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# Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions

M. Jazar [a,*,1], R. Kiwan [b]

[a] *Mathematics department, Lebanese University, P.O. Box 826, Tripoli, Lebanon*
[b] *Laboratoire de Mathématiques et Physique Théorique, Université de Tours, UMR 6083 du CNRS, Parc de Grandmont, 37200 Tours, France*

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**Abstract**

In this paper we give a positive answer to the conjecture proposed in [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39] by El Soufi et al. concerning the finite time blow-up for solutions of the problem (1), (2) below. More precisely, we give a direct proof of [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39, Theorem 1.1] and the conjecture given for the case $p > 2$.


**Résumé**

Dans cet article on donne une réponse positive à la conjecture proposeé dans [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39] par El Soufi et al. concernant l'explosion en temps fini des solutions du problème (1), (2) ci-dessous. Plus précisément, on donne une preuve directe du [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39, Theorem 1.1] ainsi que la conjecture énoncée pour le cas $p > 2$.


Consider a bounded regular (of class $C^2$) domain $\Omega$ in $\mathbb{R}^N$, and study the solutions $u(x,t)$ of the following equation for some $p > 1$ (denoting the mean value $\frac{1}{|\Omega|}\int_\Omega f$ by $\fint_\Omega f$ for a general function $f$):

$$\begin{cases} u_t - \Delta u = |u|^p - \fint_\Omega |u|^p & \text{on } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

* Corresponding author.
*E-mail addresses:* mjazar@ul.edu.lb (M. Jazar), kiwan@lmpt.univ-tours.fr (R. Kiwan).
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with the initial condition

$$\begin{cases} u(x,0) = u_0(x) \quad \text{on } \Omega, \\ \text{with} \displaystyle\fint_\Omega u_0 = 0. \end{cases} \tag{2}$$

Without loss of generality, we may assume that the measure of $\Omega$ is 1: $|\Omega| = 1$ (see [1]).

In this short note we show the following theorem:

**Theorem 1** *(Sufficient condition for blow-up, $p > 1$). Let $p > 1$ and let $u$ be a solution of* (1), (2) *with $u_0 \in C(\overline{\Omega})$, $u_0 \not\equiv 0$. If the energy of $u_0$,*

$$E(u_0) := \int_\Omega \left[\frac{1}{2}|\nabla u_0|^2 - \frac{1}{p+1}|u_0|^{p+1}\right] \mathrm{d}x$$

*is non-positive, that is*

$$E(u_0) \leqslant 0, \tag{3}$$

*then the solution does not exist in $L^\infty((0,T); L^2(\Omega))$ for all $T > 0$.*

We refer to [1] for the motivation and references concerning the study of Eq. (1) above. In particular, we have the following local existence result

**Theorem 2** *(Local existence result, $1 < p < +\infty$).* [1, Theorem 2.2] *For every $u_0 \in C(\overline{\Omega})$ there is a $0 < t_{\max} \leqslant \infty$ such that the problem* (1), (2) *has a unique mild solution, i.e. a unique solution*

$$u \in C\big([0, t_{\max}); C\big(\overline{\Omega}\big)\big) \cap C^1\big((0, t_{\max}); C\big(\overline{\Omega}\big)\big) \cap C\big((0, t_{\max}); D\big)$$

*of the following integral equation*

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} f\big(u(s)\big)\, \mathrm{d}s$$

*on* $[0, t_{\max})$*, with*

$$f\big(u(s)\big) = \big|u(s)\big|^p - \fint_\Omega \big|u(s)\big|^p.$$

*Moreover, we have*

$$\int_\Omega u(t) = 0 \quad \text{for all } t \in [0, t_{\max}) \tag{4}$$

*and if $t_{\max} < \infty$ then*

$$\lim_{t \uparrow t_{\max}} \big\|u(t)\big\|_{L^\infty(\Omega)} = \infty.$$

**Proof of Theorem 1.** We start with the following

**Lemma 1.** *For all $t$ we have*

$$E\big(u(t)\big) = E(u_0) - \int_0^t \int_\Omega u_t^2. \tag{5}$$

**Proof.** A direct computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E\big(u(t)\big) = -\int_\Omega u_t\big(\Delta u + u^p\big) = -\int_\Omega u_t^2.$$

Thus, integrating from 0 to $t$ one gets (5). □

**Lemma 2.** *Define the two functions*

$$m(t) := \frac{1}{2}\int_\Omega u^2(t)\,\mathrm{d}x \quad \textit{and} \quad h(t) := \int_0^t m(s)\,\mathrm{d}s.$$

*Then the two functions satisfy*

$$m'(t) \geqslant (p+1)\int_0^t\int_\Omega u_t^2\,\mathrm{d}x, \tag{6}$$

$$m'(t) \geqslant \lambda(p-1)m(t), \tag{7}$$

$$\frac{p+1}{2}\big(h'(t)-h'(0)\big)^2 \leqslant h(t)h''(t). \tag{8}$$

**Proof.** We have

$$\begin{aligned} m'(t) &= \int_\Omega u_t u \\ &= \int_\Omega u\left(\Delta u + |u|^p - \int |u|^p\right) \\ &= -\int_\Omega \nabla u^2 + \int_\Omega u|u|^p - \int_\Omega u \int_\Omega |u|^p, \end{aligned}$$

$$m'(t) = -(p+1)E(u) + \frac{p-1}{2}\int_\Omega \nabla u^2. \tag{9}$$

From one side, this last equality implies that

$$m'(t) \geqslant -(p+1)E(u) = -(p+1)E(0) + (p+1)\int_0^t\int_\Omega u_t^2$$

by using Lemma 1, and by the hypothesis (3) we get (6).

On the other side, (9), (3) and Poincaré's inequality gives

$$m'(t) \geqslant \left(\frac{p-1}{2}\right)\int_\Omega \nabla u^2 \geqslant \frac{\lambda(p-1)}{2}\int_\Omega u^2 = \lambda(p-1)m(t),$$

which is (7). Now,

$$\begin{aligned} h'(t) - h'(0) &= \int_0^t m'(s)\,\mathrm{d}s = \int_0^t\int_\Omega u u_t \\ &\leqslant \left(\int_0^t\int_\Omega u^2\right)^{1/2}\left(\int_0^t\int_\Omega u_t^2\right)^{1/2} \end{aligned}$$

$$\leqslant \left(\frac{2}{p+1}\right)^{1/2} \big(h(t)\big)^{1/2} \big(m'(t)\big)^{1/2}$$
$$= \left(\frac{2}{p+1}\right)^{1/2} \big(h(t)\big)^{1/2} \big(h''(t)\big)^{1/2};$$

but, by (6)

$$h'(t) - h'(0) = \int_0^t m'(s)\,\mathrm{d}s \geqslant (p+1) \int_0^t\!\!\int_\Omega u_t^2 \geqslant 0.$$

Therefore

$$\frac{p+1}{2}\big(h'(t) - h'(0)\big)^2 \leqslant h(t)h''(t). \qquad \square$$

*Proof of the theorem.* Assume that the solution exists for all $t$, then (7) implies that

$$\lim_{t\to\infty} h'(t) = \lim_{t\to\infty} m(t) = +\infty. \tag{10}$$

Thus, for all $0 < B < p+1$, there exists $T_B$ such that, for all $t \geqslant T_B$

$$Bh'(t)^2 \leqslant (p+1)\big[h'(t) - h'(0)\big]^2.$$

Hence, by (8), we have for all $t \geqslant T_B$

$$Bh'(t)^2 \leqslant 2h(t)h''(t).$$

Consider now the function $H(t) = [h(t)]^{-q}$ for some $q > 0$ to be determined later. We have

$$H''(t) = qh(t)^{-(q+2)}\big[(q+1)h'^2(t) - h(t)h''(t)\big]$$
$$\leqslant qh(t)^{-(q+2)}\left[\frac{2(q+1)}{B} - 1\right]h(t)h''(t),$$

for $t \geqslant T_B$. In order to choose $H$ concave on $[T_B, \infty)$, take $B$ and $q > 0$ such that $q < \frac{p-1}{2}$ and $2(q+1) < B < p+1$. Consequently, $H$ is a positive concave function on $[T_B, +\infty[$ and, by (10), $\lim_{t\to\infty} H(t) = 0$, which is a contradiction. $\square$

## References

[1] A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39.