $\square$

Let us put some additional assumptions on behaviour of the functional Φ at infinity. We would like to say something more about behaviour of closed connected components of $(\nabla_u\Phi)^{-1}(0)$ at infinity. Suppose that the functional $\Phi\in C^2_{\mathrm{SO}(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ satisfies assumption (c1) and the following assumption:

(c2) $\Phi(u,\lambda)=\frac{1}{2}\langle u,u\rangle_{\mathbb{H}}-\frac{1}{2}\langle K_\infty(\lambda)u,u\rangle_{\mathbb{H}}-\eta_\infty(u,\lambda)$, where
- (i) $K_\infty(\lambda):\mathbb{H}\to\mathbb{H}$ is a linear, SO(2)-equivariant, self-adjoint, operator for every $\lambda\in\mathbb{R}$,
- (ii) the mapping $\mathbb{H}\times\mathbb{R}\ni(u,\lambda)\mapsto K_\infty(\lambda)u\in\mathbb{H}$ is compact,
- (iii) $\nabla_u\eta_\infty:\mathbb{H}\times\mathbb{R}\to\mathbb{H}$ is a SO(2)-equivariant, compact operator such that $\nabla_u\eta_\infty(u,\lambda)=\mathrm{o}(\|u\|)$, as $\|u\|\to\infty$ uniformly on bounded λ-intervals.

For $\lambda\in\mathbb{R}$ define $\nabla^2_u\Phi(\infty,\lambda)=\mathrm{Id}-K_\infty(\lambda)$. Fix arbitrary $\lambda_0\in\mathbb{R}$ and assume that $\ker\nabla^2_u\Phi(\infty,\lambda_0)\neq\{0\}$. Choose $\varepsilon>0$, define $\lambda_\pm=\lambda_0\pm\varepsilon$ and assume that the following condition is fulfilled

$$\big\{\lambda\in[\lambda_-,\lambda_+]:\ \nabla^2_u\Phi(\infty,\lambda)\text{ is not an isomorphism}\big\}=\{\lambda_0\}. \tag{3.5}$$

It is easy to see that under the above assumptions there exists $\gamma>0$ such that condition (3.2) is satisfied.

Definition 3.3. We say that an unbounded closed connected set C meets (∞, λ_0), if for every $\delta, \gamma > 0$

$$C \cap \{(\mathbb{H} \setminus B_\gamma(\mathbb{H})) \times [\lambda_0 - \delta, \lambda_0 + \delta]\} \neq \emptyset. \tag{3.6}$$

In the following theorem we localize points at which closed connected sets of solutions of Eq. (3.1) meet infinity.

Theorem 3.2. *Let potential $\Phi \in C^2_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ satisfy assumption* (c2)*. Choose $\varepsilon, \gamma > 0$, $\lambda_0, \lambda_\pm \in \mathbb{R}$ such that* (3.2) *and* (3.5) *hold true. If* $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \neq \Theta \in U(SO(2))$*, then the statement of Theorem* 3.1 *holds true. Moreover, C meets (∞, λ_0).*

Proof. The existence of an unbounded closed connected component C of $\nabla_u \Phi^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ satisfying $C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$, is a direct consequence of Theorem 3.1. It remains to prove that C meets (∞, λ_0). Note that it is sufficient to show, that condition (3.6) holds true just for large $\varrho > 0$ and small $\delta > 0$. Choose any $\delta > 0$ such that $\delta < \varepsilon$. By assumption, for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 - \delta) \cup (\lambda_0 + \delta, \lambda_0 + \varepsilon]$, $\nabla^2 \Phi(\infty, \lambda)$ is an isomorphism. Moreover, by (c2) we obtain $\nabla_u \Phi(u, \lambda) = \nabla_u^2 \Phi(\infty, \lambda) u + \nabla_u \eta_\infty(u, \lambda)$, where $\nabla_u \eta_\infty(u, \lambda) = o(\|u\|)$, as $\|u\| \to \infty$ uniformly on bounded λ-intervals i.e.

$$\forall_{\epsilon > 0}\, \exists_{R_\epsilon > 0}\, \forall_{\lambda \in [a,b] \subset \mathbb{R}}\, \forall_{u \in \mathbb{H}} \|u\| > R_\epsilon \quad \Rightarrow \quad \|\nabla_u \eta_\infty(u, \lambda)\| < \epsilon \|u\|.$$

Put $\epsilon = \|\nabla_u^2 \Phi(\infty, \lambda)^{-1}\|^{-1}/4$. Hence, for $\|u\| > R_\epsilon$, we obtain

$$\begin{aligned}\|\nabla_u \Phi(u, \lambda)\| &= \|\nabla_u^2 \Phi(\infty, \lambda) u + \nabla_u \eta_\infty(u, \lambda)\| \geqslant \|\nabla_u^2 \Phi(\infty, \lambda) u\| - \|\nabla_u \eta_\infty(u, \lambda)\| \\ &\geqslant \frac{\|\nabla_u^2 \Phi(\infty, \lambda)^{-1}\|^{-1}}{2} \|u\| - \frac{\|\nabla_u^2 \Phi(\infty, \lambda)^{-1}\|^{-1}}{4} \|u\| \\ &\geqslant \frac{\|\nabla_u^2 \Phi(\infty, \lambda)^{-1}\|^{-1}}{4} \|u\| > 0.\end{aligned}$$

Hence, for every $\varrho > R_\epsilon$,

$$C \cap \big((\mathbb{H} \setminus B_\varrho(\mathbb{H}, \infty)) \times [\lambda_0 - \varepsilon, \lambda_0 - \delta) \cup (\lambda_0 + \delta, \lambda_0 + \varepsilon]\big) = \emptyset.$$

Since C is unbounded, $C \cap (\mathbb{H} \setminus B_\varrho(\mathbb{H}, \infty)) \times [\lambda_0 - \delta, \lambda_0 + \delta] \neq \emptyset$, which completes the proof. □

The principal significance of the lemma below is that it allows one to control the isotropy groups of solutions of Eq. (3.1) sufficiently close to infinity.

Lemma 3.3. *Let $\Phi \in C^2_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ satisfy assumption* (c2)*. Then for every $\lambda_0 \in \mathbb{R}$ there exist $\gamma > 0, \delta > 0$ such that if $(u, \lambda) \in (\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H} \setminus B_\gamma(\mathbb{H})) \times [\lambda_0 - \delta, \lambda_0 + \delta]$, then there exists $v \in \ker(\mathrm{Id} - K_\infty(\lambda_0))$ such that* $SO(2)_u = SO(2)_v$.

Proof. Fix $\lambda_0 \in \mathbb{R}$. By the SO(2)-equivariant version of the implicit function theorem at infinity (see Theorem 3.2 of [7]), we obtain that solutions of $\nabla_u \Phi(u, \lambda) = 0$ in a neighborhood of (∞, λ_0) are of the form $(v, \omega(v, \lambda), \lambda)$, where $v \in \ker \nabla_u^2 \Phi(\infty, \lambda_0) \setminus \mathrm{cl}(B_\gamma(\ker \nabla_u^2 \Phi(\infty, \lambda_0)))$, $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ for some $\gamma, \delta > 0$ and the map $(v, \lambda) \to \omega(v, \lambda) \in \mathrm{im}\, \nabla_u^2 \Phi(\infty, \lambda_0)$ is an SO(2)-equivariant C^1-mapping. The rest of the proof is the same as the proof of Lemma 3.2. □

Remark 3.5. If moreover, assumptions of Theorem 3.2 are satisfied, then without loss of generality one can assume that $\delta \leqslant \varepsilon$.

Below we present some useful corollaries of Theorem 3.2. First of them is a counterpart of Corollary 3.1 at infinity, also based on Corollary 3.1 of [7].

Corollary 3.3. *Let assumptions of Theorem* 3.2 *be satisfied. Additionally suppose that* $\ker((\nabla_u^2 \Phi(\infty, \lambda_0))) \cap \mathbb{H}^{SO(2)} = \{0\}$*. Then the statement of Theorem* 3.2 *holds true. Moreover, for closed connected set C either phenomenon of symmetry breaking occurs or there exists at least one nontrivial solution of Eq.* (3.1) *such that $(u, \lambda) \in C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_-, \lambda_+\})$.*

Proof. Note that by assumption and Lemma 3.3, the isotropy group of any solution of Eq. (3.1) close to (∞, λ_0) is different from SO(2). Thus $(\mathbb{H}^{SO(2)} \times [\lambda_-, \lambda_+]) \cap (\nabla_u \Phi)^{-1}(0)$ is bounded and by Theorem 3.2 and Corollary 3.1 the proof is completed. □

Definition 3.4. Let $\mathbb{V}$ and $\mathbb{W}$ be SO(2)-representations. We say that SO(2)-representation $\mathbb{V}$ is not consistent with SO(2)-representation $\mathbb{W}$, if $SO(2)_v \neq SO(2)_w$ for every $v \in \mathbb{V} \setminus \{0\}$, $w \in \mathbb{W} \setminus \{0\}$.

Remark 3.6. SO(2)-representation $\mathbb{V} = \bigoplus_{i=1}^{p} \mathbb{R}[k_i, m_i]$ is not consistent with SO(2)-representation $\mathbb{W} = \bigoplus_{j=1}^{q} \mathbb{R}[k'_j, m'_j]$, if $\gcd(m'_{i_1}, \ldots, m'_{i_r}) \neq \gcd(m'_{j_1}, \ldots, m'_{j_s})$, for every $\{i_1, \ldots, i_r\} \subset \{1, \ldots, p\}$, $\{j_1, \ldots, j_s\} \subset \{1, \ldots, q\}$.

Corollary 3.4. *Let assumptions of Theorem* 3.2 *be satisfied. Additionally, suppose that*

(i) $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H}^{SO(2)} \times [\lambda_-, \lambda_+]) = \{u_1, \ldots, u_q\} \times [\lambda_-, \lambda_+]$,
(ii) $(\nabla_u \Phi)^{-1}(0) \cap (B_\gamma(\mathbb{H}) \times \{\lambda_\pm\}) = \{u_1, \ldots, u_q\} \times \{\lambda_\pm\}$,
(iii) $\{(u, \lambda) \in \{u_1, \ldots, u_q\} \times [\lambda_-, \lambda_+]$: $\nabla_u^2 \Phi(u, \lambda)$ *is not an isomorphism*$\} = \{(u_{i_1}, \lambda_{i_1}), \ldots, (u_{i_d}, \lambda_{i_d})\}$,
(iv) $\ker(\nabla_u^2 \Phi(u_{i_k}, \lambda_{i_k}))$ *is not consistent with* $\ker(\nabla_u^2 \Phi(\infty, \lambda_0))$ *for every* $k = 1, \ldots, d$.

Then the statement of Theorem 3.2 *holds true. Moreover, for C phenomenon of symmetry breaking occurs.*

Proof. By Theorem 3.2 we obtain an unbounded closed connected component C of $(\nabla \Phi)^{-1}(0) \cap (\mathbb{H} \times [\lambda_-, \lambda_+])$ such that $C \cap (B_\gamma(\mathbb{H}) \times \{\lambda_\pm\}) \neq \emptyset$. From assumption (ii) it follows that $C \cap (\{u_1, \ldots, u_k\} \times [\lambda_-, \lambda_+]) \neq \emptyset$. Moreover, by assumption (iii) we obtain $C \cap (\{u_1, \ldots, u_k\} \times [\lambda_-, \lambda_+]) \subset \{(u_{i_1}, \lambda_{i_1}), \ldots, (u_{i_d}, \lambda_{i_d})\}$. The rest of the proof is a direct consequence of assumption (iv) and Lemmas 3.2, 3.3. □

One can also proof the following slight generalization of Corollary 3.4. Since the proof of the following corollary is similar to the proof of Corollary 3.4 we omit it.

Corollary 3.5. *Let assumptions of Theorem* 3.1 *be satisfied. Additionally, suppose that*

(i) $(\nabla_u \Phi)^{-1}(0) \cap (\mathbb{H}^{SO(2)} \times [\lambda_-, \lambda_+]) = \bigcup_{j=1}^{q} \{u_j\} \times [\lambda_-, \lambda_+]$,
(ii) $(\nabla_u \Phi)^{-1}(0) \cap (B_\gamma(\mathbb{H}) \times \{\lambda_\pm\}) = \bigcup_{j=1}^{q} \{u_j\} \times \{\lambda_\pm\}$,
(iii) $\{(u, \lambda) \in \{u_1, \ldots, u_q\} \times [\lambda_-, \lambda_+]$: $\ker \nabla_u^2 \Phi(u, \lambda) \neq \{0\}\} = \bigcup_{j=1}^{d} \{(u_{i_j}, \lambda_{i_j})\}$,
(iv) $\{\lambda \in [\lambda_-, \lambda_+]$: $\ker \nabla_u^2 \Phi(\infty, \lambda) \neq \{0\}\} = \bigcup_{j=1}^{p} \{\lambda_j^\infty\} \subset (\lambda_-, \lambda_+)$,
(v) $\ker(\nabla_u^2 \Phi(u_{i_k}, \lambda_{i_k}))$ *is not consistent with* $\ker(\nabla_u^2 \Phi(\infty, \lambda_j^\infty))$ *for every* $k = 1, \ldots, d$ *and* $j = 1, \ldots, p$.

Then the statement of Theorem 3.2 *holds true. Moreover,*

(a) *there is* $j_0 \in \{1, \ldots, p\}$ *such that C meets* (∞, λ_{j_0}),
(b) *for C the phenomenon of symmetry breaking occurs.*

4. Connected sets of periodic solutions bifurcating from infinity

In this section we study continuation of 2π-periodic solutions of family of autonomous second order Hamiltonian systems of the form

$$(E_\lambda) \quad \begin{cases} \ddot{u}(t) = -\nabla_u V\big(u(t), \lambda\big), \\ u(0) = u(2\pi), \\ \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{4.1}$$

where

(a1) $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$,
(a2) $V(x,\lambda) = \frac{1}{2}(A(\lambda)x, x) + \eta(x,\lambda)$, where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^n$.
(a3) $A(\lambda)$ is real symmetric matrix for every $\lambda \in \mathbb{R}$,
(a4) $\nabla_x \eta(x,\lambda) = \mathrm{o}(\|x\|)$, as $\|x\| \to \infty$ uniformly on bounded λ-intervals.

Define a separable Hilbert space

$$\mathbb{H}^1_{2\pi} = \{u : [0,2\pi] \to \mathbb{R}^n:\ u \text{ is abs. cont.}, u(0) = u(2\pi),\ \dot{u} \in L^2([0,2\pi], \mathbb{R}^n)\}$$

with a scalar product given by the formula $\langle u, v\rangle_{\mathbb{H}^1_{2\pi}} = \int_0^{2\pi} (\dot{u}(t), \dot{v}(t)) + (u(t), v(t))\,\mathrm{d}t$. The space $(\mathbb{H}^1_{2\pi}, \langle\cdot,\cdot\rangle_{\mathbb{H}^1_{2\pi}})$ is an orthogonal SO(2)-representation with the SO(2)-action given by shift in time.

It is well known that solutions of system (4.1) are in one to one correspondence with critical points of an SO(2)-invariant C^2-functional $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ given by the formula

$$\Phi_V(u,\lambda) = \frac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2\,\mathrm{d}t - \int_0^{2\pi} V(u(t),\lambda)\,\mathrm{d}t. \tag{4.2}$$

Moreover, it is known that $\nabla^2_u \Phi_V(\infty,\lambda) = \mathrm{Id} - L_{A(\lambda)}$, where $L_{A(\lambda)} : \mathbb{H}^1_{2\pi} \to \mathbb{H}^1_{2\pi}$ is a linear, self-adjoint, SO(2)-equivariant and compact operator defined by the formula $\langle L_{A(\lambda)}(u), v\rangle_{\mathbb{H}^1_{2\pi}} = \int_0^{2\pi} (u(t) + A(\lambda)u(t), v(t))\,\mathrm{d}t$. By Corollary 5.1.1. of [7], $\nabla^2_u \Phi_V(\infty,\lambda)$ is an isomorphism iff $\sigma(A(\lambda)) \cap \{k^2:\ k \in \mathbb{N} \cup \{0\}\} = \emptyset$. Note that $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ satisfies assumptions (c1), (c2) of the previous section.

Let us put two additional assumptions:

(a5) assume that there exist $\lambda_-, \lambda_+ > 0$ such that the set of solutions of $(E_{\lambda_\pm})$ is bounded in $\mathbb{H}^1_{2\pi}$, i.e. there exists $\gamma > 0$ such that

$$\nabla_u \Phi_V(\cdot,\lambda_\pm)^{-1}(0) \cap \big(\big(\mathbb{H}^1_{2\pi} \setminus B_\gamma(\mathbb{H}^1_{2\pi})\big) \times \{\lambda_\pm\}\big) = \emptyset, \tag{4.3}$$

(a6) assume that
- $\sigma(A(\lambda_-)) \cap \{k^2:\ k \in \mathbb{N}\} = \{(k_1^-)^2, \ldots, (k_r^-)^2\}$,
- $\sigma(A(\lambda_+)) \cap \{k^2:\ k \in \mathbb{N}\} = \{(k_1^+)^2, \ldots, (k_s^+)^2\}$.

Put

$$\mathbb{K} = \bigcup_{\{i_1,\ldots,i_l\} \in \{1,\ldots,r\}} \{\gcd(k^-_{i_1}, \ldots, k^-_{i_l})\} \cup \bigcup_{\{i_1,\ldots,i_m\} \in \{1,\ldots,s\}} \{\gcd(k^+_{i_1}, \ldots, k^+_{i_m})\}.$$

If $\sigma(A(\lambda_\pm)) \cap \{k^2:\ k \in \mathbb{N}\} = \emptyset$, then it is understood that $\mathbb{K} = \emptyset$. For $\alpha \in \mathbb{R}$ we will denote by $\mu_A(\alpha)$ the multiplicity of α considered as an eigenvalue of matrix A. If $\alpha \notin \sigma(A)$ then it is understood that $\mu_A(\alpha) = 0$. For every $k \in \mathbb{N} \cup \{0\}$ define

(1) $\sigma_k(A, 2\pi) = \sigma(A) \cap (k^2, +\infty)$,
(2) $j_k(A, 2\pi) = \sum_{\alpha \in \sigma_k(A,2\pi)} \mu_A(\alpha)$.

Put $\mathrm{ind}(-\nabla_x V(\cdot,\lambda_\pm), \infty) = \lim_{\alpha\to\infty} \deg_{\mathrm{B}}(-\nabla_x V(\cdot,\lambda_\pm), B_\alpha(\mathbb{R}^n, 0), 0)$, where $\deg_{\mathrm{B}}$ denotes the Brouwer degree.

Theorem 4.1. *Let assumptions* (a1)–(a6) *be satisfied. Additionally, suppose that one of the following conditions holds*:

(i) $\mathrm{ind}(\nabla_x V(\cdot,\lambda_+), \infty) \neq \mathrm{ind}(\nabla_x V(\cdot,\lambda_-), \infty)$,
(ii) $\mathrm{ind}(\nabla_x V(\cdot,\lambda_+), \infty) = \mathrm{ind}(\nabla_x V(\cdot,\lambda_-), \infty) \neq 0$ *and there exists* $k \in \mathbb{N} \setminus \mathbb{K}$ *such that* $j_k(A(\lambda_+), 2\pi) \neq j_k(A(\lambda_-), 2\pi)$.

Then there exists an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [\lambda_-, \lambda_+]$ *of solutions of system* (4.1) *such that* $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$.

Proof. First of all notice that $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ given by formula (4.2) satisfies condition (c1).

(i) By Lemma 5.2.3. of [7],

$$\nabla_{SO(2)}\text{-deg}_{SO(2)}\big(\nabla_u \Phi_V(\cdot, \lambda_\pm), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big) = \text{ind}\big(-\nabla_x V(\cdot, \lambda_\pm), \infty\big).$$

That is why we obtain

$$\begin{aligned}&\text{Bif}_{SO(2)}\big(\infty, [\lambda_-, \lambda_+]\big)\\ &\quad= \nabla_{SO(2)}\text{-deg}_{SO(2)}\big(\nabla_u \Phi_V(\cdot, \lambda_+), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big) - \nabla_{SO(2)}\text{-deg}_{SO(2)}\big(\nabla_u \Phi_V(\cdot, \lambda_-), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big)\\ &\quad= \text{ind}\big(-\nabla_x V(\cdot, \lambda_+), \infty\big) - \text{ind}\big(-\nabla_x V(\cdot, \lambda_-), \infty\big) \neq 0.\end{aligned}$$

(ii) By Lemma 5.2.3. of [7],

$$\nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}\big(\nabla_u \Phi_V(\cdot, \lambda_\pm), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big) = \text{ind}\big(-\nabla_x V(\cdot, \lambda_\pm), \infty\big) \cdot j_k\big(A(\lambda_\pm), 2\pi\big).$$

Therefore we have

$$\begin{aligned}&\text{Bif}_{\mathbb{Z}_k}\big(\infty, [\lambda_-, \lambda_+]\big)\\ &\quad= \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}\big(\nabla_u \Phi_V(\cdot, \lambda_+), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big) - \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}\big(\nabla_u \Phi_V(\cdot, \lambda_-), B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big)\\ &\quad= \text{ind}\big(-\nabla_x V(\cdot, \lambda_+), \infty\big) \cdot j_k\big(A(\lambda_+), 2\pi\big) - \text{ind}\big(-\nabla_x V(\cdot, \lambda_-), \infty\big) \cdot j_k\big(A(\lambda_-), 2\pi\big) \neq 0.\end{aligned}$$

Since $\text{Bif}(\infty, [\lambda_-, \lambda_+]) \neq \Theta \in U(SO(2))$, the rest of the proof is a direct consequence of Theorem 3.1. □

Theorem 4.2. *Theorem* 4.1 *remains true if the assumption* (a5) *is replaced by*

(a) $\mathbb{K} = \emptyset$,
(b) $(\nabla_x V(\cdot, \lambda_\pm)^{-1}(0) \cap ((\mathbb{R}^n \setminus B_\gamma(\mathbb{R}^n)) \times \{\lambda_\pm\}) = \emptyset$.

Proof. Notice that $(\mathbb{H}^1_{2\pi})^{SO(2)} = \mathbb{R}[n, 0]$ and that $\nabla_u \Phi_V(\cdot, \lambda_\pm)^{SO(2)} = -\nabla_x V(\cdot, \lambda_\pm)$. From Lemma 3.3 it follows that for every $(u, \lambda) \in (\nabla_u \Phi_V)^{-1}(0)$ close to $(\infty, \lambda_\pm)$, there exists $v \in \ker \nabla^2_u \Phi_V(\infty, \lambda_\pm)$ such that $SO(2)_{(u,\lambda)} = SO(2)_v$. Combining the assumptions with Lemma 5.1.1 and Corollary 5.1.1. of [7] we obtain that $\ker \nabla^2_u \Phi_V(\infty, \lambda_\pm) \subset \mathbb{R}[n, 0]$. Therefore $SO(2)_{(u,\lambda)} = SO(2)$ and $\nabla_u \Phi_V(u, \lambda) = 0$ iff $\nabla_u V(u, \lambda) = 0$. □

Definition 4.1. We say that $2\pi \geqslant T > 0$ is a period of function $u \in \mathbb{H}^1_{2\pi}$ if $u(t + T) = u(t)$ for every $t \in [0, 2\pi]$. We say that $T_{\min} \geqslant 0$ is a minimal period of function $u \in \mathbb{H}^1_{2\pi}$ if $T_{\min} = \inf\{T > 0:\ u(t + T) = u(t)$ for every $t \in [0, 2\pi]\}$.

Remark 4.1. Notice that if $u \in (\mathbb{H}^1_{2\pi})^{SO(2)}$, i.e. $u = \text{const}$, $T_{\min} = 0$ and therefore $T_{\min}$ is not a period of function u. Nevertheless, we call $T_{\min} = 0$ the minimal period of a constant function u.

Corollary 4.1. *Let assumptions of Theorem* 4.1 *be satisfied. If additionally* $(\nabla_x V)^{-1}(0) \cap (\mathbb{R}^n \times [\lambda_-, \lambda_+])$ *is bounded, then conclusion of Theorem* 4.1 *holds true. Moreover, continuum C emanates from the set of stationary solutions and contains solutions with different minimal periods or there exists at least one non-stationary solution (u, λ) of system* (4.1) *such that* $(u, \lambda) \in (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-, \lambda_+\}) \cap C$.

Proof. Note that $(\mathbb{H}^1_{2\pi})^{SO(2)} = \mathbb{R}[n, 0]$. It is clear that solutions with different isotropy group have different minimal periods. Since all the assumptions of Corollary 3.1 are satisfied, we obtain our assertion. □

Remark 4.2. Under assumptions of Corollary 4.1, if moreover equations $(E_{\lambda_\pm})$ possesses only stationary periodic solutions then continuum C contains solutions with different minimal periods.

Corollary 4.2. *Let assumptions of Theorem* 4.1 *be satisfied. Additionally, suppose that* $\ker \nabla^2_u \Phi_V(u, \lambda) \subset (\mathbb{H}^1_{2\pi})^{SO(2)} = \mathbb{R}[n, 0]$ *for every* $u \in (\mathbb{H}^1_{2\pi})^{SO(2)}$ *and* $\lambda \in [\lambda_-, \lambda_+]$, *then conclusion of Theorem* 4.1 *holds true. Moreover, either*

$C \subset (\mathbb{H}^1_{2\pi})^{SO(2)} \times [\lambda_-, \lambda_+]$ or C contains only non-stationary solutions. If additionally equations $(E_{\lambda_\pm})$ possesses only stationary periodic solutions then C consists of stationary solutions of system (4.1).

Proof. Immediate consequence of Corollary 3.2. □

Let us put the following assumption

(a7) fix $\lambda_0 \in \mathbb{R}$ and choose $\lambda_- < \lambda_+$ such that

$$\{\lambda \in [\lambda_-, \lambda_+]\colon \sigma(A(\lambda)) \cap \{k^2\colon k \in \mathbb{N} \cup \{0\}\} \neq \emptyset\} = \{\lambda_0\}. \tag{4.4}$$

Combining assumption (4.4) with Corollary 5.1.1 of [7] we obtain that $\nabla^2_u \Phi(\infty, \lambda_\pm) : \mathbb{H}^1_{2\pi} \to \mathbb{H}^1_{2\pi}$ is a linear isomorphism. Therefore assumption (a5) is satisfied.

Theorem 4.3. *Let assumptions* (a1)–(a4), (a7) *be satisfied. Additionally, suppose that at least one of the following conditions holds*:

(i) $(-1)^{j_0(A(\lambda_+),2\pi)} \neq (-1)^{j_0(A(\lambda_-),2\pi)}$,
(ii) *there exists $k \in \mathbb{N}$ such that $j_k(A(\lambda_+), 2\pi) \neq j_k(A(\lambda_-), 2\pi)$.*

Then there exists an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [\lambda_-, \lambda_+]$ of solutions of system (4.1) *such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$. Moreover, C meets* (∞, λ_0).

Proof. Note that $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ given by formula (4.2) satisfies (c2).

(i) By Lemma 5.2.2. and Remark 5.2.2. of [7],

$$\nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla_u \Phi_V(\cdot, \lambda_\pm), B_\gamma(\mathbb{H}^1_{2\pi})) = (-1)^{j_0(A(\lambda_\pm),2\pi)}.$$

Therefore we obtain

$$\begin{aligned}
&\mathrm{Bif}_{SO(2)}(\infty, [\lambda_-, \lambda_+]) \\
&\quad = \nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla_u \Phi_V(\cdot, \lambda_+), B_\gamma(\mathbb{H}^1_{2\pi})) - \nabla_{SO(2)}\text{-deg}_{SO(2)}(\nabla_u \Phi_V(\cdot, \lambda_-), B_\gamma(\mathbb{H}^1_{2\pi})) \\
&\quad = (-1)^{j_0(A(\lambda_+),2\pi)} - (-1)^{j_0(A(\lambda_-),2\pi)} \neq 0.
\end{aligned}$$

(ii) By Lemma 5.2.2. and Remark 5.2.2. of [7],

$$\nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}(\nabla_u \Phi_V(\cdot, \lambda_\pm), B_\gamma(\mathbb{H}^1_{2\pi})) = (-1)^{j_0(A(\lambda_\pm),2\pi)} \cdot j_k(A(\lambda_\pm), 2\pi).$$

That is why we have

$$\begin{aligned}
\mathrm{Bif}_{\mathbb{Z}_k}(\infty, [\lambda_-, \lambda_+]) &= \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}(\nabla_u \Phi_V(\cdot, \lambda_+), B_\gamma(\mathbb{H}^1_{2\pi})) - \nabla_{SO(2)}\text{-deg}_{\mathbb{Z}_k}(\nabla_u \Phi_V(\cdot, \lambda_-), B_\gamma(\mathbb{H}^1_{2\pi})) \\
&= (-1)^{j_0(A(\lambda_+),2\pi)} \cdot j_k(A(\lambda_+), 2\pi) - (-1)^{j_0(A(\lambda_-),2\pi)} \cdot j_k(A(\lambda_-), 2\pi).
\end{aligned}$$

Summing up, $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \neq \Theta$. The rest of the proof is a direct consequence of Theorem 3.2. □

Recall that by Corollary 5.1.2. of [7]

$$\ker \nabla^2_u \Phi_V(\infty, \lambda_0) = \ker(\mathrm{Id} - L_{A(\lambda_0)}) \approx \bigoplus_{k=0}^{\infty} \mathbb{R}[\mu_{A(\lambda_0)}(k^2), k].$$

Note that for almost every $k \in \mathbb{N} \cup \{0\}$, $k^2 \notin \sigma(A(\lambda_0))$ and hence $\mu_{A(\lambda_0)}(k^2) = 0$. Since $\mathbb{R}[0, k] = \{0\}$, $\dim \ker \nabla^2_u \Phi_V(\infty, \lambda_0) < \infty$.

Corollary 4.3. *Let assumptions of Theorem* 4.3 *be satisfied. Suppose that*

$$\sigma\big(A(\lambda_0)\big) \cap \big\{k^2\colon\ k \in \mathbb{N} \cup \{0\}\big\} = \big\{k_0^2, k_1^2, \dots, k_r^2\big\},$$

where $0 \leqslant k_0 < k_1 \cdots < k_r$.

(i) *If* $\det A(\lambda_0) = 0$*, then for every solution* (u, λ) *of system* (4.1) *in* $\mathbb{H}^1_{2\pi} \times [\lambda_-, \lambda_+]$ *sufficiently close to* (∞, λ_0) *its minimal period* $T_{\min}$ *is equal to zero* ($u = const$) *or to* $2\pi/\gcd(k_{i_1}, \dots, k_{i_s})$ *for some* $\{k_{i_1}, \dots, k_{i_s}\} \subset \{k_1, \dots, k_r\}$.
(ii) *If* $\det A(\lambda_0) \neq 0$*, then for every solution* (u, λ) *of system* (4.1) *in* $\mathbb{H}^1_{2\pi} \times [\lambda_-, \lambda_+]$ *sufficiently close to* (∞, λ_0) *its minimal period* $T_{\min}$ *is equal to* $2\pi/\gcd(k_{i_1}, \dots, k_{i_s})$ *for some* $\{k_{i_1}, \dots, k_{i_s}\} \subset \{k_0, \dots, k_r\}$.

Proof. By assumption and Corollary 5.1.2. of [7] we have

$$\ker \nabla_u^2 \Phi_V(\infty, \lambda_0) = \ker(\mathrm{Id} - L_{A(\lambda_0)}) \approx \bigoplus_{i=0}^{r} \mathbb{R}\big[\mu_{A(\lambda_0)}\big(k_i^2\big), k_i\big].$$

By Lemma 3.3 any solution (u, λ) of system (4.1) sufficiently close to (∞, λ_0) has the same isotropy group as some element of $\ker \nabla_u^2 \Phi_V(\infty, \lambda_0)$. Therefore if $\det A(\lambda_0) = 0$, then the possible isotropy group of any solution is equal to $\mathrm{SO}(2)$ or $\mathbb{Z}_{\gcd(k_{i_1}, \dots, k_{i_s})}$ for some $\{k_{i_1}, \dots, k_{i_s}\} \subset \{k_1, \dots, k_r\}$, which completes the proof of (i). Otherwise, it is equal to $\mathbb{Z}_{\gcd(k_{i_1}, \dots, k_{i_s})}$ for some $\{k_{i_1}, \dots, k_{i_s}\} \subset \{k_0, \dots, k_r\}$, which completes the proof of (ii). □

Corollary 4.4. *Let assumptions of Theorem* 4.3 *be satisfied. If additionally* $\det A(\lambda_0) \neq 0$ *then conclusion of Theorem* 4.3 *holds true. Moreover, continuum* C *emanates from the set of stationary solutions and contains solutions with different minimal periods or there exists at least one non-stationary solution such that* $(u, \lambda) \in (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-, \lambda_+\}) \cap C$.

Proof. By Lemma 5.1.1 and Corollary 5.1.1 we obtain $\ker \nabla_u^2 \Phi_V(\infty, \lambda_0) \cap (\mathbb{H}^1_{2\pi})^{\mathrm{SO}(2)} = \{0\}$ iff $\det A(\lambda_0) \neq 0$. The rest of the proof is a direct consequence of Lemma 3.3. □

From now on we consider special case of system (4.1). Namely, we consider system

$$\begin{cases} \ddot{u}(t) = -\lambda^2 \nabla V\big(u(t)\big), \\ u(0) = u(2\pi), \\ \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{4.5}$$

where

(b1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$,
(b2) $V(x) = \frac{1}{2}(Ax, x) + \eta(x)$,
(b3) A is a real symmetric matrix,
(b4) $\nabla\eta(x) = \mathrm{o}(\|x\|)$, as $\|x\| \to \infty$,
(b5) $(\nabla V)^{-1}(0)$ is bounded,
(b6) $\mathrm{ind}(\nabla V, \infty) \neq 0$.

It is easy to show that $\nabla_u^2 \Phi_V(\infty, \lambda)$ is not an isomorphism if and only if

$$\lambda \in \left\{\frac{k}{\sqrt{\alpha}}\colon\ k \in \mathbb{N},\ \alpha \in \sigma_+(A)\right\} \quad \text{or} \quad \det A \neq 0.$$

Lemma 4.1. *Fix* $k_0 \in \mathbb{N}, \alpha_0 \in \sigma_+(A)$ *and choose* $\lambda_- < \lambda_+$ *such that*

$$[\lambda_-, \lambda_+] \cap \left\{\frac{k}{\sqrt{\alpha}}\colon\ k \in \mathbb{N},\ \alpha \in \sigma_+(A)\right\} = \left\{\frac{k_0}{\sqrt{\alpha_0}}\right\}.$$

Then $\mathrm{Bif}(\infty, [\lambda_-, \lambda_+]) \in U(\mathrm{SO}(2))$ *is well-defined. Moreover,*

$$\mathrm{Bif}_{\mathbb{Z}_{k_0}}\big(\infty, [\lambda_-, \lambda_+]\big) = \mathrm{ind}(-\nabla V, \infty) \cdot \mu_A(\alpha_0).$$

Proof. Since $(\nabla_u \Phi(\cdot,\lambda_\pm))^{-1}(0) \subset \mathbb{H}^1_{2\pi}$ is bounded, $\mathrm{Bif}(\infty,[\lambda_-,\lambda_+]) \in U(\mathrm{SO}(2))$ is well-defined. Applying Lemma 5.2.2 of [7], we obtain:

$$\begin{aligned}\mathrm{Bif}_{\mathbb{Z}_{k_0}}\big(\infty,[\lambda_-,\lambda_+]\big) &= \nabla_{\mathrm{SO}(2)}\text{-}\deg_{\mathbb{Z}_{k_0}}\big(\mathrm{Id}-L_{\lambda_+^2 A}, B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big) - \nabla_{\mathrm{SO}(2)}\text{-}\deg_{\mathbb{Z}_{k_0}}\big(\mathrm{Id}-L_{\lambda_-^2 A}, B_\gamma\big(\mathbb{H}^1_{2\pi}\big)\big)\\ &= \mathrm{ind}\big(-\lambda_+^2\nabla V,\infty\big)\cdot j_{k_0}\big(\lambda_+^2 A,2\pi\big) - \mathrm{ind}\big(-\lambda_-^2\nabla V,\infty\big)\cdot j_{k_0}\big(\lambda_-^2 A,2\pi\big)\\ &= \mathrm{ind}(-\nabla V,\infty)\cdot\big(j_{k_0}\big(\lambda_+^2 A,2\pi\big) - j_{k_0}\big(\lambda_-^2 A,2\pi\big)\big)\\ &= \mathrm{ind}(-\nabla V,\infty)\cdot\Bigg(\sum_{\alpha\in\sigma_{k_0}(\lambda_+^2 A,2\pi)}\mu_{(\lambda_+^2 A)}(\alpha) - \sum_{\alpha\in\sigma_{k_0}(\lambda_-^2 A,2\pi)}\mu_{(\lambda_-^2 A)}(\alpha)\Bigg)\\ &= \mathrm{ind}(-\nabla V,\infty)\cdot\mu_A(\alpha_0). \qquad \square\end{aligned}$$

The following theorem is a consequence of Theorem 4.3.

Theorem 4.4. *Let assumptions* (b1)–(b6) *be fulfilled. Then for every*

$$\lambda_0 \in \left\{\frac{k}{\sqrt{\alpha}}:\ k\in\mathbb{N},\ \alpha\in\sigma_+(A)\right\}$$

there exists an unbounded closed connected component $C(\lambda_0) \subset \mathbb{H}^1_{2\pi}\times[\lambda_-,\lambda_+]$ of solutions of system (4.5) *such that $C(\lambda_0)\cap(B_\gamma(\mathbb{H}^1_{2\pi})\times\{\lambda_-,\lambda_+\}) \neq \emptyset$, where $\lambda_- < \lambda_+$ satisfy*

$$[\lambda_-,\lambda_+]\cap\left\{\frac{k}{\sqrt{\alpha}}:\ k\in\mathbb{N},\ \alpha\in\sigma_+(A)\right\} = \{\lambda_0\}.$$

Moreover, $C(\lambda_0)$ meets (∞,λ_0).

Fix $\lambda_0 = \frac{k_0}{\sqrt{\alpha_0}}$ for some $k_0\in\mathbb{N}$, $\alpha_0\in\sigma_+(A)$.

Corollary 4.5. *Let assumptions of Theorem* 4.4 *be satisfied. Assume additionally that*

(i) $(\nabla V)^{-1}(0) = \{u_1,\dots,u_q\}$,
(ii) *the only periodic solutions of $(E_{\lambda_\pm^2})$ are the critical points of V,*
(iii) $\{(u,\lambda)\in\{u_1,\dots,u_q\}\times[\lambda_-,\lambda_+]:\ \sigma(\lambda^2\nabla^2 V(u_i))\cap\{k^2:\ k\in\mathbb{N}\cup\{0\}\}\neq\emptyset\} = \{(u_{i_1},\lambda_{i_1}),\dots,(u_{i_d},\lambda_{i_d})\}$,
(iv) $\ker(\nabla_u^2\Phi_V(u_{i_k},\lambda_{i_k}))$ *is not consistent with* $\ker(\nabla_u^2\Phi_V(\infty,\lambda_0))$ *for all $k=1,\dots,d$.*

Then there exists an unbounded closed connected component $C(\lambda_0)\subset\mathbb{H}^1_{2\pi}\times[\lambda_-,\lambda_+]$ of solutions of system (4.5) *such that $C(\lambda_0)\cap\big(B_\gamma(\mathbb{H}^1_{2\pi})\times\{\lambda_-,\lambda_+\}\big)\neq\emptyset$ and $C(\lambda_0)$ meets (∞,λ_0). Moreover, $C(\lambda_0)$ contains solutions with different minimal periods.*

Proof. Note that $\sigma(\lambda^2\nabla^2 V(u_{i_k}))\cap\{k^2:\ k\in\mathbb{N}\cup\{0\}\} = \emptyset$ implies that$\nabla_u^2\Phi_V(u_{i_k},\lambda)$ is an isomorphism for every $k=1,\dots,d$. Therefore applying Corollary 3.4 we complete the proof. $\square$

5. Examples

In this section we discuss three examples of potentials in order to illustrate results proved in the previous section. We consider system (4.1) with simple potential V and show that assumptions of our theorems are satisfied.

Example 5.1. Define potential $V:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ as follows

$$V(x,\lambda) = \frac{1}{2}\big(A(\lambda)x,x\big) + W(x,\lambda) = \frac{1}{2}\big(A(\lambda)x,x\big) + \frac{-\lambda^2}{\sqrt{\|x\|^2+a}}, \tag{5.1}$$

where $a > 0$ and $A(\lambda)$ is a real symmetric $(n \times n)$-matrix for every $\lambda \in \mathbb{R}$. Consider system (4.1) with potential (5.1). Put $n = 4$, $a = 1$, $\lambda_{\pm} = \pm 1$ and define

$$A(\lambda) = \begin{bmatrix} \lambda^2 - 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} + \lambda & 0 & 0 \\ 0 & 0 & \lambda - \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{5} + \lambda \end{bmatrix}.$$

Systems $(E_{\pm 1})$ are resonant at infinity because

$$\sigma\big(A(\pm 1)\big) \cap \big\{k^2\colon k \in \mathbb{N} \cup \{0\}\big\} = \{0\}. \tag{5.2}$$

Notice that assumptions (a1)–(a4), (a6) of Theorem 4.1 are satisfied. Moreover,

(1) $(\nabla_x V(\cdot, \pm 1))^{-1}(0)$ is bounded because $\#(\nabla_x V(\cdot, \pm 1))^{-1}(0) < \infty$ (consequence of Lemma 6.2 of [7]),
(2) $\mathbb{K} = \emptyset$ (consequence of (5.2)).

Applying Theorem 4.2 we show that assumption (a5) of Theorem 4.1 is fulfilled.

Moreover,

(1) $\mathrm{ind}(-\nabla_x V(\cdot, \pm 1), \infty) = (-1)^{n - m^-(A(\pm 1))} = (-1)^{4-1} = -1$ (consequence of Lemma 6.4 of [7]),
(2) $j_1(A(+1), 2\pi) = 2 \neq 1 = j_1(A(-1), 2\pi)$.

Applying Theorem 4.1 we obtain an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [-1, +1]$ of solutions of system (4.1) such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{-1, +1\}) \neq \emptyset$.

Additionally, taking into consideration that

(1) $(\nabla_x V(\cdot, \pm 1))^{-1}(0)$ is bounded,
(2) $\{\lambda \in (-1, +1)\colon \sigma(A(\lambda)) \cap \{k^2\colon k \in \mathbb{N} \cup \{0\}\} \neq \emptyset\} = \{\lambda_0 = 1 - \sqrt{2}\}$,
(3) $\sigma(A(1 - \sqrt{2})) = \{1\}$,

and Corollary 4.3 we obtain that the continuum meets $(\infty, 1 - \sqrt{2})$ and that any solution $(u, \lambda) \in C$ of system (4.1) sufficiently close to $(\infty, 1 - \sqrt{2})$ has minimal period equal to 2π.

Example 5.2. Define potential $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as follows

$$V(x, \lambda) = \frac{1}{2}\big(A(\lambda)x, x\big) + W(x, \lambda) = \frac{1}{2}\big(A(\lambda)x, x\big) + \frac{-1}{\sqrt{\|x\|^2 + a}}, \tag{5.3}$$

where $a > 0$ and $A(\lambda)$ is a real symmetric $(n \times n)$-matrix for every $\lambda \in \mathbb{R}$.

Consider system (4.1) with potential (5.3). Put $n = 4$, $a = 1$ and define

$$A(\lambda) = \begin{bmatrix} 4 + \lambda & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

System (E_0) is resonant at infinity because

$$\sigma\big(A(0)\big) \cap \big\{k^2\colon k \in \mathbb{N} \cup \{0\}\big\} = \{4\}. \tag{5.4}$$

Moreover, put $\lambda_{\pm} = \pm(1/2)$ and notice that

$$\sigma\big(A(\lambda)\big) \cap \big\{k^2\colon k \in \mathbb{N} \cup \{0\}\big\} = \emptyset \tag{5.5}$$

for every $\lambda \in [-1/2, +1/2] \setminus \{0\}$.

Since $j_2(A(\frac{1}{2})) = 1 \neq 0 = j_2(A(-\frac{1}{2}))$, all the assumptions of Theorem 4.3 are fulfilled. Therefore there exists an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [-1/2, +1/2]$ of solutions of system (4.1) such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{-1/2, +1/2\}) \neq \emptyset$ and that C meets $(\infty, 0)$.

Properties of potential V have been precisely studied in [7]. Stationary solutions of system (4.1) have the following properties:

(1) $(\nabla_x V)^{-1}(0) \cap (\mathbb{R}^4 \times [-1/2, +1/2]) = \{0\} \times [-1/2, +1/2]$ (consequence of Lemma 6.2 of [7]),
(2) $\nabla^2_{xx} V(0, \lambda) = A(\lambda) + \mathrm{Id}$, for every $\lambda \in [-1/2, +1/2]$ (consequence of Lemma 6.1 of [7]),
(3) $\sigma(\nabla^2_{xx} V(0, \lambda)) \cap \{k^2\colon k \in \mathbb{N} \cup \{0\}\} = \emptyset$ for every $\lambda \in [-1/2, +1/2]$ (consequence of (2)).

Moreover, by (5.4), (5.5) and Corollary 4.3 we obtain that any solution $(u, \lambda) \in C$ of system (4.1) sufficiently close to $(\infty, 0)$ has minimal period equal to π. Additionally, from (3) and Remark 3.3 it follows that continuum C consist of non-stationary solutions.

Example 5.3. Consider system (4.1) with potential (5.3). Put $n = 5$, $a = 1$, $\lambda_\pm = \pm 1$ and define

$$A(\lambda) = \begin{bmatrix} 4 + \frac{\lambda^2}{2} & 0 & 0 & 0 & 0 \\ 0 & \lambda^3 - \sqrt{10} & 0 & 0 & \\ 0 & 0 & 9 + \frac{\lambda^2}{2} & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 + \sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & 25 + \frac{\lambda^2}{2} \end{bmatrix}.$$

It is easy to see that

(1) $\sigma(A(\lambda)) \cap \{k^2\colon k \in \mathbb{N} \cup \{0\}\} = \emptyset$ for every $\lambda \in [-1, 1] \setminus \{0\}$,
(2) $\sigma(A(0)) \cap \{k^2\colon k \in \mathbb{N} \cup \{0\}\} = \{4, 9, 25\}$.

Hence assumptions (a1)–(a4), (a7) of Theorem 4.3 are fulfilled.

Since $j_2(A(1), 2\pi) = 4 \neq 3 = j_2(A(-1), 2\pi)$, all the assumption of Theorem 4.3 are satisfied. Therefore there exists an unbounded closed connected component C of solutions of system (4.1) in $\mathbb{H}^1_{2\pi} \times [-1, 1]$ such that $C \cap \big(B_\gamma(\mathbb{H}^1_{2\pi}) \times \{-1, 1\}\big) \neq \emptyset$ and C meets $(\infty, 0)$. Moreover, by (2) and Corollary 4.3(ii) any solution $(u, \lambda) \in C$ sufficiently close to $(\infty, 0)$ possesses the minimal period $T_{\min} \in \{2\pi, \pi, \frac{2\pi}{3}, \frac{2\pi}{5}\}$.

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