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Isoperimetric profile and uniqueness for Neumann problems

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Abstract

Given a connected compact Riemannian surface (M,g), f an absolutely continuous function satisfying $f \geqslant f' > 0$ and a real parameter α, we deal with classical solutions of

$$\begin{cases} -\Delta_g u = f(u) - \alpha & \text{in } M, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M. \end{cases}$$

We prove that any non-constant solution of the above problem satisfies

$$\int_M f(u) \geqslant 8\pi \inf_{s \in (0, \mathrm{vol}(M))} \left\{ \frac{I_M^2(s)}{I_{\mathbb{S}_M}^2(s)} \right\},$$

where I_M and $I_{\mathbb{S}_M}$ denote respectively the isoperimetric profile of M and of the standard two-dimensional sphere having same measure than M (see Definition 2.1 below). This inequality is applied to derive new uniqueness results for mean field type equations. A similar result for linear problems is established and gives lower bounds for the first non-zero Neumann eigenvalue.

Résumé

Étant donnée une surface Riemannienne compacte connexe (M,g), f une fonction absolument continue satisfaisant $f \geqslant f' > 0$ et un paramètre réel α, nous considérons les solutions classiques du problème de Neumann décrit ci-dessus. Nous prouvons que toute solution non-constante vérifie

$$\int_M f(u) \geqslant 8\pi \inf_{s \in (0, \mathrm{vol}(M))} \left\{ \frac{I_M^2(s)}{I_{\mathbb{S}_M}^2(s)} t \right\},$$

où I_M et $I_{\mathbb{S}_M}$ dénotent respectivement les profils isopérimétriques de M et de la sphère canonique bi-dimensionelle ayant même mesure que M. Cette inégalité est appliquée pour dériver de nouveaux résultats d'unicité pour des équations du type champs moyen. Un résultat similaire est établi pour des problèmes linéaires et permet de dériver diverses bornes inférieures sur la première valeure propre non-nulle.

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1. Introduction

Consider a Riemannian surface (M, g) and a function $f : \mathbb{R} \to (0, \infty)$ satisfying the assumptions:

(H1) (M, g) is a connected compact oriented two-dimensional Riemannian manifold of class C^2 with a piecewise C^1 boundary ∂M (possibly empty);
(H2) f is absolutely continuous, strictly increasing and $f > 0$.

Under these requirements the present paper deals with the problem

$$\begin{cases} -\Delta_g u = f(u) - \alpha, \\ u \in C^2(M \setminus \partial M) \cap C^1(M), \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial M, \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$, and Δ_g stands for the Laplace–Beltrami operator given in local coordinates by

$$\Delta_g u = |g|^{-1/2} \frac{\partial}{\partial x_i} \left(|g|^{1/2} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

with $|g| = \det g_{ij}$. The surface M can be with or without boundary, and in the second case the Neumann boundary condition in (1.1) is irrelevant. Integrating Eq. (1.1) on the surface M, we get the natural identity

$$\alpha |M| = \int_M f(u). \tag{1.2}$$

Relation (1.2) shows that any constant function trivially solves problem (1.1). The aim of the present work is to find, within the class of non-constant solutions, a lower bound for (1.2) which depends on the geometry of the surface M.

Our motivation for getting such a priori bound is driven by the wish to understand the uniqueness of solutions to non-local problems of the type

$$-\Delta_g u = \lambda \left(\frac{\mathrm{e}^u}{\int_M \mathrm{e}^u} - \frac{1}{|M|} \right), \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial M, \tag{1.3}$$

or

$$-\Delta_g u = \lambda \left(\frac{[u^2 + 1]}{\int_M [u^2 + 1]} - \frac{1}{|M|} \right), \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial M, \quad u \geqslant 0. \tag{1.4}$$

In these problems the constant functions are solutions, and one would like to know for which range of the parameter λ they are the only solutions. Notice that a given non-constant function u solving the non-local problem (1.3) can be considered as a solution of (1.1) by defining

$$C := \frac{\lambda}{\int_M \mathrm{e}^u}, \qquad f(s) := C\mathrm{e}^s, \qquad \alpha := \frac{\lambda}{|M|},$$

and similarly for (1.4). Hence a lower a priori estimate for (1.2) in the class of non-constant solutions gives immediately a necessary condition on λ to ensure existence of non-trivial solutions in problems (1.3)–(1.4).

Problem (1.3) is of relevance in several fields. Non-linear PDE involving exponential non-linearity had already been brought to our attention by Liouville in [34], who was certainly interested in the problem of prescribing constant curvature on a domain of $\mathbb{R}^2$. On the standard 2-sphere, with $\lambda = 8\pi$, Eq. (1.3) is precisely the problem of finding the conformal metrics $\bar{g} = \mathrm{e}^u g$ having constant Gaussian curvature. This is a special case of the commonly called "Nirenberg Problem" which has been extensively studied by Moser [39], Kazdan and Warner [27] and others; we refer to [8] for a more detailed discussion. In statistical mechanics Caglioti et al. [7] and independently Kiessling [28] have shown that the asymptotical behavior of the Gibbs measure associated to a system of particles contained in a domain of $\mathbb{R}^2$ and having logarithmic interactions leads to the equation

$$-\Delta u = \lambda \frac{\mathrm{e}^u}{\int_\Omega \mathrm{e}^u}, \quad u \in H_0^1(\Omega). \tag{1.5}$$

But similar consideration on manifold lead naturally to Eq. (1.3) (see [29]). Both problems (1.3) and (1.4) turn out also to be of interest in several models of chemotaxis [22–24].

In both works [28] and [7] it has been noted that problem (1.5) always admits a solution in the range $\lambda < 8\pi$. For simply connected domains of $\mathbb{R}^2$, Suzuki proved in [45] that uniqueness holds in the range $\lambda < 8\pi$. The crucial point in his arguments is the property that the first eigenvalue of the linearized operator is greater or equal to 8π. Suzuki's arguments use the Dirichlet boundary conditions and rely mainly on the *Bol's inequality* (see [2,45]), an isoperimetric inequality available for simply connected domain. At the critical value 8π, the problem does not always have a solution, but Chang et al. [10] have improved Suzuki's result by showing that there is at most one solution, still under the assumption that the domain is simply connected. A careful inspection of the arguments of [45] and [10] shows that the semilinear equation (1.3) with zero boundary condition on a bounded simply connected domain of $\mathbb{R}^2$ admits only $u \equiv 0$ as a solution whenever $\lambda \leqslant 8\pi$. In [36] we were able to prove this same result without restriction on the topology of the domain $M \Subset \mathbb{R}^2$.

On a manifold or when the boundary condition is of Neumann type, the proof of uniqueness requires new arguments. On the standard sphere the successive works of Onofri [41], Hong [21], Chanillo and Kiessling [11] and Lin [31] have shown that the constant functions are the unique solution for problem (1.3) whenever $\lambda < 8\pi$. In [32], we proved that for some flat torus the constants are the unique solutions whenever $\lambda \leqslant 8\pi$. Typically the result holds when the period cell is a square. Concerning problem (1.4) nothing seems to be known about uniqueness of solutions.

Problems (1.3) and (1.4) are quite different in nature. For example the first is the Euler–Lagrange equation of the functional

$$J : E \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_M |\nabla u|^2 - \lambda \log\left(\frac{1}{|M|}\int_M \mathrm{e}^u\right), \tag{1.6}$$

where $E := \{u \in H^1(M)\colon \int_M u = 0\}$. The fact that the functional (1.6) is well-defined and smooth is a consequence of the Moser–Trudinger inequality [39,17]. But this is not anymore the case for problem (1.4) which does not admit any variational formulation. Nevertheless we shall see how the approach used in [36,32] can be extended and successfully used to handle in a unified way the question of uniqueness for both problems (1.3) and (1.4).

In order to state our results we introduce the surface

$$\mathbb{S}_M = \text{ canonical 2-sphere having same measure than } M,$$

and denote by I_M the isoperimetric profile of M whose definition is recalled in the next section (see Definition 2.1). With these notations the main result of this paper reads as follows:

Theorem 1.1. *Assume* (H1), (H2) *hold. Then any non-constant solutions u of problem* (1.1) *satisfies*

$$\int_M f(u) \geqslant 8\pi \frac{\int_{-\infty}^{\infty} f(t) I_M^2(V(t))\,dt}{\int_{-\infty}^{\infty} f'(t) I_{\mathbb{S}_M}^2(V(t))\,dt}, \tag{1.7}$$

where $V(t) := |\{x \in M\colon u(x) > t\}|$.

Theorem 1.1 immediately implies the following result that will have interesting consequences:

Proposition 1.2. *Assume* (H1), (H2) *hold and*

$$f(s) \geqslant f'(s) > 0, \quad \forall s \in \mathbb{R}. \tag{1.8}$$

Then any non-constant solution of problem (1.1) *satisfies*

$$\int_M f(u) \geqslant 8\pi \inf_{s\in(0,|M|)} \left\{\frac{I_M^2(s)}{I_{\mathbb{S}_M}^2(s)}\right\}. \tag{1.9}$$

Note that in the above results the derivative of f exists almost everywhere since in (H2) we assume f to be absolutely continuous.

In [32] a weaker version of Theorem 1.1 has been established for the specific problem (1.3) on a flat torus. In this previous work the analyticity of the exponential function and of the manifold (a torus) was used. The purpose of this paper is to refine these arguments in order to assume only (H1)–(H2), and to present several applications of Theorem 1.1 that go much beyond the case considered in [32]. For example if the surface is a sphere S_κ of curvature κ, Proposition 1.2 implies

$$\int_{S_\kappa} f(u) \geqslant 8\pi,$$

for any non-constant solution u of problem (1.1). In particular we recover the result of [41,21,11,31] stating that, on the standard sphere, the constant functions are the unique solutions of problem (1.3) whenever $\lambda < 8\pi$. But the present paper shows that this uniqueness result is not restricted to the exponential non-linearity but applies also to (1.4). Actually by making use of the Lévy–Gromov inequality [20], we are even able to generalize these previous results as follows:

Proposition 1.3. *Let M be a surface without boundary satisfying* (H1) *with Gauss curvature bounded from below by $\kappa > 0$. Assume f satisfies* (H2) *and* (1.8). *Then any non-constant solution u of problem* (1.1) *satisfies*

$$\int_M f(u) \geqslant 2\kappa|M|. \tag{1.10}$$

Furthermore if equality holds in (1.10) *for some non-constant solution u, then M is isometric to the sphere $\mathbb{S}_M$.*

Note that in the case of a standard two-dimensional sphere, the lower bound in (1.10) is precisely 8π. In this paper we will present beside Proposition 1.3 other examples for which similar explicit lower bounds can be derived. On bounded domain of $\mathbb{R}^2$ we will treat among other things discs, rectangles and triangles. In this latter case we shall see that our conclusion is optimal.

But our approach has also quite interesting consequences on the linear problem

$$\begin{cases} -\Delta_g u = \lambda u, \\ u & \in C^2(M \setminus \partial M) \cap C^1(M), \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial M. \end{cases} \tag{1.11}$$

Though this problem does not fit directly in the framework of Theorem 1.1, by exploiting the linearity one can derive from inequality (1.7) the following:

Theorem 1.4. *Let M be a surface satisfying* (H1). *Then any eigenfunction φ with associated eigenvalue $\lambda > 0$ satisfies*

$$\lambda|M| \geqslant 8\pi \frac{\int_{-\infty}^{\infty} I_M^2(V(t))\,dt}{\int_{-\infty}^{\infty} I_{\mathbb{S}_M}^2(V(t))\,dt}, \tag{1.12}$$

where $V(t) := |\{x \in M\colon \varphi(x) > t\}|$.

As a consequence we derive a lower bound on the first non-zero eigenvalue:

Proposition 1.5. *Assume* (H1) *holds and let λ be a positive eigenvalue of problem* (1.11). *Then*

$$\lambda|M| \geqslant 8\pi \inf_{s\in(0,|M|)} \left\{ \frac{I_M^2(s)}{I_{\mathbb{S}_M}^2(s)} \right\}. \tag{1.13}$$

Applying Proposition 1.5 within the class of closed compact surface with Gauss curvature bounded from below by 1, we get in dimension two an alternative proof of the well known Lichnerowicz–Obata Theorem. This and other consequences will be discussed in detail in Section 5. An interesting feature of the present work is that the lower bounds obtained in Propositions 1.2 and 1.5 for $\int_M f(u)$ and respectively $\lambda|M|$ are the same.

The paper is organized as follows. In Section 2 we derive a differential inequality which involves the distribution function of u, the function $\int_{\{u>t\}} f(u)$, and the isoperimetric profile of the surface. Based on this result we prove in Section 3 the inequality stated in Theorem 1.1 and give a more precise version of Proposition 1.2. This main theorem is applied in Section 4 on several manifolds and contains in particular the proof of Proposition 1.3. In a final section we derive Theorem 1.4 from Theorem 1.1, and apply it to get a lower bound on the first non-zero Neumann eigenvalue. We conclude with several results of uniqueness for the non-local problem (1.3) and discuss also in that case the question of existence of non-trivial solutions.

2. A differential inequality

Denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure in M. Given $\omega \subset M$, its perimeter relative to $M^o := M \setminus \partial M$ is defined as

$$\mathcal{P}(\omega, M^o) := \mathcal{H}^1(\partial\omega \cap M^o),$$

and its area $\mathcal{H}^2(\omega)$ will be denoted by $|\omega|$.

Definition 2.1. Consider the class $\mathcal{M}$ of open set $\omega \subset M$ such that $\partial\omega \cap M^o$ is a 1-submanifold of class C^1. The "isoperimetric profile" of M is the function $I_M : [0, |M|] \to (0, \infty)$ defined as

$$I_M(s) := \inf\{\mathcal{P}(\omega, M^o)\colon\ \omega \in \mathcal{M},\ |\omega| = s\}, \quad \forall s \in (0, |M|],$$

and we set $I_M(0) = 0$.

Let us emphasize that the possible part of the boundary of ω lying in ∂M is *not* taken into consideration in the above definition. We mention two properties of the isoperimetric profile that will be used in the sequel:

$$I_M(s) = I_M(|M| - s), \quad \forall s \in [0, |M|]; \tag{2.1}$$

$$I_M(s) = 0 \quad \Longleftrightarrow \quad s = 0 \text{ or } s = |M|. \tag{2.2}$$

The symmetry property (2.1) readily follows from the definition of isoperimetric profile, and for (2.2) we refer to [19].

In order to simplify the notations, we shall write $\{u > t\}$, $\{u < t\}$, $\{u = t\}$ instead of $\{x \in M^o\colon\ u(x) > t\}$, $\{x \in M^o\colon\ u(x) < t\}$ and $\{x \in M^o\colon\ u(x) = t\}$. Given a solution u of (1.1), we also introduce the following notations:

$$F(t) := \int_{\{u>t\}} f(u), \qquad V(t) := \int_{\{u>t\}} 1, \tag{2.3}$$

$$\widetilde{F}(t) := \int_{\{u<t\}} f(u), \qquad \widetilde{V}(t) := \int_{\{u<t\}} 1. \tag{2.4}$$

We will need the following lemma.

Lemma 2.2.

(a) *Assume* (H1) *is satisfied and* $f \in C^0(\mathbb{R})$. *Given a non-constant solution u of problem* (1.1) *we have*

$$\mathcal{H}^2(\{u = t\}) = 0, \quad \textit{whenever } f(t) \neq \alpha. \tag{2.5}$$

In particular the functions defined in (2.3) *and* (2.4) *are continuous on any closed interval of* $\mathbb{R} \setminus f^{-1}(\alpha)$.

(b) *Given a function* $\Psi \in C([a, b], \mathbb{R})$ *which is differentiable a.e. with* $\Psi' \geqslant 0$. *Then* Ψ *is monotone increasing.*

Proof. (a) Given a fixed $t \in \mathbb{R}$, the set

$$\{x \in M^o\colon\ u(x) = t,\ du(x) \neq 0\}$$

is by the implicit function theorem a C^1-submanifold and so has measure zero.

To prove (2.5) we only need to study the set of critical points:

$$K_0 := \{x \in M^o\colon\ du(x) = 0,\ f(u(x)) \neq \alpha\}.$$

Let $p \in K_0$. Consider a system of Riemannian normal coordinates $\varphi : U \subset M^o \to B(0,\epsilon)$, with $\varphi(p) = 0$. In these coordinates we have

$$\Delta_g u(p) = \left(\frac{\partial^2 (u \circ \varphi^{-1})}{\partial x_1^2} + \frac{\partial^2 (u \circ \varphi^{-1})}{\partial x_2^2}\right)(0) = \alpha - f(u(p)) \neq 0.$$

Therefore without loss of generality we may assume $\frac{\partial^2 (u\circ\varphi^{-1})}{\partial x_1^2}(0) \neq 0$. The implicit function theorem implies that the set $\{x \in B(0,\epsilon)\colon\ \frac{\partial (u\circ\varphi^{-1})}{\partial x_1}(x) = 0\}$ is a 1-submanifold in a neighborhood of 0. Hence the set K_0 is a countable union of 1-submanifold and so statement (2.5) follows.

(b) For the second statement we refer to [18] (p. 19 and p. 97). $\square$

The main results of this section are the two differential inequalities given in the next proposition.

Proposition 2.3. *Let M be a surface satisfying* (H1) *and $f \in C^0(\mathbb{R})$ a non-negative function differentiable a.e. Then any solution u of* (1.1) *satisfies the following inequalities*:

$$\{\alpha f V^2 - F^2\}'(t) \geqslant \alpha f'(t) V^2(t) + 2 f(t) I_M^2(V(t)) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}, \tag{2.6}$$

$$\{\alpha f \widetilde{V}^2 - \widetilde{F}^2\}'(t) \geqslant \alpha f'(t) \widetilde{V}^2(t) + 2 f(t) I_M^2(\widetilde{V}(t)) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}, \tag{2.7}$$

where $\mathcal{C} := \{u(x)\colon\ x \in M^o, du(x) = 0\}$, the functions $V, F, \widetilde{V}, \widetilde{F}$ are defined by (2.3)–(2.4) *and I_M stands for the isoperimetric profile of M.*

Proof. Note that since $u \in C^2(M^o)$ and M^o is assumed to be a C^2-manifold, Sard's Theorem ensures that its set of critical value $\mathcal{C}$ has Lebesgue measure zero in $\mathbb{R}$.

Let us first prove (2.6). By Lemma 2.2, the functions F and V are continuous on $\mathbb{R} \setminus f^{-1}(\alpha)$. Therefore, by using co-area formula (see [16], Proposition 3, p. 118), we obtain

$$V'(t) = -\int_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^1 \quad \forall t \in \mathbb{R} \setminus \mathcal{C}, \tag{2.8}$$

$$F'(t) = -\int_{\{u=t\}} \frac{f(u)}{|\nabla u|}\, d\mathcal{H}^1 = f(t) V'(t) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}. \tag{2.9}$$

Secondly, by integrating Eq. (1.1) on the set $\{u > t\}$ and using Stokes's Theorem, we obtain

$$\int_{\partial\{u>t\}} |\nabla u|\, d\mathcal{H}^1 = F(t) - \alpha V(t) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}. \tag{2.10}$$

Since

$$\begin{aligned} \partial\{u > t\} &= (\partial\{u > t\} \cap M^o) \cup (\partial\{u > t\} \cap \partial M) \\ &= \{u = t\} \cup (\partial\{u > t\} \cap \partial M), \end{aligned}$$

and furthermore ∂M is either empty or u satisfies a *Neumann boundary condition*, the left-hand side of (2.10) can be rewritten as:

$$\int_{\partial\{u>t\}} |\nabla u|\, d\mathcal{H}^1 = \int_{\{u=t\}} |\nabla u|\, d\mathcal{H}^1.$$

Based on this observation, Eq. (2.10) yields

$$F(t) = \alpha V(t) + \int_{\{u=t\}} |\nabla u|\, d\mathcal{H}^1 \quad \forall t \in \mathbb{R} \setminus \mathcal{C}. \tag{2.11}$$

Using (2.11), (2.9) and Schwarz inequality together with the assumption $f \geqslant 0$, we derive:

$$\begin{aligned}
-F(t)F'(t) &= \Big(\alpha V(t) + \int\limits_{\{u=t\}} |\nabla u|\Big) \int\limits_{\{u=t\}} \frac{f(t)}{|\nabla u|}, \\
&= f(t)\Big(\int\limits_{\{u=t\}} |\nabla u|\Big)\Big(\int\limits_{\{u=t\}} \frac{1}{|\nabla u|}\Big) - \alpha f(t)V(t)V'(t), \\
&\geqslant f(t)\Big(\int\limits_{\{u=t\}} d\mathcal{H}^1\Big)^2 - \alpha f(t)V(t)V'(t).
\end{aligned} \tag{2.12}$$

The definition of perimeter and of isoperimetric profile of M now implies

$$\int\limits_{\{u=t\}} d\mathcal{H}^1 = \mathcal{P}\big(\{u > t\}, M^o\big) \geqslant I_M\big(V(t)\big). \tag{2.13}$$

Hence (2.12) together with (2.13) yield

$$-F(t)F'(t) \geqslant f(t)I_M^2\big(V(t)\big) - \alpha f(t)V(t)V'(t) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}. \tag{2.14}$$

Since

$$FF' = \frac{(F^2)'}{2} \quad \text{and} \quad fVV' = f\frac{(V^2)'}{2} = \frac{1}{2}\big\{(fV^2)' - f'V^2\big\},$$

we may rewrite (2.14) as:

$$\frac{1}{2}\{\alpha f V^2 - F^2\}'(t) \geqslant f(t)I_M^2\big(V(t)\big) + \frac{\alpha}{2}f'(t)V^2(t) \quad \forall t \in \mathbb{R} \setminus \mathcal{C}.$$

This proves (2.6).

The proof of (2.7) follows the same line, but instead of (2.8), (2.9) and (2.11) one needs to use the identities

$$\widetilde{V}'(t) = \int\limits_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^1, \qquad \widetilde{F}'(t) = \int\limits_{\{u=t\}} \frac{f(t)}{|\nabla u|}\, d\mathcal{H}^1,$$

$$\widetilde{F}(t) = \alpha\widetilde{V}(t) + \int\limits_{\{u=t\}} |\nabla u|\, d\mathcal{H}^1. \qquad \square$$

Remark 2.4.

(i) The idea of deriving a differential inequality for F and V as in the proof of Proposition 2.3 is inspired by some arguments found in [2], and especially from the proof of Bol's inequality as given by Suzuki in (Proposition 3, [45]). In both of these works, the aim was to get estimates on functions v satisfying $-\Delta v \leqslant e^v$ in a domain of $\mathbb{R}^2$.

(ii) When $f(s) = e^s$ we were in [32] (also in [36]) considering the level sets of e^u (as in [2,45]). Here instead we have worked with the level sets of the solution u. This allows us to relax the assumptions on the non-linearity f.

3. Comparison with the profile of the sphere

All our results will rely on the following integral inequality.

Theorem 3.1. *Assume* (H1)–(H2) *hold. Given a solution u of* (1.1)*, consider V defined in* (2.3)*. Then*

$$\frac{\alpha}{2}\int_{-\infty}^{\infty} f'(t)V(t)\big(|M| - V(t)\big)\,dt \geqslant \int_{-\infty}^{\infty} f(t)I_M^2\big(V(t)\big)\,dt. \tag{3.1}$$

Proof. Let us set

$$t_0 := \min_M u, \quad t_1 := \max_M u, \quad F_M := \int_M f(u).$$

The result will follow by integrating the differential inequalities (2.6) and (2.7) on the interval (t_0, t_1) and then by summing the both relations obtained in this way. More specifically consider the functions

$$\Psi := \alpha f V^2 - F^2 \quad \text{and} \quad \widetilde{\Psi} := \alpha f \widetilde{V}^2 - \widetilde{F}^2.$$

Let us remind that by Lemma 2.2 the functions Ψ and $\widetilde{\Psi}$ are continuous on any interval $[a,b] \subset \mathbb{R} \setminus f^{-1}(\alpha)$. But at the value $f^{-1}(\alpha)$ these functions may be discontinuous, which creates a technical difficulty when integrating (2.6) or (2.7).

Step 1. Setting $a := f^{-1}(\alpha)$ and $\Gamma_a := \{u = a\}$, we claim

$$\Psi(a^+) - \Psi(a^-) = 2f(a)|\Gamma_a|\big(F(a) - f(a)V(a)\big) \geqslant 0, \tag{3.2}$$

$$\widetilde{\Psi}(a^+) - \widetilde{\Psi}(a^-) = 2f(a)|\Gamma_a|\big(f(a)\widetilde{V}(a) - \widetilde{F}(a)\big) \geqslant 0, \tag{3.3}$$

where $\Psi(a^{\pm}) := \lim_{\epsilon \downarrow 0} \Psi(a \pm \epsilon)$, $\widetilde{\Psi}(a^{\pm}) := \lim_{\epsilon \downarrow 0} \widetilde{\Psi}(a \pm \epsilon)$.

Indeed we check easily that (remind also $f(a) = \alpha$):

$$\Psi(a^+) = \alpha f(a)V(a)^2 - F(a)^2 = f(a)^2 V(a)^2 - F(a)^2, \tag{3.4}$$

$$\Psi(a^-) = f(a)^2\big(V(a) + |\Gamma_a|\big)^2 - \big(F(a) + f(a)|\Gamma_a|\big)^2. \tag{3.5}$$

Relations (3.4), (3.5) and the assumption that f is increasing yield (3.2). The proof of (3.3) is similar.

Step 2. Let us prove that

$$\int_{t_0}^{t_1} \Psi' \leqslant F_M^2 - \alpha f(t_0)|M|^2, \tag{3.6}$$

$$\int_{t_0}^{t_1} \widetilde{\Psi}' \leqslant \alpha f(t_1)|M|^2 - F_M^2. \tag{3.7}$$

We only prove (3.6) since the arguments for (3.7) are similar. Choose $\varepsilon > 0$. On one hand by Lemma 2.2 and assumption (H2) the function Ψ is continuous in $[t_0, t_1] \setminus f^{-1}(\alpha)$. Furthermore inequality (2.6) and the hypothesis $f' \geqslant 0$ show that $\Psi' \geqslant 0$ a.e. in $[t_0, t_1]$. Hence Lemma 2.2 implies that Ψ is monotone increasing on the intervals $[t_0, a - \varepsilon]$ and $[a + \varepsilon, t_1]$. By applying now ([35], Theorem 7.2.3, p. 159) we deduce that

$$\int_{a+\varepsilon}^{t_1} \Psi' \leqslant \Psi(t_1) - \Psi(a+\varepsilon) = -\Psi(a+\varepsilon), \tag{3.8}$$

$$\int_{t_0}^{a-\varepsilon} \Psi' \leqslant \Psi(a-\varepsilon) - \Psi(t_0) = \Psi(a-\varepsilon) - \big(\alpha f(t_0)|M|^2 - F_M^2\big). \tag{3.9}$$

Therefore by adding (3.8) with (3.9), letting ϵ tend to zero and using (3.2), we obtain

$$\int_{t_0}^{t_1} \Psi' \leqslant \Psi(a^-) - \Psi(a^+) + \big(F_M^2 - \alpha f(t_0)|M|^2\big) \leqslant F_M^2 - \alpha f(t_0)|M|^2,$$

which concludes the proof of inequality (3.6). The same arguments yield (3.7).

We may now conclude the proof of the statement (3.1). Integrating (2.6) on the interval (t_0, t_1) and taking into consideration (3.6), we deduce

$$F_M^2 - \alpha f(t_0)|M|^2 \geqslant \int_{t_0}^{t_1} \big\{\alpha f'(t)V^2(t) + 2f(t)I_M^2\big(V(t)\big)\big\}\,dt. \tag{3.10}$$

Similarly, integrating (2.7) on the interval (t_0, t_1) together with (3.7) yield

$$\alpha f(t_1)|M|^2 - F_M^2 \geqslant \int_{t_0}^{t_1} \left\{\alpha f'(t)\widetilde{V}^2(t) + 2f(t)I_M^2\big(\widetilde{V}(t)\big)\right\} dt. \tag{3.11}$$

Therefore, by adding (3.10) with (3.11), writing $\widetilde{V} = |M| - V$ and using that $I_M(V) = I_M(|M| - V)$ we derive

$$\begin{aligned}
&\alpha|M|^2\big(f(t_1) - f(t_0)\big) \\
&\quad\geqslant \int_{t_0}^{t_1} \alpha f'(t)\left\{V^2(t) + \big(|M| - V(t)\big)^2\right\} dt + 4\int_{t_0}^{t_1} f(t)I_M^2\big(V(t)\big)\,dt, \\
&\quad= \alpha|M|^2 \int_{t_0}^{t_1} f'(t)\,dt + 2\alpha \int_{t_0}^{t_1} f'(t)V(t)\big(V(t) - |M|\big)\,dt + 4\int_{t_0}^{t_1} f(t)I_M^2\big(V(t)\big)\,dt.
\end{aligned} \tag{3.12}$$

Since f is absolutely continuous, inequality (3.12) yields:

$$0 \geqslant \alpha \int_{t_0}^{t_1} f'(t)V(t)\big(V(t) - |M|\big)\,dt + 2\int_{t_0}^{t_1} f(t)I_M^2\big(V(t)\big)\,dt,$$

or equivalently

$$\alpha \int_{t_0}^{t_1} f'(t)V(t)\big(|M| - V(t)\big)\,dt \geqslant 2\int_{t_0}^{t_1} f(t)I_M^2\big(V(t)\big)\,dt,$$

which proves the proposition. □

Remark 3.2. It is not clear if the functions (2.3), (2.4) are absolutely continuous (see [4]). But if this is the case the proof of previous theorem goes through for Lipschitz function without the restriction that f is strictly increasing.

So far all our results are, up to some obvious modifications, available in higher dimension. But in dimension two, one term appearing in the conclusion of Theorem 3.1 can be identified as the isoperimetric profile of the canonical sphere having same volume than the surface M. This leads to our Theorem 1.1:

Proof of Theorem 1.1. The isoperimetric profile of the two-dimensional canonical sphere $S\kappa$ of curvature κ is explicitly given by (see [42,12]):

$$I_{S_\kappa}^2(s) = s(4\pi - \kappa s) \quad \forall s \in \big[0, |S_\kappa|\big]. \tag{3.13}$$

Therefore, in dimension two, the integrand appearing in the left-hand side of (3.1) can be rewritten using (3.13) as follows:

$$V(t)\big(|M| - V(t)\big) = \frac{|M|}{4\pi} V(t)\left(4\pi - \frac{4\pi}{|M|}V(t)\right) = \frac{|M|}{4\pi} I_{S_\kappa}^2\big(V(t)\big),$$

with $\kappa = \frac{|M|}{4\pi}$ and note also that $|S_k| = |M|$. Therefore by considering the canonical two-dimensional sphere S_κ of constant curvature κ having same volume than M, inequality (3.1) is equivalent to

$$\frac{\alpha|M|}{8\pi} \int_{-\infty}^{\infty} f'(t)I_{S_\kappa}^2\big(V(t)\big)\,dt \geqslant \int_{-\infty}^{\infty} f(t)I_M^2\big(V(t)\big)\,dt. \tag{3.14}$$

We claim that the right-hand side of (3.14) is strictly positive. Indeed by (H2) we have $f > 0$, and since u is assumed non-constant we easily derive from (2.2) that $I_M(V(t)) > 0$ for $t \in (\min u, \max u)$. Therefore the left-hand side of (3.14) is also strictly positive and conclusion (1.7) follows. □

The previous theorem readily implies Proposition 1.2. We actually have the following more precise result:

Proposition 3.3. *Let* (H1)–(H2) *be satisfied with $f \geqslant f' > 0$. Assume the isoperimetric profile satisfies for some $\gamma > 0$ the inequality*:

$$I_M^2(s) \geqslant \gamma s\big(|M| - s\big), \quad a.e.\ s \in \big[0, |M|\big]. \tag{3.15}$$

Then any non-constant *solution u of problem* (1.1) *satisfies*

$$\int_M f(u) \geqslant 2\gamma |M|,$$

and the inequality is strict whenever (3.15) *is strict a.e.*

In dimension strictly greater than two we cannot except to verify a condition like (3.15). Indeed under mild assumptions it is known that $I_M(s) = \mathrm{O}(s^{\frac{n-1}{n}})$, and therefore in such a case we get

$$\inf_{s \in (0,|M|)} \frac{I_M^2(s)}{s(|M| - s)} = 0.$$

4. Isoperimetric bounds and L^1-estimates

In this section we illustrate how Theorem 3.3 can be applied to estimate $\int_M f(u)$ in problem (1.1). We first recall one definition.

Definition 4.1. Let $\mathcal{M}$ be the class of open sets $\omega \subset M$ such that $\partial\omega \cap M^o$ is a 1-submanifold of class C^1. The "relative isoperimetric constant" $\mathbf{i}_M$ is defined by:

$$\mathbf{i}_M := \inf_{\omega \in \mathcal{M}} \frac{\mathcal{P}(\omega, M^o)}{(\min\{|\omega|, |M \setminus \omega|\})^{1/2}}.$$

The next proposition gives several cases where it is possible to compare the isoperimetric profile of a surface with the one of a sphere. Due to the symmetry of the isoperimetric profile, it is enough to state such bounds on the interval $(0, |M|/2)$. Other interesting estimates on I_M can be found in [3].

Proposition 4.2. *Let M be a surface satisfying* (H1).

(a) *If $\mathbf{i}_M > 0$ then*

$$\left(\frac{I_M(s)}{s}\right)^2 > \frac{I_M^2(s)}{s(|M| - s)} > \frac{\mathbf{i}_M^2}{|M|}, \quad \forall s \in \left(0, \frac{|M|}{2}\right). \tag{4.1}$$

(b) *Assume M is simply connected, $\partial M = \emptyset$ and denote by K its Gauss curvature. Then*

$$\frac{I_M^2(s)}{s(|M| - s)} \geqslant \left(\frac{8\pi}{|M|} - \sup_M K\right), \quad \forall s \in \left(0, \frac{|M|}{2}\right], \tag{4.2}$$

and inequality is strict whenever $\sup_M K > \frac{4\pi}{|M|}$.

(c) Lévy–Gromov inequality. *Assume $\partial M = \emptyset$ and that its Gauss curvature is bounded from below by $\kappa > 0$. Then,*

$$\frac{I_M^2(s)}{s(|M| - s)} \geqslant \kappa, \quad \forall s \in \left(0, \frac{|M|}{2}\right]. \tag{4.3}$$

If equality holds for some s, then M is isometric to the canonical 2-sphere of curvature $\frac{4\pi}{|M|}$.

(d) *For a flat torus T with shortest closed geodesic length ℓ, we have*

$$\frac{I_T^2(s)}{s(|T| - s)} > \begin{cases} \frac{4\pi}{|T|} & \textit{if } \frac{|T|}{\ell^2} \leqslant \frac{4}{\pi}, \\ 16\frac{\ell^2}{|T|^2} & \textit{if } \frac{|T|}{\ell^2} \geqslant \frac{4}{\pi}, \end{cases} \quad \forall s \in \left(0, \frac{|T|}{2}\right). \tag{4.4}$$

(e) *If M is a rectangle $R=[0,a]\times[0,b]$ in $\mathbb{R}^2$ ($a\geqslant b$), then*

$$\frac{I_R^2(s)}{s(|R|-s)} > \begin{cases} \frac{\pi}{|R|} & \text{if } \frac{a}{b}\leqslant\frac{4}{\pi},\\ \frac{4}{a^2} & \text{if } \frac{a}{b}\geqslant\frac{4}{\pi}, \end{cases} \quad \forall s\in\left(0,\frac{|R|}{2}\right). \tag{4.5}$$

(f) *For a disc $D\subset\mathbb{R}^2$ we have*

$$\frac{I_D^2(s)}{s(|D|-s)} > \frac{16}{\pi|D|}, \quad \forall s\in\left(0,\frac{|D|}{2}\right). \tag{4.6}$$

Furthermore relations (4.4)–(4.6) *become an equality at $s=|M|/2$.*

Proof. (a) The first inequality in (4.1) is obvious. Concerning the second, we note that the isoperimetric profile of a surface can be bounded from below as follows:

$$I_M(s)\geqslant \mathbf{i}_M|s|^{1/2}, \quad \forall s\in\left(0,\frac{|M|}{2}\right]. \tag{4.7}$$

Using the lower approximation (4.7), for each $s\in(0,\frac{M}{2}]$ we get

$$\frac{I_M^2(s)}{s(|M|-s)}\geqslant\frac{\mathbf{i}_M^2}{|M|-s}>\frac{\mathbf{i}_M^2}{|M|}. \tag{4.8}$$

(b) Set $\kappa:=\sup_M K$. For a two-dimensional simply connected surface, recall that the following isoperimetric inequality holds for any simply connected domain with C^1-boundary [42]:

$$\left[\mathcal{H}^1(\partial w)\right]^2\geqslant|w|\big(4\pi-\kappa|\omega|\big). \tag{4.9}$$

Using the fact that the surface is simply connected and arguing as in (Lemma 4.2, [10]), inequality (4.9) holds also for non-simply connected domain. Hence the isoperimetric profile of M can be bounded from below as follows:

$$I^2(s)\geqslant s(4\pi-\kappa s). \tag{4.10}$$

Therefore

$$\frac{I^2(s)}{s(|M|-s)}\geqslant\frac{4\pi-\kappa s}{|M|-s}. \tag{4.11}$$

By applying Gauss–Bonnet Theorem we note that

$$4\pi=\int_M K\leqslant\kappa|M|.$$

Hence

(i) if $\kappa|M|>4\pi$ the function in the right-hand side of (4.11) is strictly decreasing on the interval $(0,\frac{|M|}{2})$;
(ii) if $\kappa|M|=4\pi$, the right-hand side of (4.11) is identically equal to $\frac{4\pi}{|M|}$.

Thus in case (i) we get

$$\frac{I^2(s)}{s(|M|-s)}>\frac{8\pi-\kappa|M|}{|M|}, \quad \forall s\in\left(0,\frac{|M|}{2}\right),$$

and in the second case (ii),

$$\frac{I^2(s)}{s(|M|-s)}\geqslant\frac{4\pi}{|M|}, \quad \forall s\in\left(0,\frac{|M|}{2}\right).$$

(c) See Appendix C in [20], and we also refer to [3].

(d) The isoperimetric profile of a flat torus is given by (see [25]):

$$I_T(s) := \begin{cases} \sqrt{4\pi s} & \text{if } s \in \left[0, \frac{\ell^2}{\pi}\right], \\ 2\ell & \text{if } s \in \left(\frac{\ell^2}{\pi}, \frac{|T|}{2}\right], \end{cases}$$

and is symmetric with respect to $s = \frac{|T|}{2}$. Therefore,

$$\frac{I_T^2(s)}{s(|T|-s)} = \begin{cases} \frac{4\pi}{|T|-s} & \text{if } s \in \left(0, \frac{\ell^2}{\pi}\right], \\ \frac{4\ell^2}{s(|T|-s)} & \text{if } s \in \left(\frac{\ell^2}{\pi}, \frac{|T|}{2}\right]. \end{cases} \tag{4.12}$$

Thus the function in (4.12) is strictly increasing on the interval $(0, \frac{\ell^2}{\pi}]$ and strictly decreasing on the interval $(\frac{\ell^2}{\pi}, \frac{|T|}{2}]$. Therefore we get the strict inequality

$$\frac{I_T^2(s)}{s(|T|-s)} > \begin{cases} \frac{4\pi}{|T|} & \text{if } \frac{\ell^2}{|T|} \geqslant \frac{\pi}{4}, \\ 16\frac{\ell^2}{|T|^2} & \text{if } \frac{\ell^2}{|T|} \leqslant \frac{\pi}{4}, \end{cases} \quad \forall s \in \left(0, \frac{|T|}{2}\right).$$

(e) The isoperimetric profile of $R := [0, a] \times [0, b]$ ($a \geqslant b$) is given by (see [25]):

$$I_R(s) := \begin{cases} \sqrt{\pi s} & \text{if } s \in \left[0, \frac{b^2}{\pi}\right], \\ b & \text{if } s \in \left(\frac{b^2}{2\pi}, \frac{|R|}{2}\right]. \end{cases}$$

Hence

$$\frac{I_R^2(s)}{s(|R|-s)} = \begin{cases} \frac{\pi}{|R|-s} & \text{if } s \in \left(0, \frac{b^2}{\pi}\right], \\ \frac{b^2}{s(|R|-s)} & \text{if } s \in \left(\frac{b^2}{2\pi}, \frac{|R|}{2}\right]. \end{cases}$$

Arguing as we did for the flat torus (part (d)) we deduce

$$\frac{I_R^2(s)}{s(|R|-s)} > \begin{cases} \frac{\pi}{|R|} & \text{if } \frac{a}{b} \leqslant \frac{4}{\pi}, \\ \frac{4}{a^2} & \text{if } \frac{a}{b} \geqslant \frac{4}{\pi}, \end{cases} \quad \forall s \in \left(0, \frac{|R|}{2}\right).$$

(f) Since (4.6) is invariant by dilation, it is enough to consider a disc of radius 1 centered at the origin. Referring to [5, 18.1.3], the isoperimetric regions E of a disc[1] are well-known and are given by:

(i) either $E = D^o \cap B$ where B is an open ball whose boundary intersects ∂D orthogonally when $s \in (0, \frac{|D|}{2})$;
(ii) or E is the intersection of a half-plane with D^o when $s = \frac{|D|}{2}$.

For each ball $B(A, r)$ whose boundary meets ∂D orthogonally, consider the intersections points $\{A_1, A_2\} := \partial B(A, r) \cap \partial D$ and denote by $\theta \in [0, \frac{\pi}{2})$ the (positive) angle defined by the vectors OA and OA_1. By setting

$$L(\theta) := \mathcal{H}^1\big(\partial B(A, r) \cap D^\circ\big), \qquad V(\theta) := \big|B(a, r) \cap D^\circ\big|,$$

simple geometrical arguments show that

$$L(\theta) = 2\left(\frac{\pi}{2} - \theta\right)\tan\theta, \tag{4.13}$$

$$V(\theta) = \left(\frac{\pi}{2} - \theta\right)\tan^2\theta - \tan\theta + \theta. \tag{4.14}$$

Using (4.13) and (4.14), let us prove that

$$\Psi := L^2 - \frac{16}{\pi^2}V(\pi - V) > 0 \quad \text{in } \left(0, \frac{\pi}{2}\right). \tag{4.15}$$

[1] These are defined as open sets $E \subset D^\circ$ satisfying $I_D(|E|) = \mathcal{P}(E, D^\circ)$.

Notice that

$$\Psi(0) = \lim_{\theta \to \frac{\pi}{2}} \Psi(\theta) = 0. \tag{4.16}$$

We claim that Ψ' has a unique zero θ_0 in the interval $(0, \frac{\pi}{2})$, and

$$\Psi' > 0 \quad \text{in } (0, \theta_0), \qquad \Psi' < 0 \quad \text{in } \left(\theta_0, \frac{\pi}{2}\right). \tag{4.17}$$

With the aim of proving (4.17) we calculate the derivatives L', V', V''. A straight calculation and the convexity of the function $\theta \mapsto \tan\theta$ in the interval $[0, \frac{\pi}{2})$ imply

$$\frac{L'}{2} = \left(\frac{\pi}{2} - \theta\right)(1 + \tan^2\theta) - \tan\theta > 0. \tag{4.18}$$

Furthermore

$$\begin{aligned} V' &= 2\tan\theta\left\{\left(\frac{\pi}{2} - \theta\right)(1 + \tan^2\theta) - \tan\theta\right\} \\ &= L'\tan\theta, \end{aligned} \tag{4.19}$$

and also

$$\begin{aligned} \frac{V''}{2} &= (1 + \tan^2\theta)\left\{3\left(\frac{\pi}{2} - \theta\right)\tan^2\theta - 3\tan\theta + \left(\frac{\pi}{2} - \theta\right)\right\} \\ &= 3(1 + \tan^2\theta)\tan^2\theta \underbrace{\left[\frac{\pi}{2} - \theta - \frac{1}{\tan\theta} + \frac{\frac{\pi}{2} - \theta}{3\tan^2\theta}\right]}_{f(\theta)}. \end{aligned} \tag{4.20}$$

We readily check that $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ satisfies both following properties:

$$\lim_{\theta \to \frac{\pi}{2}} f(\theta) = 0,$$

$$f'(\theta) = \frac{2}{3\tan^3\theta}\left\{\tan\theta - \left(\frac{\pi}{2} - \theta\right)(1 + \tan^2\theta)\right\} = -\frac{L'(\theta)}{3\tan^3\theta} < 0,$$

the last strict inequality following from (4.18). Therefore f is strictly positive in the interval $(0, \frac{\pi}{2})$. As a consequence, relation (4.20) shows that $V'' > 0$ in $[0, \frac{\pi}{2})$. Summarizing above information, we get in the interval $(0, \frac{\pi}{2})$

$$L' > 0, \quad V' = 2L'\tan\theta, \quad V'' > 0. \tag{4.21}$$

Calculating the derivative of Ψ yields

$$\begin{aligned} \Psi'(\theta) &= 2LL' - \frac{16}{\pi^2}(\pi - 2V)V' \\ &= 4L'\tan\theta\left\{\left(\frac{\pi}{2} - \theta\right) - \frac{8}{\pi^2}\left(\frac{\pi}{2} - V\right)\right\}. \end{aligned} \tag{4.22}$$

Since $L' > 0$ and V is strictly convex in $(0, \frac{\pi}{2})$ we check easily that the function (4.22) vanishes at exactly one value θ_0 and so (4.17) follows. Properties (4.17) and (4.16) readily give conclusion (4.15). □

The above bounds on the isoperimetric profile allows us to derive the following L^1-apriori estimates in problem (1.1).

Proposition 4.3. *Assume* (H1)–(H2) *hold, and* $f \geqslant f'$. *Then for any* non-constant *solution* u *of* (1.1) *the following statements hold.*

(a) *If* $\mathbf{i}_M > 0$ *then* $\int_M f(u) > 2\mathbf{i}_M^2$.

(b) *Assume M is simply connected without boundary and denote by K its Gauss curvature. Then*

$$\int_M f(u) \geqslant 2\Big(8\pi - \sup_M\big\{K|M|\big\}\Big),$$

and inequality is strict if $\sup_M\{K|M|\} > 4\pi$.

(c) *Assume $\partial M = \emptyset$ and that its Gauss curvature is bounded from below by $\kappa > 0$. Then, $\int_M f(u) \geqslant 2\kappa|M|$ and if equality holds for some non-constant solution u then M is isometric to the sphere $\mathbb{S}_M$.*

(d) *If M is a flat torus T with shortest closed geodesic length ℓ, then*

$$\int_T f(u) > \begin{cases} 8\pi & \text{if } \frac{|T|}{\ell^2} \leqslant \frac{4}{\pi}, \\ 32\frac{\ell^2}{|T|} & \text{if } \frac{|T|}{\ell^2} \geqslant \frac{4}{\pi}. \end{cases}$$

(e) *If M is a rectangle $R = [0,a] \times [0,b]$ in $\mathbb{R}^2$ $(a \geqslant b)$, then*

$$\int_R f(u) > \begin{cases} 2\pi & \text{if } \frac{a}{b} \leqslant \frac{4}{\pi}, \\ 8\frac{b}{a} & \text{if } \frac{a}{b} \geqslant \frac{4}{\pi}. \end{cases}$$

(f) *If $D \subset \mathbb{R}^2$ is a disc we have $\int_D f(u) > \frac{32}{\pi}$.*

Proof. Each of these statements is proved by applying Theorem 3.3 together with the bounds on I_M^2 as obtained in Proposition 4.2. □

For domain $\Omega \Subset \mathbb{R}^2$ with Lipschitz boundary it is known that $\mathbf{i}_\Omega > 0$ (see [38, Corollary 3.2.1], [16, Section 5.6]), but the lower bound obtained in this way could be very rough. For example in the case of a rectangle $R = [0,a] \times [0,b]$ of $\mathbb{R}^2$ $(a \geqslant b)$, the isoperimetric constant is given by $\mathbf{i}_R^2 = 2\frac{b}{a}$ (see [14]). Hence claim (a) of Proposition 4.3 gives $\int_R f(u) \geqslant 4\frac{b}{a}$ which is far from being optimal by comparing with part (e) of Proposition 4.3. Nevertheless, such an estimate could be useful when the isoperimetric profile is too complicated and can also in some cases give sharp results (see the next section). By using the specific knowledge of the relative isoperimetric constant for some domains of the plane (see [14]), we give a sample of explicit lower bounds that can be derived thanks to part (a) of Proposition 4.3.

Corollary 4.4. *Given f satisfying* (H2)*, consider a non-constant solution u of* (1.1)*. Then the following statements hold.*

(a) *Let K be a bounded convex domain of $\mathbb{R}^2$ which is symmetric about the origin and denote by W_K its width. Then $\int_K f(u) > 4\frac{W_K^2}{|K|}$.*

(b) *Let $E_{a,b}$ be the ellipse whose axes have lengths a and b $(a \geqslant b)$. Then $\int_{E_{a,b}} f(u) > \frac{16}{\pi}\frac{b}{a}$.*

(c) *Let P_k be a regular polygon of $\mathbb{R}^2$ with k-sides, k an even integer. By setting $\theta := \pi/k$ we get $\int_{P_k} f(u) > \frac{32}{k}\frac{\cos^2\theta}{\sin 2\theta}$.*

(d) *Let T be a triangle of $\mathbb{R}^2$ with smallest angle θ_0. Then $\int_T f(u) > 4\theta_0$.*

5. Uniqueness results

5.1. Lower bounds on the spectrum

Let us see how our results can be used to derive lower bounds on the first non-zero Neumann eigenvalue $\lambda_1(M)$ of the linear problem (1.11). We need first to adapt our Theorem 1.1.

Proof of Theorem 1.4. We would like to apply Theorem 3.1 with the linear function $f(s) = \lambda s$. But this latter changes sign and so the assumption (H2) is not satisfied. To overcome this difficulty, we consider for each $\epsilon > 0$ the function

$$\varphi_\epsilon := \epsilon\Big(\varphi - \min_M \varphi\Big) + 1,$$

which is non-constant and satisfies

$$-\Delta\varphi_\epsilon = \lambda\varphi_\epsilon - \alpha_\epsilon, \quad \varphi_\epsilon > 1, \tag{5.1}$$

where

$$\alpha_\epsilon = \lambda\Big(1 - \epsilon \min_M \varphi\Big).$$

Consider the linear function $f(s) = \lambda s$ restricted to the interval $(1, \infty)$ and extend f in order that (H2) is satisfied. Then by defining

$$V_\epsilon(t) := \big|\{\varphi_\epsilon > t\}\big| = \left|\left\{\varphi > \frac{t-1}{\epsilon} + \min_M \varphi\right\}\right|,$$

Theorem 3.1 applied to (5.1) with $f(s) = \lambda s$ on $(1, \infty)$ yields

$$\frac{\alpha_\epsilon}{2} \int\limits_{\min\varphi_\epsilon}^{\max\varphi_\epsilon} V_\epsilon(t)\big(|M| - V_\epsilon(t)\big)\,dt \geqslant \int\limits_{\min\varphi_\epsilon}^{\max\varphi_\epsilon} t I_M^2\big(V_\epsilon(t)\big)\,dt. \tag{5.2}$$

Making the change of variable $s := \frac{t-1}{\epsilon} + \min_M \varphi$ and setting $V(s) := |\{\varphi > s\}|$ we get

$$\frac{\alpha_\epsilon}{2} \int\limits_{\min\varphi}^{\max\varphi} V(s)\big(|M| - V(s)\big)\,ds \geqslant \int\limits_{\min\varphi}^{\max\varphi} \Big(\epsilon[s - \min_M \varphi] + 1\Big) I^2(V(s)).$$

By letting ϵ tend to zero and recalling that $I_{S_\kappa} = s(4\pi - \kappa s)$ we obtain (1.12). $\square$

As an immediate consequence of Proposition 1.4 we get

Corollary 5.1. *Assume* (H1) *holds and consider the constant*

$$\mathcal{I}_M := \inf_{s\in(0,|M|/2)} \frac{I_M^2(s)}{s(|M| - s)}. \tag{5.3}$$

Then $\lambda_1(M) \geqslant 2\mathcal{I}_M$.

Let us compare the constant (5.3) with the “Cheeger's constant” $\mathbf{h}_M$ defined as

$$\mathbf{h}_M := \inf_{\omega\in\mathcal{M}} \frac{\mathcal{P}(\omega, M^o)}{\min\{|\omega|, |M \setminus \omega|\}}.$$

By using (4.1) we always have $\mathcal{I}_M \leqslant \mathbf{h}_M^2$. Furthermore Cheeger's inequality states that $\lambda_1(M) \geqslant (\frac{\mathbf{h}_M}{2})^2$ (see [13,12]). But the constant $\mathcal{I}_M$ may give a better lower bound. For example in the square $Q := [0,1] \times [0,1]$ of $\mathbb{R}^2$ we have:

$$\left(\frac{\mathbf{h}_Q}{2}\right)^2 = 1, \quad 2\mathcal{I}_Q = 2\pi, \quad \lambda_1(Q) = \pi^2. \tag{5.4}$$

The first equality in (5.4) follows from the fact that the function $\frac{I(s)}{s}$ is decreasing (see [3]), and the second is contained in our Proposition 4.2.

Other lower bounds can be obtained by applying Proposition 1.4 together with the estimates stated in Proposition 4.2. For example by using the second inequality in (4.1) we get

$$\lambda_1(M)|M| \geqslant 2\mathbf{i}_M^2. \tag{5.5}$$

But this estimate is not so interesting since the better inequality $\lambda_1(M)|M| \geqslant 4\mathbf{i}_M^2$ is known (see [16]). More interesting are the following lower bounds.

Corollary 5.2. *Let M be a surface without boundary satisfying* (H1)*, and denote by K its Gauss curvature.*

(a) *If M is simply connected then*

$$\lambda_1(M) \geqslant \frac{8\pi}{|M|} - \max_M K,$$

and inequality is strict whenever $\max_M K > \frac{4\pi}{|M|}$.

(b) (Lichnerowicz, Obata). *Assume K is bounded from below by some constant $\kappa > 0$. Then $\lambda_1(M) \geqslant 2\kappa$ and equality holds if and only if M is isometric to the standard 2-sphere of curvature* $\frac{4\pi}{|M|}$.

The second statement of Corollary 5.2 holds in any dimension by assuming a positive lower bound on the Ricci curvature. This was proved by Lichnerowicz [30] and Obata [40] (see [12, Chapter III. 4]). It is quite interesting that for surfaces such a result can be recovered from our inequality (1.12).

5.2. *Uniqueness for a mean field equation*

Let us now apply our results got in previous section to derive several new uniqueness results for problem (1.3). Since this latter is invariant by adding a constant to a solution, we define

$$\mathring{H}(M) := \left\{ u \in H^1(M): \int_T u = 0 \right\},$$

and consider the equivalent problem

$$-\Delta_g u = \lambda\left(\frac{\mathrm{e}^u}{\int_M \mathrm{e}^u} - \frac{1}{|M|}\right), \quad u \in \mathring{H}(M). \tag{5.6}$$

Clearly $u \equiv 0$ solves (5.6) and the question arises if there are non-trivial solutions. In most cases problem (5.6) admits a variational formulation and is the Euler–Lagrange equation of the functional:

$$J(\lambda, \cdot): \mathring{H}(M) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_M |\nabla u|^2 - \lambda \log\left(\frac{1}{|M|}\int_M \mathrm{e}^u\right). \tag{5.7}$$

Indeed, the two-dimensional Moser–Trudinger inequality [39] and its various extensions assert the existence of a constant $\beta_M > 0$ such that

$$\sup_{u \in H^1(M)} \int_\Omega \mathrm{e}^{\beta_M\left(\frac{|u-\bar{u}|}{\|\nabla u\|_2}\right)^2} < \infty, \tag{5.8}$$

in each of the following situations:

(i) for compact manifolds without boundary, and in this case $\beta_M = 4\pi$ (see [17]);
(ii) for bounded domains of $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary, and in this case we have $\beta_M = 2\pi$ (see [15, Theorem 1.1]);
(iii) for bounded, finitely connected domains of $\mathbb{R}^2$ whose boundary is C^2-piecewise with finite number of vertices; in this case $\beta = 2\theta_0$, where θ_0 is the minimum interior angle at the vertices (see [9, Proposition 2.3], [15]).

Whenever (5.8) holds for some constant $\beta_M > 0$, we derive that the functional (5.7) has the following properties (see [1]):

(i) it is of class C^∞ and its critical points solve (5.6);
(ii) it admits a minimizer for each $\lambda < 2\beta_M$.

But such a minimizer could just be the trivial solution $u \equiv 0$. This is the case when $\lambda \leqslant 0$, since in this range of the parameter the functional (5.7) is strictly convex. For $\lambda > 0$, the study of uniqueness becomes more subtle. Our previous results can be applied to establish an explicit range of the parameter where $u \equiv 0$ is the unique solution of problem (5.6). More precisely we have

Proposition 5.3. *Let (λ, u) be a* non-zero *solution of problem* (5.6). *Then the following statements hold.*

(a) *On any surface satisfying* (H1), *we have* $\lambda > 2\mathbf{i}_M^2$.
(b) *Assume* M *satisfies* (H1), $\partial M = \emptyset$ *and is simply-connected. Denoting by* K *its Gauss curvature we have*

$$\lambda \geqslant 2\Big(8\pi - \sup_M\big\{K|M|\big\}\Big),$$

and inequality is strict whenever $\sup_M\{K|M|\} > 4\pi$.
(c) *Assume* M *satisfies* (H1), $\partial M = \emptyset$ *and that its Gauss curvature is bounded from below by* κ. *Then*, $\lambda \geqslant 2\kappa|M|$ *and equality can hold if and only if* M *is isometric to the canonical* 2*-sphere of curvature* $\frac{4\pi}{|M|}$.
(d) *If* M *is a flat torus* T *with shortest closed geodesic length* ℓ, *then*

$$\lambda > \begin{cases} 8\pi & \text{if } \frac{|T|}{\ell^2} \leqslant \frac{4}{\pi}, \\ 32\frac{\ell^2}{|T|} & \text{if } \frac{|T|}{\ell^2} \geqslant \frac{4}{\pi}. \end{cases}$$

(e) *If* $M = [0,a] \times [0,b]$ *is a rectangle in* $\mathbb{R}^2$ $(a \geqslant b)$, *then*

$$\lambda > \begin{cases} 2\pi & \text{if } \frac{a}{b} \leqslant \frac{4}{\pi}, \\ 8\frac{b}{a} & \text{if } \frac{a}{b} \geqslant \frac{4}{\pi}. \end{cases}$$

(f) *For a disc* $D \subset \mathbb{R}^2$, *then* $\lambda > \frac{32}{\pi}$.

Proof. Apply Theorem 4.3 with

$$C := \frac{\lambda}{\int_M \mathrm{e}^u}, \quad f(s) := C\mathrm{e}^s, \quad \alpha := \frac{\lambda}{|M|}.$$

Concerning statement (c), we only need to mention that on the 2-sphere existence of a family of non-trivial solution at $\lambda = 8\pi$ is known by the work of [41]. □

Remark 5.4.

(i) When M is a sphere it was proved by Onofri that in the range $\lambda < 8\pi$, $u \equiv 0$ is the unique minimizer of the functional (5.7). An alternative proof of this fact has been obtained by Hong [21]. The arguments of Chanillo and Kiessling [11] and also Lin [31] have strengthen this result by showing that actually $u \equiv 0$ is the unique solution of (5.6) whenever $\lambda < 8\pi$. Our statement (c) of Proposition 5.3 extends these previous results to the class of closed oriented compact surfaces with curvature bounded from below by $\kappa > 0$.
(ii) Part (d) of Proposition 5.3 has been obtained in [32]. As shown by the present paper, this result fits actually in a much more general framework.
(iii) A result due to Weinberger [46] states that among domains of prescribed volume the first positive Neumann eigenvalue is *maximized* by a ball. In particular $\lambda_1(\Omega)|\Omega| < 4\pi$ for any $\Omega \Subset \mathbb{R}^2$. Hence for domain with $C^{1,\alpha}$-boundary one easily derives that the functional (5.7) admits a non-trivial minimizer whenever $\lambda \in (\lambda_1(\Omega)|\Omega|, 4\pi)$ (see [43]). On the other hand our results show that $u \equiv 0$ is the unique solution when $\lambda \leqslant 2\mathbf{i}_\Omega^2$, and note that $2\mathbf{i}_\Omega^2 < \lambda_1(\Omega)|\Omega|$ (remind (5.5)). We expect $u \equiv 0$ to be the unique solution also when $\lambda \in (2\mathbf{i}_\Omega^2, \lambda_1(\Omega)|\Omega|)$.
In the particular case of a ball our uniqueness result holds up to the value $\frac{32}{\pi}$ which is strictly less than $\lambda_1(B)|B|$ (we have $\lambda_1(B)|B| - \frac{32}{\pi} = 0.461$).

Let us emphasize some situations where Proposition 5.3 is sharp. Define the set of non-trivial solutions

$$\mathcal{S}_M(\lambda) := \big\{u \in \overset{\circ}{H}(M)\colon\ (\lambda, u) \text{ solves (5.6)}, u \not\equiv 0\big\}. \tag{5.9}$$

Proposition 5.5. *Let* T *be a triangle of* $\mathbb{R}^2$ *with smallest angle* θ_0, *then the following hold.*

(a) $\mathcal{S}_T(\lambda) = \emptyset$ *whenever* $\lambda \leqslant 4\theta_0$.

(b) *For a.e.* $\lambda \in (4\theta_0, \lambda_1(T)|T|)$ *we have* $\mathcal{S}_T(\lambda) \neq \emptyset$. *Furthermore for any sequence* (λ_n, u_n) *with*

$$u_n \in \mathcal{S}_T(\lambda_n), \quad \lambda_n > 4\theta_0, \quad \lambda_n \to 4\theta_0, \tag{5.10}$$

we have

$$\liminf_{\lambda_n \to 8\pi} \|\nabla u_n\|_{L^2} = \infty. \tag{5.11}$$

(c)
$$\int_T e^u \leqslant e^{\frac{1}{8\theta_0}\int_T |\nabla u|^2}, \quad \forall u \in \overset{\circ}{H}(T). \tag{5.12}$$

Proof. The first statement follows from part (a) of Proposition 5.3 and the fact that the relative isoperimetric constant of a triangle is given by $\mathbf{i}_T^2 = 2\theta_0$ (see [14]).

For the second statement, we prove as in [44] that the associated functional $J(\lambda, \cdot)$ defined by (5.7) has a "mountain pass" structure for each $\lambda \in (4\theta_0, \lambda_1(T)|T|)$. Without loss of generality we assume that the vertex of the minimum angle is located at the origin. Consider the family of functions

$$\delta_\mu(x) = \log \frac{8\mu^2}{(1+\mu^2|x|^2)^2}, \quad \mu \geqslant 1,$$

which solve $-\Delta u = e^u$ on $\mathbb{R}^2$ and concentrate at the origin as $\mu \to \infty$. The functions $\bar{\delta}_\mu := \delta_\mu - \frac{1}{|T|}\int_T \delta_\mu$ belong to the space $\overset{\circ}{H}(T)$ and a direct calculation shows that

$$\int_T |\nabla \bar{\delta}_\mu|^2 = 8\theta_0 \log \mu^2 + O(1), \tag{5.13}$$

$$\int_T e^{\bar{\delta}_\mu} = O(1), \tag{5.14}$$

$$\int_T \bar{\delta}_\mu = \log \mu^2 + O(1). \tag{5.15}$$

Therefore using (5.13)–(5.15) we obtain

$$J(\lambda, \bar{\delta}_\mu) = (4\theta_0 - \lambda)\log \mu^2 + O(1).$$

Hence for $\lambda \in (4\theta_0, \lambda_1(T)|T|)$ we get $\lim_{\mu\to\infty} J(\lambda, \bar{\delta}_\mu) = -\infty$. Furthermore as in [44] we see that $u \equiv 0$ is a local minimizer of $J(\lambda, \cdot)$ whenever $\lambda < \lambda_1(T)|T|$. This shows that the functional has a mountain pass structure for each λ in the interval $(4\theta_0, \lambda_1(T)|T|)$. Note that such an interval is non-empty, thanks for example to (5.5).

It is known that the Palais–Smale condition may fail for this problem. Nevertheless one may follow [44], or alternatively apply [26], or [37, Proposition 1.2], to deduce the existence of a non-trivial solution for almost every $\lambda \in (4\theta_0, \lambda_1(T)|T|)$. The fact that these solutions "blow up" as $\lambda \downarrow 4\theta_0$, namely that (5.11) holds, follows from standard arguments by noting that $(4\theta_0, 0) \in \mathbb{R}\times \overset{\circ}{H}(T)$ cannot be a bifurcation point since $4\theta_0$ is strictly less than the first eigenvalue of the linearized problem given by $\lambda_1(T)|T|$.

To prove the last statement, we note that $J(\lambda, 0) = 0$, and the minimizer is achieved whenever $\lambda < 4\theta_0$ (by [9]). So by applying part (a), we deduce that $J(\lambda, u) \geqslant 0$ whenever $\lambda \leqslant 4\theta_0$ and conclusion (5.12) follows immediately. □

Remark 5.6.

(i) To complete the statement (b) of above proposition, one needs to make a blow-up analysis in order to get existence of solutions in the full interval under consideration. But this analysis is not yet available, and we will give the details elsewhere.
(ii) Above proposition also implies that the conclusion stated in part (a) of Proposition 5.3 is optimal.
(iii) By applying Moser–Trudinger's inequality (5.8) one deduces existence of a minimizer for the functional (5.7) in the range $\lambda < 2\beta_M$. At the critical value $2\beta_M$ the discussion of existence for a minimizer becomes very delicate

(for homogeneous Dirichlet condition see [10]). In the case of a triangle an answer to this question is given by Proposition 5.5, which shows that $J(4\theta_0, \cdot)$ admits indeed a minimizer which is furthermore unique and given by $u \equiv 0$.

Other cases where Proposition 5.5 yield optimal results are the following.

(i) The flat torus T whose shortest length ℓ satisfied $\frac{|T|}{\ell^2} \leqslant \frac{4}{\pi}$. This has been discussed in detailed in [32] where one can find the analogue of Proposition 5.5. For rectangular torus others uniqueness results have been obtained in [6] and [33].
(ii) For rectangle $[0, a] \times [0, b]$ with $\frac{a}{b} \leqslant \frac{4}{\pi}$, Proposition 5.5 holds by replacing the constant θ_0 with $\frac{\pi}{4}$.

As a conclusion, let us mention that for closed surface with Gauss curvature bounded from below by $\kappa > 0$, our result is also optimal and gives the following extension of Onofri's inequality:

$$\int_M e^u \leqslant e^{\frac{1}{4\kappa|M|}\int_M |\nabla u|^2}, \quad \forall u \in \overset{\circ}{H}(M).$$

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