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The Schrödinger–Maxwell system with Dirac mass ☆

Le système de Schrödinger–Maxwell avec masse de Dirac

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Abstract

We study a nonrelativistic charged quantum particle moving in a bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary under the action of a zero-range potential. In the electrostatic case the standing wave solution takes the form $\psi(t,x) = u(x)e^{-i\omega t}$ where u formally satisfies $-\Delta u + \alpha\varphi u - \frac{1}{\beta}\delta_{x_0} u = \omega u$ and the electric potential φ is given by $-\Delta\varphi = u^2$. We give a rigorous definition of this problem and show that it has a weak nontrivial solution.

Résumé

Nous étudions une particule quantique chargée et non relativiste se déplaçant dans un domaine ouvert borné $\Omega \subset \mathbb{R}^3$ au contour régulier sous l'action d'un potentiel concentré en un point. Dans le cas electrostatique, l'onde solution prends la forme $\psi(t,x) = u(x)e^{-i\omega t}$ où u satisfait formellement $-\Delta u + \alpha\varphi u - \frac{1}{\beta}\delta_{x_0} u = \omega u$ et le potentiel électrique φ est donné par $-\Delta\varphi = u^2$. Nous donnons une définition rigoureuse du problème et montrons qu'il possède une solution faible non triviale.

MSC: primary 35D05, 35Q40; secondary 35J65

Keywords: Schrödinger–Maxwell system; Point interaction

1. Introduction

Consider a nonrelativistic charged quantum particle moving in a bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary under the action of a short-range potential centered at $x_0 \in \Omega$. Denote by $\psi = \psi(t,x)$, $\varphi = \varphi(t,x)$ and $A = A(t,x)$ the wave function, the electric potential and the vector potential, respectively, generated by the particle.

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We consider only standing wave solutions in the electrostatic case, i.e.,

$$\psi(t,x)=u(x)\mathrm{e}^{-\mathrm{i}\omega t},\quad \varphi=\varphi(x),\quad A=0,$$

where

$$u:\Omega\longrightarrow\mathbb{R},\quad \omega\in\mathbb{R}.$$

Arguing as in [12] we have that u, φ are solutions of the following system

$$\begin{cases}-\Delta u+\alpha\varphi u-\dfrac{1}{\beta}\delta_{x_0}u=\omega u, & \text{in } \Omega,\\ -\Delta\varphi=u^2, & \text{in } \Omega,\\ u=\varphi=0, & \text{on } \partial\Omega,\end{cases}\tag{1.1}$$

where δ_{x_0} is Dirac's delta function located at x_0 and

$$\alpha,\beta>0 \text{ are constants.}\tag{1.2}$$

For the sake of notational simplicity we have scaled all physical constants into α, β.

Recall that the case of uncharged particles (i.e., $\alpha=0$) is treated in [5], while the case of charged particles with $\beta=\infty$ is discussed in [8], and a charged particle in the full space $\Omega=\mathbb{R}^3$ and $\beta=\infty$ can be found in [10,12]. Furthermore, the case of a nonlinear external vector field, $\beta=\infty$, $\Omega=\mathbb{R}^3$ is analyzed in [11]. Under the assumption $\beta=\infty$ the case of a relativistic particle is considered in [9,14]. A nonlinear version of (1.1) with $\alpha=0$, $\beta=\beta(\psi)$ is studied in [1–3]. Finally, classical papers on the coupling of the Maxwell and the Schrödinger equations are [6,7,13].

Eq. (1.1) is not well-defined as it stands due to the presence of Dirac's delta function. However, one can rigorously define the operator $-\Delta-\frac{1}{\beta}\delta_{x_0}$ as a self-adjoint operator on $L^2(\Omega)$ by suitably rescaling the coupling constant multiplying the delta function. See [5, Chapter I.1, Appendix K.2.1] for an extensive discussion of this. The actual definition of the self-adjoint operator depends strongly on the dimension of the underlying physical space $\Omega\subset\mathbb{R}^3$. Let $\mathcal{L}_{\beta,x_0}$ denote this self-adjoint operator. Thus the rigorous version of (1.1) reads

$$\mathcal{L}_{\beta,x_0}u+\alpha\varphi u=\omega u,\quad -\Delta\varphi=u^2,\tag{1.3}$$

with vanishing Dirichlet boundary conditions $u=\varphi=0$ on $\partial\Omega$. The operator can be derived in several different ways: One can define $\mathcal{L}_{\beta,x_0}$ as a (strong resolvent) limit of short range Schrödinger operators

$$\mathcal{L}^\epsilon=-\Delta+\frac{\lambda(\epsilon)}{\epsilon^2}V\left(\frac{x-x_0}{\epsilon}\right)$$

as $\epsilon\to0$ where $\lambda(\epsilon)=1+\mathrm{o}(\epsilon)$. Here the potential should have short range, for instance, have compact support. Observe that

$$\frac{\lambda(\epsilon)}{\epsilon^2}V\left(\frac{x-x_0}{\epsilon}\right)=\epsilon\lambda(\epsilon)\frac{1}{\epsilon^3}V\left(\frac{x-x_0}{\epsilon}\right),$$

thus we see that, formally speaking, the coupling constant multiplying the Dirac delta function is infinitely weak. Alternatively, one can define the operator $\mathcal{L}_{\beta,x_0}$ by studying self-adjoint extensions of the symmetric operator $-\Delta|_{C_0^\infty(\Omega\setminus\{x_0\})}$ since the two operators $\mathcal{L}_{\beta,x_0}$ and $-\Delta$ formally coincide on $C_0^\infty(\Omega\setminus\{x_0\})$. Further discussions and rigorous definitions can be found in [5].

It turns out that it is considerably easier to work with the resolvent of $\mathcal{L}_{\beta,x_0}$ than $\mathcal{L}_{\beta,x_0}$ itself. Indeed, we have (precise domains are defined in the next section)

$$\mathcal{L}_{\beta,x_0}u=-\Delta v\quad\text{if and only if}\quad u=v-\frac{v(x_0)}{\beta}G(\cdot,x_0),\tag{1.4}$$

where $G=G(x,y)$ is the Green's function of $-\Delta$ on Ω with homogeneous boundary conditions and $v\in H^2(\Omega)\cap H_0^1(\Omega)$.

Thus we will work with

$$\begin{cases}-\Delta v+\alpha u\displaystyle\int_\Omega G(\cdot,y)u^2(y)\,\mathrm{d}y=\omega u, & \text{in } \Omega,\\ u=v-v(x_0)G(\cdot,x_0)\beta^{-1}, & \text{in } \Omega,\\ v=0, & \text{on } \partial\Omega.\end{cases}$$

Our main result says that if $\omega > \omega_0$, where ω_0 is the lowest eigenvalue of $-\Delta$ on Ω, then (1.1) admits a nontrivial weak solution. The result is obtained by construction an iterative approximation that is proved to converge to a weak solution. An additional argument is needed to show that the solution is nontrivial.

2. Mathematical preliminaries

Arguing as in [5, Theorem 1.1.3, Appendix K.2.1], we define the linear operator $\mathcal{L}_{\beta,x_0}$ by[1]

$$\mathcal{D}(\mathcal{L}_{\beta,x_0}) := \{v - v(x_0)G(\cdot,x_0)\beta^{-1} \mid v \in H^2(\Omega) \cap H_0^1(\Omega)\}, \tag{2.1}$$

$$\mathcal{L}_{\beta,x_0}u = -\Delta v, \quad u = v - \frac{v(x_0)}{\beta}G(\cdot,x_0) \in \mathcal{D}(\mathcal{L}_{\beta,x_0}), \tag{2.2}$$

where $G = G(x,y)$ is the Green's function of $-\Delta$ on Ω with homogeneous boundary conditions. Recall that

$$H^2(\Omega) \subset C(\overline{\Omega}),$$

and observe that

$$\mathcal{D}(\mathcal{L}_{\beta,x_0}) \subset L^p(\Omega) \cap W_0^{1,q}(\Omega), \quad 1 \leqslant p < 3,\ 1 \leqslant q < \frac{3}{2}. \tag{2.3}$$

Definition 2.1. Let $u, \varphi : \Omega \to \mathbb{R}$ be maps. We say that (u,φ) is a weak solution of (1.1) if

$$u \in \mathcal{D}(\mathcal{L}_{\beta,x_0}), \quad \varphi \in H_0^1(\Omega) \cap W^{2,p}(\Omega), \quad 1 \leqslant p < \frac{3}{2}, \tag{2.4}$$

and

$$\mathcal{L}_{\beta,x_0}u + \alpha\varphi u = \omega u, \quad -\Delta\varphi = u^2, \quad \text{in } \Omega. \tag{2.5}$$

The main result of this paper is the following:

Theorem 2.1. *Assume* (1.2). *Let*

$$\omega > \omega_0, \tag{2.6}$$

where ω_0 is the lowest eigenvalue of $-\Delta$ on Ω. Then (1.1) *admits a nontrivial weak solution in the sense of Definition* 2.1.

Let $u \in \mathcal{D}(\mathcal{L}_{\beta,x_0})$. The unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = u^2, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

is

$$\varphi(x) = \int_\Omega G(x,y)u^2(y)\,\mathrm{d}y, \quad x \in \Omega. \tag{2.8}$$

Hence, (1.1) is equivalent to

$$\begin{cases} \mathcal{L}_{\beta,x_0}u + \alpha u \displaystyle\int_\Omega G(\cdot,y)u^2(y)\,\mathrm{d}y = \omega u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

[1] In [5] the definition reads $(\mathcal{L}_{\beta,x_0} - z)u = (-\Delta - z)v$ for a z in the resolvent set of $-\Delta$, and where $u = v - \frac{v(x_0)}{\beta - \mathrm{i}\sqrt{z}/(4\pi)}G_z(\cdot,x_0)$. Here G_z is the resolvent of $-\Delta - z$ on $L^2(\Omega)$ with Dirichlet boundary conditions. Since the lowest eigenvalue of $-\Delta$ is positive, we can let $z \to 0$.

or from (2.2)

$$\begin{cases} -\Delta v + \alpha u \displaystyle\int_\Omega G(\cdot, y) u^2(y)\, \mathrm{d}y = \omega u, & \text{in } \Omega, \\ u = v - v(x_0) G(\cdot, x_0)\beta^{-1}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Finally, we can rewrite the equation in (2.10) only in terms of v

$$\begin{aligned} &-\Delta v(x) + \alpha \int_\Omega G(x, y) v^2(y) v(x)\, \mathrm{d}y - 2v(x_0)\frac{\alpha}{\beta}\int_\Omega G(x, y) G(y, x_0) v(y) v(x)\, \mathrm{d}y \\ &\quad + v^2(x_0)\frac{\alpha}{\beta^2}\int_\Omega G(x, y) G^2(y, x_0) v(x)\, \mathrm{d}y - v(x_0)\frac{\alpha}{\beta}\int_\Omega G(x, y) G(x, x_0) v^2(y)\, \mathrm{d}y \\ &\quad + 2v^2(x_0)\frac{\alpha}{\beta^2}\int_\Omega G(x, y) G(y, x_0) G(x, x_0) v(y)\, \mathrm{d}y - v^3(x_0)\frac{\alpha}{\beta^3}\int_\Omega G(x, y) G^2(y, x_0) G(x, x_0)\, \mathrm{d}y \\ &= \omega v(x) - v(x_0)\frac{\omega}{\beta} G(x, x_0). \end{aligned} \tag{2.11}$$

We continue with a preliminary regularity result for (2.7).

Lemma 2.1. *Assume* (1.2)*. Let $u \in \mathcal{D}(\mathcal{L}_{\beta,x_0})$. Then*

$$\varphi = \int_\Omega G(\cdot, y) u^2(y)\, \mathrm{d}y \in W^{2,p}(\Omega) \cap H_0^1(\Omega), \quad 1 \leqslant p < \frac{3}{2}. \tag{2.12}$$

In particular

$$\varphi = \int_\Omega G(\cdot, y) u^2(y)\, \mathrm{d}y \in L^q(\Omega), \quad 1 \leqslant q < \infty, \tag{2.13}$$

and

$$\|\varphi\|_{H_0^1(\Omega)}, \|\varphi\|_{W^{2,p}(\Omega)} \leqslant C_0\big(\|v\|^2_{H_0^1(\Omega)} + \big|v(x_0)\big|^2\big), \quad 1 \leqslant p < \frac{3}{2}, \tag{2.14}$$

where $C_0 > 0$ is a constant and v is the unique map in $H^2(\Omega) \cap H_0^1(\Omega)$ such that $u = v - v(x_0)G(\cdot, x_0)\beta^{-1}$, see (2.2).

This result is based on the classical Agmon regularity result (see [4, Theorem 8.2]).

Lemma 2.2. *Let $1 < p < \infty$ and $f \in L^p(\Omega)$ be a given function. Let ϕ be the unique solution of the Dirichlet problem*

$$\begin{cases} -\Delta\phi = f, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Then

$$\phi \in W^{2,p}(\Omega), \tag{2.15}$$

and

$$\|\nabla\phi\|_{L^p(\Omega)}, \big\|D^2\phi\big\|_{L^p(\Omega)} \leqslant C_1\big(\|f\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)}\big), \tag{2.16}$$

for some constant $C_1 > 0$, where $D^2\phi$ is the Hessian matrix of ϕ.

Proof of Lemma 2.1. Let $u \in \mathcal{D}(\mathcal{L}_{\beta,x_0})$. From (2.2) and [5, Theorem 1.1.3], we know that there exists a unique $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $u = v - v(x_0)G(\cdot, x_0)\beta^{-1}$.

We begin by proving that

$$\varphi \in H_0^1(\Omega), \quad \|\varphi\|_{H_0^1(\Omega)} \leqslant C_0\big(\|v\|^2_{H_0^1(\Omega)} + \big|v(x_0)\big|^2\big). \tag{2.17}$$

Since φ solves (2.7) and $G(\cdot, x_0) \in L^{12/5}(\Omega)$, using the Sobolev and Hölder inequalities, we get

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x = \int_\Omega \varphi u^2\,\mathrm{d}x &\leqslant \|\varphi\|_{L^6(\Omega)}\|u^2\|_{L^{6/5}(\Omega)}\\ &\leqslant c_1\|\nabla\varphi\|_{L^2(\Omega)}\|u\|^2_{L^{12/5}(\Omega)}\\ &\leqslant c_2\|\nabla\varphi\|_{L^2(\Omega)}\big(\|v\|^2_{L^{12/5}(\Omega)} + \big|v(x_0)\big|^2\big)\\ &\leqslant c_3\|\nabla\varphi\|_{L^2(\Omega)}\big(\|v\|^2_{H_0^1(\Omega)} + \big|v(x_0)\big|^2\big),\end{aligned}$$

for some constants $c_1, c_2, c_3 > 0$. Clearly, this gives (2.17).

Finally, we have to prove

$$\varphi \in W^{2,p}(\Omega), \quad \|\varphi\|_{W^{2,p}(\Omega)} \leqslant C_0\big(\|v\|^2_{H_0^1(\Omega)} + \big|v(x_0)\big|^2\big), \quad 1 \leqslant p < \frac{3}{2}. \tag{2.18}$$

Due to (2.3), (2.2) and the Sobolev inequality we know that

$$u^2 \in L^p(\Omega), \quad \big\|u^2\big\|_{L^p(\Omega)} \leqslant c_4\big(\|v\|^2_{H_0^1(\Omega)} + \big|v(x_0)\big|^2\big), \quad 1 \leqslant p < \frac{3}{2},$$

for some constant $c_4 > 0$. Then (2.18) is consequence of Lemma 2.2 and of the Sobolev embedding theorem. □

Our argument is based on the following recursive approximation of (2.11). We begin by fixing a map $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, for each $n \in \mathbb{N}$, we define $v_n \in H^2(\Omega) \cap H_0^1(\Omega)$ as one of the *minimal energy solutions* of the following nonlinear problem:

$$\begin{aligned}&-\Delta v_n(x) + \alpha \int_\Omega G(x,y)v_n^2(y)v_n(x)\,\mathrm{d}y - 2v_{n-1}(x_0)\frac{\alpha}{\beta}\int_\Omega G(x,y)G(y,x_0)v_n(y)v_n(x)\,\mathrm{d}y\\ &\quad + v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_\Omega G(x,y)G^2(y,x_0)v_n(x)\,\mathrm{d}y - v_{n-1}(x_0)\frac{\alpha}{\beta}\int_\Omega G(x,y)G(x,x_0)v_n^2(y)\,\mathrm{d}y\\ &\quad + 2v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_\Omega G(x,y)G(y,x_0)G(x,x_0)v_n(y)\,\mathrm{d}y - v_{n-1}^3(x_0)\frac{\alpha}{\beta^3}\int_\Omega G(x,y)G^2(y,x_0)G(x,x_0)\,\mathrm{d}y\\ &= \omega v_n(x) - v_{n-1}(x_0)\frac{\omega}{\beta}G(x,x_0).\end{aligned} \tag{2.19}$$

Observe that, denoting

$$u_n := v_n - \frac{v_{n-1}(x_0)}{\beta}G(\cdot, x_0), \quad n \in \mathbb{N}, \tag{2.20}$$

we can rewrite (2.19) in the following form

$$-\Delta v_n(x) + \alpha u_n(x)\int_\Omega G(x,y)u_n^2(y)\,\mathrm{d}y = \omega u_n(x). \tag{2.21}$$

3. Approximate solutions

In this section we prove that the sequence $\{v_n\}_{n\in\mathbb{N}}$ exists. Arguing by induction, it is enough to fix $n \in \mathbb{N}$ and show that we can find v_n using v_{n-1}.

In other words we have to prove the existence of a *minimal energy solution* for the equation

$$
\begin{aligned}
&-\Delta v(x) + \alpha \int_\Omega G(x,y) v^2(y) v(x)\,\mathrm{d}y - 2 v_{n-1}(x_0) \frac{\alpha}{\beta} \int_\Omega G(x,y) G(y,x_0) v(y) v(x)\,\mathrm{d}y \\
&\quad + v_{n-1}^2(x_0) \frac{\alpha}{\beta^2} \int_\Omega G(x,y) G^2(y,x_0) v(x)\,\mathrm{d}y - v_{n-1}(x_0) \frac{\alpha}{\beta} \int_\Omega G(x,y) G(x,x_0) v^2(y)\,\mathrm{d}y \\
&\quad + 2 v_{n-1}^2(x_0) \frac{\alpha}{\beta^2} \int_\Omega G(x,y) G(y,x_0) G(x,x_0) v(y)\,\mathrm{d}y - v_{n-1}^3(x_0) \frac{\alpha}{\beta^3} \int_\Omega G(x,y) G^2(y,x_0) G(x,x_0)\,\mathrm{d}y \\
&= \omega v(x) - v_{n-1}(x_0) \frac{\omega}{\beta} G(x,x_0),
\end{aligned}
\tag{3.1}
$$

endowed with the boundary condition

$$
v = 0, \quad \text{on } \partial\Omega. \tag{3.2}
$$

3.1. The variational approach

Consider the functional

$$
J_n : H_0^1(\Omega) \longrightarrow \mathbb{R}
$$

defined as follows

$$
\begin{aligned}
J_n(v) &= \frac{1}{2} \int_\Omega \left|\nabla v(x)\right|^2 \mathrm{d}x + \frac{\alpha}{4} \int_{\Omega\times\Omega} G(x,y) v^2(y) v^2(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - v_{n-1}(x_0) \frac{\alpha}{\beta} \int_{\Omega\times\Omega} G(x,y) G(y,x_0) v(y) v^2(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + v_{n-1}^2(x_0) \frac{\alpha}{\beta^2} \int_{\Omega\times\Omega} G(x,y) G(y,x_0) G(x,x_0) v(y) v(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + v_{n-1}^2(x_0) \frac{\alpha}{2\beta^2} \int_{\Omega\times\Omega} G(x,y) G^2(y,x_0) v^2(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - v_{n-1}^3(x_0) \frac{\alpha}{\beta^3} \int_{\Omega\times\Omega} G(x,y) G^2(y,x_0) G(x,x_0) v(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{\omega}{2} \int_\Omega v^2(x)\,\mathrm{d}x - v_{n-1}(x_0) \frac{\omega}{\beta} \int_\Omega G(x,x_0) v(x)\,\mathrm{d}x.
\end{aligned}
$$

Lemma 3.1. *Assume (1.2). Then J_n is of class C^1 on $H_0^1(\Omega)$ and its critical points are the distributional solutions of* (3.1)–(3.2).

Proof. Define

$$
\begin{aligned}
J_{n,1}(v) &= \frac{1}{2} \int_\Omega \left|\nabla v(x)\right|^2 \mathrm{d}x + v_{n-1}^2(x_0) \frac{\alpha}{2\beta^2} \int_{\Omega\times\Omega} G(x,y) G^2(y,x_0) v^2(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - v_{n-1}^3(x_0) \frac{\alpha}{\beta^3} \int_{\Omega\times\Omega} G(x,y) G^2(y,x_0) G(x,x_0) v(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{\omega}{2} \int_\Omega v^2(x)\,\mathrm{d}x - v_{n-1}(x_0) \frac{\omega}{\beta} \int_\Omega G(x,x_0) v(x)\,\mathrm{d}x,
\end{aligned}
$$

$$J_{n,2}(v) = \frac{\alpha}{4}\int\limits_{\Omega\times\Omega} G(x,y)v^2(y)v^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$J_{n,3}(v) = -v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)v(y)v^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$J_{n,4}(v) = v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)v(x)\,\mathrm{d}x\,\mathrm{d}y,$$

and observe that $J_n = J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4}$.

Let $\varepsilon \in \mathbb{R}$, $f, v \in H_0^1(\Omega)$. We begin by computing the derivative of $J_{n,1}$:

$$\begin{aligned} J_{n,1}(v+\varepsilon f) = {} & \frac{1}{2}\int\limits_{\Omega}\bigl|\nabla\bigl[v(x)+\varepsilon f(x)\bigr]\bigr|^2\,\mathrm{d}x + v_{n-1}^2(x_0)\frac{\alpha}{2\beta^2}\int\limits_{\Omega\times\Omega} G(x,y)G^2(y,x_0)\bigl[v(x)+\varepsilon f(x)\bigr]^2\,\mathrm{d}x\,\mathrm{d}y \\ & - v_{n-1}^3(x_0)\frac{\alpha}{\beta^3}\int\limits_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\bigl[v(x)+\varepsilon f(x)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\ & - \frac{\omega}{2}\int\limits_{\Omega}\bigl[v(x)+\varepsilon f(x)\bigr]^2\,\mathrm{d}x - v_{n-1}(x_0)\frac{\omega}{\beta}\int\limits_{\Omega} G(x,x_0)\bigl[v(x)+\varepsilon f(x)\bigr]\,\mathrm{d}x. \end{aligned}$$

Since

$$\begin{aligned} J'_{n,1}(v)[f] &= \frac{\mathrm{d}J_{n,1}}{\mathrm{d}\varepsilon}(v+\varepsilon f)\Big|_{\varepsilon=0} \\ &= \int\limits_{\Omega}\nabla v(x)\cdot\nabla f(x)\,\mathrm{d}x + v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int\limits_{\Omega\times\Omega} G(x,y)G^2(y,x_0)v(x)f(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - v_{n-1}^3(x_0)\frac{\alpha}{\beta^3}\int\limits_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)f(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \omega\int\limits_{\Omega} v(x)f(x)\,\mathrm{d}x - v_{n-1}(x_0)\frac{\omega}{\beta}\int\limits_{\Omega} G(x,x_0)f(x)\,\mathrm{d}x, \end{aligned}$$

we conclude

$$\begin{aligned} J'_{n,1}(v) = {} & -\Delta v(x) + v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int\limits_{\Omega} G(x,y)G^2(y,x_0)v(x)\,\mathrm{d}y \\ & - v_{n-1}^3(x_0)\frac{\alpha}{\beta^3}\int\limits_{\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\,\mathrm{d}y - \omega v(x) - v_{n-1}(x_0)\frac{\omega}{\beta}G(x,x_0). \end{aligned} \tag{3.3}$$

We continue by computing the derivative of $J_{n,2}$:

$$J_{n,2}(v+\varepsilon f) = \frac{\alpha}{4}\int\limits_{\Omega\times\Omega} G(x,y)\bigl[v(y)+\varepsilon f(y)\bigr]^2\bigl[v(x)+\varepsilon f(x)\bigr]^2\,\mathrm{d}x\,\mathrm{d}y,$$

$$\begin{aligned} \frac{\mathrm{d}J_{n,2}}{\mathrm{d}\varepsilon}(v+\varepsilon f) = {} & \frac{\alpha}{2}\int\limits_{\Omega\times\Omega} G(x,y)\bigl[v(y)+\varepsilon f(y)\bigr]f(y)\bigl[v(x)+\varepsilon f(x)\bigr]^2\,\mathrm{d}x\,\mathrm{d}y \\ & + \frac{\alpha}{2}\int\limits_{\Omega\times\Omega} G(x,y)\bigl[v(y)+\varepsilon f(y)\bigr]^2\bigl[v(x)+\varepsilon f(x)\bigr]f(x)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Using the symmetry of G (i.e., $G(x,y) = G(y,x)$)

$$
\begin{aligned}
J'_{n,2}(v)[f] &= \frac{\mathrm{d}J_{n,2}}{\mathrm{d}\varepsilon}(v+\varepsilon f)\bigg|_{\varepsilon=0} \\
&= \frac{\alpha}{2}\int\limits_{\Omega\times\Omega} G(x,y)v(y)f(y)v^2(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{\alpha}{2}\int\limits_{\Omega\times\Omega} G(x,y)v^2(y)v(x)f(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \alpha\int\limits_{\Omega\times\Omega} G(x,y)v^2(y)v(x)f(x)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

namely

$$
J'_{n,2}(v) = \alpha\int\limits_{\Omega} G(x,y)v^2(y)v(x)\,\mathrm{d}y. \tag{3.4}
$$

We pass to the derivative of $J_{n,3}$:

$$
\begin{aligned}
J_{n,3}(v+\varepsilon f) = {} & -v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)v(y)\big[v(x)+\varepsilon f(x)\big]^2\,\mathrm{d}x\,\mathrm{d}y \\
& - \varepsilon v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)f(y)\big[v(x)+\varepsilon f(x)\big]^2\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}J_{n,3}}{\mathrm{d}\varepsilon}(v+\varepsilon f) = {} & -2v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)v(y)\big[v(x)+\varepsilon f(x)\big]f(x)\,\mathrm{d}x\,\mathrm{d}y \\
& - v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)f(y)\big[v(x)+\varepsilon f(x)\big]^2\,\mathrm{d}x\,\mathrm{d}y \\
& - 2\varepsilon v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)f(y)\big[v(x)+\varepsilon f(x)\big]f(x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Due to the symmetry of G

$$
\begin{aligned}
J'_{n,3}(v)[f] &= \frac{\mathrm{d}J_{n,3}}{\mathrm{d}\varepsilon}(v+\varepsilon f)\bigg|_{\varepsilon=0} \\
&= -2v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)v(y)v(x)f(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega\times\Omega} G(x,y)G(x,x_0)f(x)v^2(y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

we get

$$
J'_{n,3}(v) = -2v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega} G(x,y)G(y,x_0)v(y)v(x)\,\mathrm{d}y - v_{n-1}(x_0)\frac{\alpha}{\beta}\int\limits_{\Omega} G(x,y)G(x,x_0)v^2(y)\,\mathrm{d}y. \tag{3.5}
$$

Finally, we look at the derivative of $J_{n,4}$:

$$
\begin{aligned}
J_{n,4}(v+\varepsilon f) = {} & v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)\big[v(x)+\varepsilon f(x)\big]\,\mathrm{d}x\,\mathrm{d}y \\
& + \varepsilon v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int\limits_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)f(y)\big[v(x)+\varepsilon f(x)\big]\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

$$\frac{\mathrm{d}J_{n,4}}{\mathrm{d}\varepsilon}(v+\varepsilon f) = v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)f(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)f(y)\big[v(x)+\varepsilon f(x)\big]\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \varepsilon v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)f(y)f(x)\,\mathrm{d}x\,\mathrm{d}y.$$

Again by the symmetry of G we find

$$J'_{n,4}(v)[f] = \frac{\mathrm{d}J_{n,4}}{\mathrm{d}\varepsilon}(v+\varepsilon f)\bigg|_{\varepsilon=0}$$
$$= 2v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)f(x)\,\mathrm{d}x\,\mathrm{d}y,$$

and thus we obtain

$$J'_{n,4}(v) = 2v_{n-1}^2(x_0)\frac{\alpha}{\beta^2}\int_{\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)\,\mathrm{d}y. \tag{3.6}$$

The claim is direct consequence of (3.3)–(3.6). □

The next step in the analysis of (3.1) is a discussion of the topological properties of the functional J_n.

Lemma 3.2 *(Weak lower semicontinuity). Assume* (1.2). *The functional J_n is weakly lower semicontinuous on $H_0^1(\Omega)$, namely*

$$v_k \rightharpoonup v \text{ weakly in } H_0^1(\Omega) \Longrightarrow \liminf_k J_n(v_k) \geqslant J_n(v). \tag{3.7}$$

In particular, the functionals

$$I_{1,n}(v) = \int_{\Omega\times\Omega} G(x,y)v^2(y)v^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$I_{2,n}(v) = \int_{\Omega\times\Omega} G(x,y)G(y,x_0)v(y)v^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$I_{3,n}(v) = \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(y)v(x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$I_{4,n}(v) = \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)v^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$I_{5,n}(v) = \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)v(x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$I_{6,n}(v) = \int_{\Omega} v^2(x)\,\mathrm{d}x,$$
$$I_{7,n}(v) = \int_{\Omega} G(x,x_0)v(x)\,\mathrm{d}x,$$

are all weakly continuous, i.e.,

$$v_k \rightharpoonup v \text{ weakly in } H_0^1(\Omega) \Longrightarrow \lim_k I_{i,n}(v_k) = I_{i,n}(v),\ i\in\{1,\dots,7\}. \tag{3.8}$$

Proof. Let $\{v_k\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$ such that

$$v_k \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega). \tag{3.9}$$

We know that

$$\liminf_k \int_\Omega \left|\nabla v_k(x)\right|^2 \mathrm{d}x \geqslant \int_\Omega \left|\nabla v(x)\right|^2 \mathrm{d}x. \tag{3.10}$$

The compact embeddings of $H_0^1(\Omega)$

$$v_k \longrightarrow v \quad \text{strongly in } L^p(\Omega),\ 1 \leqslant p < 6, \tag{3.11}$$

imply

$$\lim_k I_{6,n}(v_k) = I_{6,n}(v). \tag{3.12}$$

Moreover, since $G(\cdot, x_0) \in L^2(\Omega)$ we have also

$$\lim_k I_{7,n}(v_k) = I_{7,n}(v). \tag{3.13}$$

Observe that

$$I_{5,n}(v) = \int_\Omega f_1(x)v(x)\,\mathrm{d}x,$$

where

$$f_1 := \int_\Omega G(\cdot, y)G^2(y, x_0)G(\cdot, x_0)\,\mathrm{d}y \in L^{3/2}(\Omega),$$

indeed from (2.14) we have that

$$f_2 := \int_\Omega G(\cdot, y)G^2(y, x_0)\,\mathrm{d}y \in H_0^1(\Omega), \tag{3.14}$$

so, using the Tonelli theorem and the Hölder inequality,

$$\begin{aligned}
\|f_1\|_{L^{3/2}(\Omega)}^{3/2} &= \int_\Omega \left(\int_\Omega G(x, y)G^2(y, x_0)G(x, x_0)\,\mathrm{d}y\right)^{3/2} \mathrm{d}x \\
&= \int_\Omega G^{3/2}(x, x_0)\left(\int_\Omega G(x, y)G^2(y, x_0)\,\mathrm{d}y\right)^{3/2} \mathrm{d}x \\
&= \int_\Omega G^{3/2}(x, x_0) f_2^{3/2}(x)\,\mathrm{d}x \\
&\leqslant \left\|G^{3/2}(\cdot, x_0)\right\|_{L^{4/3}(\Omega)} \left\|f_2^{3/2}\right\|_{L^4(\Omega)} \\
&= \left\|G(\cdot, x_0)\right\|_{L^2(\Omega)}^{2/3} \|f_2\|_{L^6(\Omega)}^{2/3} < \infty.
\end{aligned} \tag{3.15}$$

Therefore, employing (3.11) and (3.15)

$$\lim_k I_{5,n}(v_k) = I_{5,n}(v). \tag{3.16}$$

From (3.11)

$$v_k^2 \longrightarrow v^2 \quad \text{strongly in } L^p(\Omega),\ 1 \leqslant p < 3, \tag{3.17}$$

so using

$$I_{4,n}(v) = \int_{\Omega} f_2(x)v(x)\,\mathrm{d}x$$

and (3.14) we conclude

$$\lim_k I_{4,n}(v_k) = I_{4,n}(v). \tag{3.18}$$

We continue with $I_{3,n}$. Observe that

$$\begin{aligned}
I_{3,n}(v_k) - I_{3,n}(v) &= \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)\big(v_k(y)v_k(x) - v(y)v(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v_k(y)\big(v_k(x) - v(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)v(x)\big(v_k(y) - v(y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega} f_{3,k}(x)G(x,x_0)\big(v_k(x) - v(x)\big)\,\mathrm{d}x + \int_{\Omega} f_4(y)G(y,x_0)\big(v_k(y) - v(y)\big)\,\mathrm{d}y,
\end{aligned}$$

where

$$f_{3,k} := \int_{\Omega} G(\cdot,y)G(y,x_0)v_k(y)\,\mathrm{d}y, \qquad f_4 := \int_{\Omega} G(x,\cdot)G(x,x_0)v(x)\,\mathrm{d}x.$$

Using the symmetry of G, (3.11) and (2.14) we have that $f_4 \in H_0^1(\Omega)$ and the sequence $\{f_{3,k}\}_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Moreover, using the Hölder inequality

$$\begin{aligned}
\left|\int_{\Omega} f_{3,k}(x)G(x,x_0)\big(v_k(x) - v(x)\big)\,\mathrm{d}x\right| &\leqslant \|f_{3,k}\|_{L^6(\Omega)} \big\|G(\cdot,x_0)(v_k - v)\big\|_{L^{6/5}(\Omega)} \\
&\leqslant \|f_{3,k}\|_{L^6(\Omega)} \big\|G^{6/5}(\cdot,x_0)\big\|_{L^2(\Omega)}^{5/6} \big\||v_k - v|^{6/5}\big\|_{L^2(\Omega)}^{5/6} \\
&= \|f_{3,k}\|_{L^6(\Omega)} \big\|G(\cdot,x_0)\big\|_{L^{12/5}(\Omega)} \|v_k - v\|_{L^{12/5}(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega} f_4(y)G(y,x_0)\big(v_k(y) - v(y)\big)\,\mathrm{d}y\right| &\leqslant \|f_4\|_{L^6(\Omega)} \big\|G(\cdot,x_0)(v_k - v)\big\|_{L^{6/5}(\Omega)} \\
&\leqslant \|f_4\|_{L^6(\Omega)} \big\|G^{6/5}(\cdot,x_0)\big\|_{L^2(\Omega)}^{5/6} \big\||v_k - v|^{6/5}\big\|_{L^2(\Omega)}^{5/6} \\
&= \|f_4\|_{L^6(\Omega)} \big\|G(\cdot,x_0)\big\|_{L^{12/5}(\Omega)} \|v_k - v\|_{L^{12/5}(\Omega)},
\end{aligned}$$

hence, (3.11) implies

$$\lim_k I_{3,n}(v_k) = I_{3,n}(v). \tag{3.19}$$

The next step is the weak semicontinuity of $I_{2,n}$. Observe that

$$\begin{aligned}
I_{2,n}(v_k) - I_{2,n}(v) &= \int_{\Omega\times\Omega} G(x,y)G(y,x_0)\big(v_k(y)v_k^2(x) - v(y)v^2(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega\times\Omega} G(x,y)G(y,x_0)v_k(y)\big(v_k^2(x) - v^2(x)\big)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$+\int_{\Omega\times\Omega} G(x,y)G(y,x_0)v^2(x)\big(v_k(y)-v(y)\big)\,dx\,dy$$
$$=\int_\Omega f_{5,k}(x)\big(v_k^2(x)-v^2(x)\big)\,dx+\int_\Omega f_6(y)\big(v_k(y)-v(y)\big)\,dy,$$

where

$$f_{5,k}:=\int_\Omega G(\cdot,y)G(y,x_0)v_k(y)\,dy,\qquad f_6:=\int_\Omega G(x,\cdot)G(\cdot,x_0)v^2(x)\,dx.$$

From (3.11) and (2.14) we have that the sequence $\{f_{5,k}\}_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Moreover $f_6\in L^{3/2}(\Omega)$, indeed

$$f_6:=G(\cdot,x_0)f_7,\quad \text{where } f_7:=\int_\Omega G(x,\cdot)v^2(x)\,dx.$$

Due to (2.14) we know that $f_7\in H_0^1(\Omega)$, so using the Hölder inequality

$$\begin{aligned}\|f_6\|_{L^{3/2}(\Omega)}^{3/2}&=\int_\Omega G^{3/2}(y,x_0)\big|f_7(y)\big|^{3/2}\,dy\\&\leqslant\big\|G^{3/2}(\cdot,x_0)\big\|_{L^{4/3}(\Omega)}\big\|f_7^{3/2}\big\|_{L^4(\Omega)}\\&=\big\|G(\cdot,x_0)\big\|_{L^2(\Omega)}^{2/3}\big\|f_7^{3/2}\big\|_{L^4(\Omega)}^{2/3}<\infty.\end{aligned}$$

Hence, (3.11) and (3.17) imply

$$\lim_k I_{2,n}(v_k)=I_{2,n}(v).\tag{3.20}$$

Finally, we consider $I_{1,n}$. Observe that

$$\begin{aligned}I_{1,n}(v_k)-I_{1,n}(v)&=\int_{\Omega\times\Omega} G(x,y)\big(v_k^2(y)v_k^2(x)-v^2(y)v^2(x)\big)\,dx\,dy\\&=\int_{\Omega\times\Omega} G(x,y)v_k^2(y)\big(v_k^2(x)-v^2(x)\big)\,dx\,dy+\int_{\Omega\times\Omega} G(x,y)v^2(x)\big(v_k^2(y)-v^2(y)\big)\,dx\,dy.\end{aligned}$$

Using the symmetry of G, (3.17) and (2.14) we have that $\int_\Omega G(x,\cdot)v^2(x)\,dx\in L^2(\Omega)$ and the sequence $\{\int_\Omega G(\cdot,y)v_k^2(y)\,dy\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Then, employing (3.17), we obtain

$$\lim_k I_{1,n}(v_k)=I_{1,n}(v).\tag{3.21}$$

Due to (3.10), (3.12), (3.13), (3.16), (3.18)–(3.21) the proof is complete. □

Lemma 3.3 *(Coercivity). Assume* (1.2). *The functional J_n is coercive in $H_0^1(\Omega)$, i.e.,*

$$\|v_k\|_{H_0^1(\Omega)}\longrightarrow\infty\Longrightarrow\lim_k J_n(v_k)=\infty.\tag{3.22}$$

Proof. Let $\{v_k\}_{k\in\mathbb{N}}\subset H_0^1(\Omega)$ such that

$$\|v_k\|_{H_0^1(\Omega)}\longrightarrow\infty.\tag{3.23}$$

We have to prove

$$\lim_k J_n(v_k)=\infty.\tag{3.24}$$

Define

$$\lambda_k:=\|v_k\|_{H_0^1(\Omega)},\qquad \tilde v_k:=\frac{v_k}{\lambda_k},$$

obviously,

$$v_k = \lambda_k \tilde{v}_k, \quad \lambda_k \longrightarrow \infty, \quad \|\tilde{v}_k\|_{H_0^1(\Omega)} = 1.$$

From the definition of J_n we get

$$J_n(v_k) = J_n(\lambda_k \tilde{v}_k) = \frac{\lambda_k^2}{2} + a_k\lambda_k^4 + b_k\lambda_k^3 + c_k\lambda_k^2 + d_k\lambda_k, \tag{3.25}$$

where

$$a_k := \frac{\alpha}{4} \int_{\Omega\times\Omega} G(x,y)\tilde{v}_k^2(y)\tilde{v}_k^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$b_k := -v_{n-1}(x_0)\frac{\alpha}{\beta} \int_{\Omega\times\Omega} G(x,y)G(y,x_0)\tilde{v}_k(y)\tilde{v}_k^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$\begin{aligned} c_k := {} & v_{n-1}^2(x_0)\frac{\alpha}{\beta^2} \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)\tilde{v}_k(y)\tilde{v}_k(x)\,\mathrm{d}x\,\mathrm{d}y \\ & + v_{n-1}^2(x_0)\frac{\alpha}{2\beta^2} \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)\tilde{v}_k^2(x)\,\mathrm{d}x\,\mathrm{d}y - \frac{\omega}{2}\int_{\Omega} \tilde{v}_k^2(x)\,\mathrm{d}x, \end{aligned}$$

$$d_k := -v_{n-1}^3(x_0)\frac{\alpha}{\beta^3} \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\tilde{v}_k(x)\,\mathrm{d}x\,\mathrm{d}y - v_{n-1}(x_0)\frac{\omega}{\beta}\int_{\Omega} G(x,x_0)\tilde{v}_k(x)\,\mathrm{d}x.$$

Due to the boundedness of $\{\tilde{v}_k\}_k$ in $H_0^1(\Omega)$, there exists $\tilde{v} \in H_0^1(\Omega)$ such that $\|\tilde{v}\|_{H_0^1(\Omega)} \leqslant 1$ and (passing to a subsequence)

$$\tilde{v}_k \rightharpoonup \tilde{v} \quad \text{weakly in } H_0^1(\Omega). \tag{3.26}$$

We distinguish two cases.

Case 1. If

$$\tilde{v} \neq 0,$$

then, employing the second part of Lemma 3.2 and (3.26),

$$a_k \longrightarrow \frac{\alpha}{4} \int_{\Omega\times\Omega} G(x,y)\tilde{v}^2(y)\tilde{v}^2(x)\,\mathrm{d}x\,\mathrm{d}y \in (0,\infty),$$

$$b_k \longrightarrow -v_{n-1}(x_0)\frac{\alpha}{\beta} \int_{\Omega\times\Omega} G(x,y)G(y,x_0)\tilde{v}(y)\tilde{v}^2(x)\,\mathrm{d}x\,\mathrm{d}y \in \mathbb{R},$$

$$\begin{aligned} c_k \longrightarrow {} & v_{n-1}^2(x_0)\frac{\alpha}{\beta^2} \int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)\tilde{v}(y)\tilde{v}(x)\,\mathrm{d}x\,\mathrm{d}y \\ & + v_{n-1}^2(x_0)\frac{\alpha}{2\beta^2} \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)\tilde{v}^2(x)\,\mathrm{d}x\,\mathrm{d}y - \frac{\omega}{2}\int_{\Omega} \tilde{v}^2(x)\,\mathrm{d}x \in \mathbb{R}, \end{aligned}$$

$$d_k \longrightarrow -v_{n-1}^3(x_0)\frac{\alpha}{\beta^3} \int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\tilde{v}(x)\,\mathrm{d}x\,\mathrm{d}y - v_{n-1}(x_0)\frac{\omega}{\beta}\int_{\Omega} G(x,x_0)\tilde{v}(x)\,\mathrm{d}x \in \mathbb{R}.$$

Clearly, from (3.25) we get

$$\lim_k \frac{J_n(v_n)}{\lambda_k^4} = \lim_k a_k = \frac{\alpha}{4} \int_{\Omega\times\Omega} G(x,y)\tilde{v}^2(y)\tilde{v}^2(x)\,\mathrm{d}x\,\mathrm{d}y > 0,$$

and then (3.24) follows.

Case 2. If

$$\tilde{v}=0,$$

then, employing the second part of Lemma 3.2 and (3.26),

$$a_k, b_k, c_k, d_k \longrightarrow 0.$$

Due to (3.23),

$$\begin{aligned}
&\lim_k\big(a_k\lambda_k^4+b_k\lambda_k^3\big)\\
&\quad=\lim_k\bigg(\frac{\alpha}{4}\int_{\Omega\times\Omega} G(x,y)v_k^2(y)v_k^2(x)\,\mathrm{d}x\,\mathrm{d}y-v_{n-1}(x_0)\frac{\alpha}{\beta}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)v_k(y)v_k^2(x)\,\mathrm{d}x\,\mathrm{d}y\bigg)\\
&\quad=\lim_k\int_{\Omega\times\Omega} G(x,y)v_k^2(x)\bigg(\frac{\alpha}{4}v_k^2(y)-v_{n-1}(x_0)\frac{\alpha}{\beta}G(y,x_0)v_k(y)\bigg)\,\mathrm{d}x\,\mathrm{d}y=\infty,
\end{aligned}$$

hence

$$\liminf_k\big(a_k\lambda_k^2+b_k\lambda_k\big)\geqslant 0.$$

By (3.25) we get

$$\liminf_k\frac{J_n(v_n)}{\lambda_k^2}=\lim_k\bigg(\frac{1}{2}+c_k+\frac{\mathrm{d}_k}{\lambda_k}\bigg)+\liminf_k\big(a_k\lambda_k^2+b_k\lambda_k\big)\geqslant\frac{1}{2},$$

from which (3.24) easily follows.

The lemma is proved. □

The two previous lemmas imply:

Lemma 3.4 *(Boundedness from below). Assume that* (1.2) *holds. The functional J_n is bounded from below on $H_0^1(\Omega)$. In particular, there exists $v_n\in H_0^1(\Omega)$ such that*

$$J_n(v_n)=\min_{v\in H_0^1(\Omega)}J_n(v),\qquad J_n'(v_n)=0. \tag{3.27}$$

We conclude this section with the following regularity result.

Lemma 3.5 *(Regularity). Assume* (1.2). *The critical points of the functional J_n belong to $H^2(\Omega)$. More precisely, if $v\in H_0^1(\Omega)$ and $J_n'(v)=0$, then $v\in H^2(\Omega)$ and*

$$\|\Delta v\|_{L^2(\Omega)}\leqslant C_1\big(\|v\|_{H_0^1(\Omega)}+\|v\|^3_{H_0^1(\Omega)}+\big|v_{n-1}(x_0)\big|+\big|v_{n-1}(x_0)\big|^3\big), \tag{3.28}$$

for some constant $C_1>0$ independent of n.

Proof. Let $v\in H_0^1(\Omega)$ be a critical point of J_n. Due to Lemma 3.1, v is a weak solution of (3.1)–(3.2). Thus (cf. (2.21), (3.1)) writing as in (2.21) we get

$$-\Delta v+\alpha\varphi u=\omega u, \tag{3.29}$$

where

$$\varphi=\int_\Omega G(\cdot,y)u^2(y)\,\mathrm{d}y,\qquad u=v-\frac{v_{n-1}(x_0)}{\beta}G(\cdot,x_0).$$

From (3.29)

$$\|\Delta v\|_{L^2(\Omega)}\leqslant\|\varphi u\|_{L^2(\Omega)}+\omega\|u\|_{L^2(\Omega)}. \tag{3.30}$$

Using the definition of u and the Poincaré inequality

$$\begin{aligned}\|u\|_{L^2(\Omega)} &\leqslant \|v\|_{L^2(\Omega)} + \frac{|v_{n-1}(x_0)|}{\beta}\big\|G(\cdot,x_0)\big\|_{L^2(\Omega)}\\ &\leqslant c_1\big(\|v\|_{H_0^1(\Omega)} + \big|v_{n-1}(x_0)\big|\big),\end{aligned} \tag{3.31}$$

for some constant $c_1 > 0$.

The Hölder inequality and (2.14) give

$$\begin{aligned}\|\varphi u\|_{L^2(\Omega)} &\leqslant \big(\big\|\varphi^2\big\|_{L^6(\Omega)}\big\|u^2\big\|_{L^{6/5}(\Omega)}\big)^{1/2}\\ &= \|\varphi\|_{L^{12}(\Omega)}\|u\|_{L^{12/5}(\Omega)}\\ &\leqslant c_2\big(\|v\|^2_{H_0^1(\Omega)} + \big|v_{n-1}(x_0)\big|^2\big)\big(\|v\|_{H_0^1(\Omega)} + \big|v_{n-1}(x_0)\big|\big)\\ &\leqslant c_3\big(\|v\|^3_{H_0^1(\Omega)} + \big|v_{n-1}(x_0)\big|^3\big),\end{aligned} \tag{3.32}$$

for some constants $c_2, c_3 > 0$.

The claim is direct consequence of (3.30)–(3.32). □

Now it is clear that given v_{n-1}, we define v_n as one of the minimizers of the functional J_n, and we say that v_n is a *minimal energy solution*. Observe that the existence of such map is in Lemma 3.4, but only with Lemma 3.5 we have the H^2 regularity; this is needed because in the equation for v_{n+1} we have the presence of $v_n(x_0)$.

4. Proof of Theorem 2.1

In this section we conclude the proof of Theorem 2.1. Our argument is based on the following two steps:

(i) The sequence of the approximated solutions $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega)\cap H_0^1(\Omega)$, thus we can extract a subsequence converging to a weak solution of (1.1) in the sense of Definition 2.1; and
(ii) the limit solution is nontrivial.

4.1. Compactness of the approximants

Here we prove the compactness of the sequence $\{v_n\}_n$ in $H^2(\Omega)\cap H_0^1(\Omega)$.

Lemma 4.1. *Assume* (1.2)*. There exists a subsequence* $\{v_{n_k}\}_{k\in\mathbb{N}}$ *and a map* $v\in H^2(\Omega)\cap H_0^1(\Omega)$ *such that*

$$v_{n_k} \rightharpoonup v \quad \text{weakly in } H^2(\Omega)\cap H_0^1(\Omega). \tag{4.1}$$

In particular,

$$v_{n_k} \longrightarrow v \quad \text{uniformly in } \Omega. \tag{4.2}$$

Proof. Clearly it suffices to prove that

$$\text{the sequence } \{v_n\}_{n\in\mathbb{N}} \text{ is bounded in } H^2(\Omega)\cap H_0^1(\Omega). \tag{4.3}$$

We begin by showing that

$$\text{the sequence } \{v_n\}_{n\in\mathbb{N}} \text{ is bounded in } H_0^1(\Omega), \text{ and} \tag{4.4}$$

$$\text{the sequence } \{v_n(x_0)\}_{n\in\mathbb{N}} \text{ is bounded in } \mathbb{R}. \tag{4.5}$$

Multiplying (2.21) by u_n (cf. (2.20)) and integrating on Ω we get

$$\int_\Omega \big|\nabla v_n(x)\big|^2\,dx + \frac{v_{n-1}(x_0)}{\beta}\int_\Omega \Delta v_n(x)G(x,x_0)\,dx + \alpha\int_{\Omega\times\Omega} G(x,y)u_n^2(y)u_n^2(x)\,dx\,dy = \omega\int_\Omega u_n^2(x)\,dx.$$

Integration by parts gives

$$\frac{v_{n-1}(x_0)}{\beta}\int_\Omega \Delta v_n(x)G(x,x_0)\,\mathrm{d}x = \frac{v_{n-1}(x_0)}{\beta}\int_\Omega v_n(x)\Delta G(x,x_0)\,\mathrm{d}x = -\frac{v_{n-1}(x_0)v_n(x_0)}{\beta},$$

thus

$$\int_\Omega \left|\nabla v_n(x)\right|^2 \mathrm{d}x + \alpha\int_{\Omega\times\Omega} G(x,y)u_n^2(y)u_n^2(x)\,\mathrm{d}x\,\mathrm{d}y = \omega\int_\Omega u_n^2(x)\,\mathrm{d}x + \frac{v_{n-1}(x_0)v_n(x_0)}{\beta}. \tag{4.6}$$

Moreover, multiplying (2.21) by $G(x,x_0)$ and integrating on Ω we have

$$-\int_\Omega G(x,x_0)\Delta v_n(x)\,\mathrm{d}x + \alpha\int_{\Omega\times\Omega} G(x,y)G(x,x_0)u_n^2(y)u_n(x)\,\mathrm{d}x\,\mathrm{d}y = \omega\int_\Omega G(x,x_0)u_n(x)\,\mathrm{d}x.$$

Integration by parts, viz.

$$-\int_\Omega G(x,x_0)\Delta v_n(x)\,\mathrm{d}x = -\int_\Omega \Delta G(x,x_0)v_n(x)\,\mathrm{d}x = v_n(x_0),$$

yields

$$v_n(x_0) = \omega\int_\Omega G(x,x_0)u_n(x)\,\mathrm{d}x - \alpha\int_{\Omega\times\Omega} G(x,y)G(x,x_0)u_n^2(y)u_n(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.7}$$

Using (4.7) in (4.6) we obtain

$$\begin{aligned}&\int_\Omega |\nabla v_n|^2\,\mathrm{d}x + \alpha\int_{\Omega\times\Omega} G(x,y)u_n^2(y)u_n^2(x)\,\mathrm{d}x\,\mathrm{d}y + \alpha\frac{v_{n-1}(x_0)}{\beta}\int_{\Omega\times\Omega} G(x,y)G(x,x_0)u_n^2(y)u_n(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \omega\int_\Omega u_n^2(x)\,\mathrm{d}x + \omega\frac{v_{n-1}(x_0)}{\beta}\int_\Omega G(x,x_0)u_n(x)\,\mathrm{d}x.\end{aligned} \tag{4.8}$$

Using the definition of u_n

$$\omega\frac{v_{n-1}(x_0)}{\beta}\int_\Omega G(x,x_0)u_n(x)\,\mathrm{d}x = \omega\frac{v_{n-1}(x_0)}{\beta}\int_\Omega G(x,x_0)v_n(x)\,\mathrm{d}x - \omega\frac{v_{n-1}^2(x_0)}{\beta^2}\int_\Omega G^2(x,x_0)\,\mathrm{d}x,$$

hence from (4.8)

$$\begin{aligned}&\int_\Omega \left|\nabla v_n(x)\right|^2 \mathrm{d}x + \omega\frac{v_{n-1}^2(x_0)}{\beta^2}\int_\Omega G^2(x,x_0)\,\mathrm{d}x + \alpha\int_{\Omega\times\Omega} G(x,y)u_n^2(y)u_n^2(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad + \alpha\frac{v_{n-1}(x_0)}{\beta}\int_{\Omega\times\Omega} G(x,y)G(x,x_0)u_n^2(y)u_n(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \omega\int_\Omega u_n^2(x)\,\mathrm{d}x + \omega\frac{v_{n-1}(x_0)}{\beta}\int_\Omega G(x,x_0)v_n(x)\,\mathrm{d}x.\end{aligned} \tag{4.9}$$

Define

$$\lambda_n := \left(\int_\Omega \left|\nabla v_n(x)\right|^2 \mathrm{d}x + \omega\frac{v_{n-1}^2(x_0)}{\beta^2}\int_\Omega G^2(x,x_0)\,\mathrm{d}x\right)^{1/2},$$

and

$$\tilde{u}_n := \frac{u_n}{\lambda_n}, \qquad \tilde{v}_n := \frac{v_n}{\lambda_n}.$$

From (4.9) we get

$$\lambda_n^2 + \lambda_n^4 a_n + \lambda_n^3 \lambda_{n-1} b_n = \lambda_n^2 c_n + \lambda_n \lambda_{n-1} d_n, \tag{4.10}$$

where

$$a_n = \alpha \int\limits_{\Omega\times\Omega} G(x,y)\tilde{u}_n^2(y)\tilde{u}_n^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$b_n = \alpha \frac{\tilde{v}_{n-1}(x_0)}{\beta} \int\limits_{\Omega\times\Omega} G(x,y)G(x,x_0)\tilde{u}_n^2(y)\tilde{u}_n(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$c_n = \omega \int\limits_{\Omega} \tilde{u}_n^2(x)\,\mathrm{d}x,$$

$$d_n = \omega \frac{\tilde{v}_{n-1}(x_0)}{\beta} \int\limits_{\Omega} G(x,x_0)\tilde{v}_n(x)\,\mathrm{d}x.$$

Assume by contradiction that $\{\lambda_n\}_{n\in\mathbb{N}}$ is not bounded, namely

$$\limsup_n \lambda_n = \infty.$$

Consider the subsequence defined inductively as follows

$$\lambda_{n_0} = \lambda_0, \qquad \lambda_{n_{k+1}} = \inf\{\lambda_n \mid n > n_k,\ \lambda_n \geqslant n_k \lambda_{n_k}\}.$$

Then we have

$$\lambda_{n_{k-1}} < \lambda_{n_k}, \quad \lim_k \lambda_{n_k} = \infty, \quad \lim_k \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}} = 0. \tag{4.11}$$

Moreover observe that the sequences $\{\tilde{v}_{n_k}\}_{k\in\mathbb{N}}$, $\{\tilde{v}_{n_k}(x_0)\}_{k\in\mathbb{N}}$ are bounded in $H_0^1(\Omega)$ and $\mathbb{R}$, respectively. Then there exists $\tilde{v} \in H_0^1(\Omega)$ and $\mu \in \mathbb{R}$ such that (passing to a subsequence)

$$\tilde{v}_{n_k}(x_0) \to \mu, \qquad \tilde{v}_{n_k} \rightharpoonup \tilde{v} \quad \text{weakly in } H_0^1(\Omega). \tag{4.12}$$

Denote

$$\tilde{u} := v - \frac{\mu}{\beta} G(\cdot, x_0).$$

Due to the compact embeddings of $H_0^1(\Omega)$ and (4.12) we have (passing to a subsequence)

$$\tilde{u}_{n_k} \longrightarrow \tilde{u} \quad \text{strongly in } L^p(\Omega),\ 1 \leqslant p < 6. \tag{4.13}$$

We distinguish two cases.

Case 1. If

$$\tilde{u} \neq 0, \tag{4.14}$$

then, employing the second part of Lemma 3.2, (4.12), (4.13),

$$a_{n_k} \longrightarrow \alpha \int\limits_{\Omega\times\Omega} G(x,y)\tilde{u}^2(y)\tilde{u}^2(x)\,\mathrm{d}x\,\mathrm{d}y \in (0,\infty),$$

$$b_{n_k} \longrightarrow \alpha \frac{\mu}{\beta} \int\limits_{\Omega\times\Omega} G(x,y)G(x,x_0)\tilde{u}^2(y)\tilde{u}(x)\,\mathrm{d}x\,\mathrm{d}y \in \mathbb{R},$$

$$c_{n_k} \longrightarrow \omega \int_{\Omega} \tilde{u}^2 \,\mathrm{d}x \in (0,\infty),$$

$$d_{n_k} \longrightarrow \omega \frac{\mu}{\beta} \int_{\Omega} G(x,x_0)\tilde{v}(x)\,\mathrm{d}x \in \mathbb{R}.$$

Since we can rewrite (4.10) in the following way

$$\frac{1}{\lambda_{n_k}^2} + a_{n_k} + \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}} b_{n_k} = \frac{c_{n_k}}{\lambda_{n_k}^2} + \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}^3} d_{n_k},$$

using (4.11), we get

$$\alpha \int_{\Omega\times\Omega} G(x,y)\tilde{u}^2(y)\tilde{u}^2(x)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

which contradicts (4.14).

Case 2. If

$$\tilde{u} = 0,$$

then, employing the second part of Lemma 3.2, (4.12), (4.13),

$$a_{n_k}, b_{n_k}, c_{n_k}, d_{n_k} \longrightarrow 0. \tag{4.15}$$

Due to the definition of u_n (see (2.20)) we have

$$\begin{aligned} a_{n_k}\lambda_{n_k}^4 + b_{n_k}\lambda_{n_k}^3\lambda_{n_{k-1}} &= \alpha \int_{\Omega\times\Omega} G(x,y)u_{n_k}^2(y)\left(u_{n_k}^2(x) + \frac{v_{n-1}(x_0)}{\beta} G(x,x_0)u_{n_k}(x)\right)\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \alpha \int_{\Omega\times\Omega} G(x,y)u_{n_k}^2(y)\left(\frac{1}{2}u_{n_k}^2(x) + \frac{v_{n-1}(x_0)}{\beta} G(x,x_0)u_{n_k}(x)\right)\mathrm{d}x\,\mathrm{d}y \\ &= \alpha \int_{\Omega\times\Omega} G(x,y)u_{n_k}^2(y)\left(\frac{1}{2}v_{n_k}^2(x) + \frac{v_{n-1}^2(x_0)}{2\beta^2} G^2(x,x_0)\right)\mathrm{d}x\,\mathrm{d}y \geqslant 0. \end{aligned}$$

So rewriting (4.10) in the following form

$$1 + \frac{1}{\lambda_{n_k}^2}\left(a_{n_k}\lambda_{n_k}^4 + \lambda_{n_{k-1}}\lambda_{n_k}^3 b_{n_k}\right) = c_{n_k} + \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}} d_{n_k},$$

we get

$$1 \leqslant c_{n_k} + \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}} d_{n_k}.$$

Using (4.11), (4.15), passing to the limit, we get $1 \leqslant 0$, which is a contradiction. This proves that we cannot have (4.11), and thus (4.4) and (4.5) hold true.

Finally, (4.3) is direct consequence of (4.4), (4.5), (3.28). □

The following result is a direct consequence of Lemma 4.1:

Corollary 4.1. *Let v be the limit map v of the previous lemma and define*

$$u := v - \frac{v(x_0)}{\beta} G(\cdot,x_0), \qquad \varphi := \int_{\Omega} G(\cdot,y)u^2(y)\,\mathrm{d}y.$$

The pair (u,φ) is a weak solution of (1.1) *in the sense of Definition* 2.1.

4.2. Nontriviality of the solution and conclusion

Lemma 4.2 *(Upper bound on the minima). Assume* (1.2), (2.6). *There exists* $c_0 > 0$ *such that*

$$\min_{v \in H_0^1(\Omega)} J_n(v) \leqslant -c_0 < 0, \tag{4.16}$$

for each $n \in \mathbb{N}$.

Proof. Let φ_0 be the normalized positive first eigenfunction of $-\Delta$ on Ω, namely φ_0 is the unique smooth map satisfying the following conditions

$$\begin{cases} -\Delta\varphi_0 = \omega_0\varphi_0, & \text{in } \Omega, \\ \varphi_0 > 0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \\ \|\varphi_0\|_{L^2(\Omega)} = 1. \end{cases}$$

Since

$$\int_\Omega \left|\nabla\varphi_0(x)\right|^2 \mathrm{d}x = \omega_0 \int_\Omega \varphi_0^2(x)\,\mathrm{d}x = \omega_0,$$

we find, by evaluating J_n in $\lambda\,\mathrm{sign}(v_{n-1}(x_0))\varphi_0$, $\lambda > 0$, that (writing $\nu_{n-1} = \mathrm{sign}(v_{n-1}(x_0))$)

$$\begin{aligned} J_n(\lambda\nu_{n-1}\varphi_0) &= \frac{\lambda^2}{2}\int_\Omega |\nabla\varphi_0|^2\,\mathrm{d}x + \lambda^4\frac{\alpha}{4}\int_{\Omega\times\Omega} G(x,y)\varphi_0^2(y)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \left|v_{n-1}(x_0)\right|\lambda^3\frac{\alpha}{\beta}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)\varphi_0(y)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + v_{n-1}^2(x_0)\lambda^2\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)\varphi_0(y)\varphi_0(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + v_{n-1}^2(x_0)\lambda^2\frac{\alpha}{2\beta^2}\int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \left|v_{n-1}^3(x_0)\right|\lambda\frac{\alpha}{\beta^3}\int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\varphi_0(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{\omega}{2}\lambda^2\int_\Omega \varphi_0^2(x)\,\mathrm{d}x - \left|v_{n-1}(x_0)\right|\lambda\frac{\omega}{\beta}\int_\Omega G(x,x_0)\varphi_0(x)\,\mathrm{d}x \\ &= \lambda^2\frac{\omega_0-\omega}{2} + \lambda^4\kappa_1 - \left|v_{n-1}(x_0)\right|\lambda^3\kappa_2 + v_{n-1}^2(x_0)\lambda^2\kappa_3 - \left|v_{n-1}^3(x_0)\right|\lambda\kappa_4 - \left|v_{n-1}(x_0)\right|\lambda\kappa_5. \end{aligned}$$

Due to the positivity of φ_0

$$\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 > 0,$$

hence

$$J_n(\lambda\nu_{n-1}\varphi_0) \leqslant \lambda^2\frac{\omega_0-\omega}{2} + \lambda^4\kappa_1 + v_{n-1}^2(x_0)\lambda^2\kappa_3 - \left|v_{n-1}(x_0)\right|\lambda\kappa_5. \tag{4.17}$$

Since the sequence $\{v_n(x_0)\}_{n\in\mathbb{N}}$ is bounded (see (4.5)), we have only the two following cases.
Case 1. If

$$\liminf_n \left|v_{n-1}(x_0)\right| = 0,$$

then, there exists n_0 such that passing to a subsequence and using (2.6),

$$v_{n-1}^2(x_0)\kappa_3 \leqslant \frac{\omega-\omega_0}{4}, \quad n > n_0.$$

Hence, from (4.17)

$$J_n(\lambda v_{n-1}\varphi_0) \leqslant \lambda^2\frac{\omega_0-\omega}{4} + \lambda^4\kappa_1, \quad n > n_0.$$

Employing (2.6)

$$\begin{aligned} J_n(v_n) &= \min_{v\in H^2(\Omega)\cap H_0^1(\Omega)} J_n(v) \leqslant \min_{\lambda>0} J_n(\lambda v_{n-1}\varphi_0) \\ &\leqslant \min_{\lambda>0}\left(\lambda^2\frac{\omega_0-\omega}{2} + \lambda^4\kappa_1\right) < 0, \quad n > n_0. \end{aligned} \tag{4.18}$$

Case 2. If

$$0 < \liminf_n \left|v_{n-1}(x_0)\right| < \infty,$$

then, there exists n_0 and $c_1, c_2 > 0$ such that

$$0 < c_1 < \left|v_{n-1}(x_0)\right| < c_2, \quad n > n_0.$$

Hence, from (4.17)

$$J_n(\lambda v_{n-1}\varphi_0) \leqslant \lambda^2\frac{\omega_0-\omega}{2} + \lambda^4\kappa_1 + c_2\lambda^2\kappa_3 - c_1\lambda\kappa_5, \quad n > n_0.$$

Since the coefficient of λ is negative

$$\begin{aligned} J_n(v_n) &= \min_{v\in H^2(\Omega)\cap H_0^1(\Omega)} J_n(v) \leqslant \min_{\lambda>0} J_n(\lambda v_{n-1}\varphi_0) \\ &\leqslant \min_{\lambda>0}\left(\lambda^2\frac{\omega_0-\omega}{2} + \lambda^4\kappa_1 + c_2\lambda^2\kappa_3 - c_1\lambda\kappa_5\right) < 0, \quad n > n_0. \end{aligned} \tag{4.19}$$

Clearly, (4.18) and (4.19) prove the lemma. □

Proof of Theorem 2.1. It is direct consequence of Lemmas 4.1, 4.2 and Corollary 4.1. □

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