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# Large solutions for the Laplacian with a power nonlinearity given by a variable exponent

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**Abstract**

In this paper we consider positive boundary blow-up solutions to the problem $\Delta u = u^{q(x)}$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. The exponent $q(x)$ is allowed to be a variable positive Hölder continuous function. The issues of existence, asymptotic behavior near the boundary and uniqueness of positive solutions are considered. Furthermore, since $q(x)$ is also allowed to take values less than one, it is shown that the blow up of solutions on $\partial\Omega$ is compatible with the occurrence of dead cores, i.e., nonempty interior regions where solutions vanish.


*Keywords:* Large solutions; Existence and uniqueness; Variable exponents

## 1. Introduction

Boundary blow-up problems for elliptic equations have been widely considered in the last few years. In general, they take the form

$$\begin{cases} \Delta u(x) = f(x, u(x)) & \text{in } \Omega, \\ u(x) = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ (say $C^{2,\eta}$) and $f(x,u)$ is a given function. By a solution of (1.1) we understand a function $u \in C^2(\Omega)$ verifying the equation in the classical sense and $u(x) \to \infty$ as $x \to \partial\Omega$. The solutions to problem (1.1) are known as "large" solutions. We refer to the pioneering papers [4,17] and [25], and to [14,23] and [26] for a large list of references.

In most of the previous works, the dependence on $x$ of $f$ was not really significative. Three types of nonlinearities have been frequently treated: $f = f(u)$, $f(x,u) = a(x)g(u)$ or $f(x,u)$ controlled in terms of a function $g(u)$ which does not depend on $x$.

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For the particular case where $f$ is increasing and does not depend on $x$, $f = f(u)$, it is well known that the so-called Keller–Osserman condition is necessary and sufficient for existence of solutions to (1.1):

$$\int_{x_0}^{\infty} \frac{ds}{\sqrt{F(s)}} < +\infty \tag{1.2}$$

for some $x_0 \in \mathbb{R}$, where $F(u) = \int_0^u f(s)\,ds$ is a primitive of $f$ (see [17] and [25]). Note that it has been recently shown that the monotonicity of $f$ is not necessary, even for large $u$ as shown in [13] (see also [10], where an existence result was obtained with a nonmonotonic $f$ which is however increasing for large $u$).

When the dependence of $f$ on $x$ is of the form $f(x,u) = a(x)g(u)$, and the weight $a(x)$ is bounded, the Keller–Osserman condition on $g$ is also necessary and sufficient for existence so that the presence of $a(x)$ is not really important. When $a(x)$ is not bounded on $\partial\Omega$ the situation is slightly different: if the growth of $a$ near $\partial\Omega$ is not too strong then solutions to (1.1) exist when $g$ satisfies (1.2) (see [5,6] and [28] for the case $g(t) = t^p$, $p > 1$). However, solutions may exist with a $g$ not satisfying (1.2), provided $a$ is singular enough on $\partial\Omega$. We refer the reader to [5] and [24].

Thus, at this point it is natural to ask what happens for a function $f(x,u)$ that depends on $x$ in such a way that condition (1.2) (where $F(u)$ is replaced by $F(x,u) = \int_0^u f(x,s)\,ds$) is satisfied at some points of $\Omega$ and not at other points. If we assume $f(x,u)$ to be continuous in $\overline{\Omega} \times \mathbb{R}$ (so the presence of an unbounded weight is ruled out), is it really needed that (1.2) is satisfied at all points of $\Omega$ to obtain existence of a solution to (1.1)?

In this direction, the problem (1.1) with $f(x,u) = -\lambda u + a(x)u^{q(x)}$ with $a > 0$ in $\overline{\Omega}$ and $q > 1$ in $\Omega$, $q = 1$ on $\partial\Omega$, was considered in the pioneering paper [21], and the existence of a maximal and a minimal positive solution was obtained (see also [9] and [22] for works dealing with nonlinearities with a variable exponent and homogeneous Dirichlet boundary conditions). Notice that condition (1.2) (with $F(x,u) = \int_0^u f(x,s)\,ds$) holds at points where $q(x) > 1$, while it ceases to be true when $q(x) \leqslant 1$. In this respect, the results in [21] show that (1.2) may fail on the boundary $\partial\Omega$, and the existence of positive solutions is still possible.

In the present paper we are considering the problem:

$$\begin{cases} \Delta u = u^{q(x)} & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the exponent $q(x)$ will be a positive Hölder continuous function. We note that a distinctive feature in this work with respect to the hypotheses in [21] is that $q < 1$ is permitted at some points in $\Omega$. In this respect, one of the contributions of the present work is to show that condition (1.2) is only needed in a neighborhood of the boundary in order to have a positive solution, while it may fail not only on the boundary, but also at interior points.

We mention in passing that the case where $q$ is constant is well understood, see [2,3,7,11,12,15,16,18,20,27], but, at the best of our knowledge, the only previous work where large solutions with nonlinearities with a variable exponent were considered is [21].

In addition to existence of positive solutions to (1.3), we also consider uniqueness and the determination of the blow-up rate of solutions near the boundary of the domain. Our techniques are mainly based on comparison, using as a reference problem (1.3) with a $q$ constant.

Now we state our results. We first show that positive solutions to (1.3) are only possible if $q \geqslant 1$ on $\partial\Omega$. When $q$ is constant, this is known to hold (see Theorem 2 in [19] and Theorem 2.2 in [8]).

**Theorem 1.** *Let $q \in C^\eta(\overline{\Omega})$ be a nonnegative function, and assume there exists $x_0 \in \partial\Omega$ such that $q(x_0) < 1$. Then problem* (1.3) *has no positive solutions. Moreover, the same conclusion holds if $q \leqslant 1$ in a whole neighborhood of $x_0 \in \partial\Omega$ relative to $\overline{\Omega}$.*

**Remark 1.** As kindly pointed out to the authors by the referee of this paper, a nonexistence result for the variable exponent related problem

$$\begin{cases} \Delta u = -\lambda u + a(x)u^{q(x)} & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

can be obtained by means of Theorem 7.1 in [21] provided the set $\{q(x) > 1\}$ is strictly contained in $\Omega$, $q = 1$ in a whole neighborhood of $\partial\Omega$ and $\lambda$ is conveniently large.

Thanks to Theorem 1 we always need $q \geqslant 1$ on $\partial\Omega$ in order to have positive solutions. We will make the assumption that $q > 1$ in a neighborhood of $\partial\Omega$, although $q$ may be 1 on $\partial\Omega$. We also remark that $q \leqslant 1$ is permitted at interior points, and still we get a solution.

**Theorem 2.** *Let $q \in C^\eta(\overline{\Omega})$ be a positive function and assume $q > 1$ in the strip $\mathcal{U}_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta > 0$. Then problem* (1.3) *admits at least a positive solution.*

It is natural to ask under which conditions the solution provided by Theorem 2 is unique. It turns out that $q > 1$ on $\partial\Omega$ is sufficient *as long as $q \geqslant 1$* in the whole $\Omega$.

**Theorem 3.** *Assume $q \in C^\eta(\overline{\Omega})$ verifies $q \geqslant 1$ in $\Omega$ and $q > 1$ on $\partial\Omega$. Then problem* (1.3) *admits a unique positive solution.*

The approach for proving Theorem 3 is to obtain the boundary behavior of all positive solutions. We remark that this is a local issue, and hence the obtained behavior is similar to that in the case where $q$ is constant, at least at points where $q > 1$. In the rest of the paper, $d(x)$ will stand for the function $\operatorname{dist}(x, \partial\Omega)$.

**Theorem 4.** *Assume $q \in C^\eta(\overline{\Omega})$ and let $x_0 \in \partial\Omega$ with $q(x_0) > 1$. If $u$ is a positive solution to* (1.3)*, then*

$$\lim_{x\to x_0} d(x)^{\alpha(x)} u(x) = \big(\alpha(x_0)\big(\alpha(x_0)+1\big)\big)^{\frac{1}{q(x_0)-1}}, \tag{1.4}$$

*where $\alpha(x) = 2/(q(x)-1)$.*

**Remark 2.** As a byproduct of the proof of Theorem 4, the exact rate of the normal derivative of $u$ can also be obtained. More precisely we have

$$\lim_{x\to x_0} d(x)^{\alpha(x)+1} \nabla u(x)\cdot \nu(\bar{x}) = \alpha(x_0)\big(\alpha(x_0)\big(\alpha(x_0)+1\big)\big)^{\frac{1}{q(x_0)-1}},$$

where $\nu$ is the outward unit normal and $\bar{x}$ is the closest point to $x$ lying on $\partial\Omega$.

Now, another natural question arises: is it essential that $q \geqslant 1$ in the whole $\Omega$ to have uniqueness? As we are showing next, the answer is no. Uniqueness of positive solutions to (1.3) also holds if $q < 1$ at interior points if we assume $q$ is large enough on $\partial\Omega$. As a technical hypothesis, we also need $q$ to be smooth in a neighborhood of $\partial\Omega$.

**Theorem 5.** *Assume $q \in C^\eta(\overline{\Omega}) \cap C^2(\mathcal{U}_\delta)$ for some $\delta > 0$ where $\mathcal{U}_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}$, $q > 0$ in $\overline{\Omega}$ and $q > 3$ on $\partial\Omega$. Then there exists a unique solution to* (1.3)*, which in addition verifies*

$$u(x) = \big(\alpha(x)\big(\alpha(x)+1\big)\big)^{\frac{1}{q(x)-1}} d(x)^{-\alpha(x)} + O\big(d(x)^\beta\big) \tag{1.5}$$

*for every $\beta \in (0, \frac{q_0-3}{q_0-1})$, where $\alpha(x) = 2/(q(x)-1)$ and $q_0 = \min_{\partial\Omega} q$.*

It should be noticed that in the case $q$ constant it was shown in [18] that $u = Ad^{-\alpha} + O(1)$ as $d \to 0$, $\alpha$ as above, $A = \alpha(\alpha+1)^{1/(q-1)}$, provided that $q \geqslant 3$ (such feature was more precisely described in [16] where a two-term asymptotic expansion for $u$ near $\partial\Omega$ was obtained). Theorem 5 provides in particular a substantial extension of the previous results covering the case where $q$ is variable.

On the other hand and as a counterpart to the uniqueness question studied in Theorem 5, the fact that $q$ achieves values less than one in $\Omega$ allows the existence of solutions $u$ of (1.3) that exhibit simultaneously a singular behavior on $\partial\Omega$ together with the presence of a dead core, i.e., a nonempty interior region $\mathcal{O}$ in $\Omega$ where $u$ vanishes. Our next result asserts that dead cores arise provided the subdomain $Q$ of $\Omega$ where $q < 1$ is large enough. We provide a statement with hypotheses that are not optimal for the sake of clarity.

**Theorem 6.** *Suppose that $q \in C^\eta(\overline{\Omega})$ is a positive functions satisfying $q > 1$ on $\partial\Omega$ while $Q := \{x \in \Omega\colon\ q(x) < 1\}$ constitutes a smooth subdomain of $\Omega$. For $\lambda > 0$ let $\Omega_\lambda = \{\lambda x\colon\ x \in \Omega\}$ and let $q_\lambda \in C^\eta(\overline{\Omega}_\lambda)$ be given by $q_\lambda(x) = q(x/\lambda)$. Then, there exists $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ all positive solutions $u_\lambda$ to*

$$\begin{cases} \Delta u = u^{q_\lambda(x)} & \text{in } \Omega_\lambda, \\ u = +\infty & \text{on } \partial\Omega_\lambda, \end{cases} \tag{1.6}$$

*possess a nonempty dead core $\mathcal{O}_\lambda := \{x \in \Omega_\lambda\colon\ u_\lambda = 0\}$. Moreover, $\mathcal{O}_\lambda$ progressively fills $Q_\lambda$ as $\lambda \to \infty$.*

**Remark 3.** It is possible to exhibit examples showing that dead cores are absent in problem (1.3) if the region $\{q < 1\}$ does not exceed a critical size.

Finally, we briefly consider the issue of boundary behavior of positive solutions to (1.3) in the case where $q = 1$ somewhere on $\partial\Omega$. Since the problem becomes linear there it is to be expected that the exact rate of divergence of the solutions $u$ cannot be obtained, as happens in [5]. It is possible however to obtain the exact behavior of the logarithm of $u$, provided $q - 1$ behaves like a nonnegative power of the distance.

**Theorem 7.** *Assume $q \in C^\eta(\overline{\Omega})$ and let $x_0 \in \partial\Omega$ be a point with $q(x_0) = 1$. If there exist positive constants $\gamma$ and $Q$ such that*

$$\lim_{x \to x_0} \frac{q(x) - 1}{d(x)^\gamma} = Q$$

*then for every positive solution to* (1.3):

$$\lim_{x \to x_0} \frac{d(x)^\gamma \log u(x)}{-\log d(x)} = \frac{2\gamma + 2}{Q}. \tag{1.7}$$

Of course it would be desirable to obtain uniqueness of solutions to (1.3), at least in the very special case $q(x) = 1 + Qd(x)^\gamma$. According to (1.7), it would be natural to deal with the equation satisfied by $v = \log u$, that is,

$$\begin{cases} \Delta v + |\nabla v|^2 = e^{Qd(x)^\gamma v} & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

However, the operator in the left-hand side of (1.8) does not have the right monotonicity, and it could even happen that uniqueness does not hold. We leave this question as an open problem.

The paper is organized as follows: in Section 2 we prove Theorems 1 and 2. Section 3 will be dedicated to prove the boundary estimates, Theorems 4 and 7. The uniqueness results, Theorems 3 and 5 will be collected in Section 3 while the issue of dead cores (Theorem 6) will be analyzed in Section 4.

## 2. Existence

In this section, we deal with the issues of existence and nonexistence of positive solutions to problem (1.3). We first show that there are no solutions if $q < 1$ somewhere on $\partial\Omega$ (alternatively, $q \leqslant 1$ in a neighborhood of a boundary point). Throughout the paper, we denote by $B(x, r)$ the ball of center $x$ and radius $r$.

**Proof of Theorem 1.** Let $r > 0$ such that $q < 1$ in $B(x_0, 3r) \cap \overline{\Omega}$, and choose a smooth subdomain $D$ of $B(x_0, 3r) \cap \Omega$ such that $\partial D \cap \partial\Omega$ contains $B(x_0, 2r) \cap \partial\Omega$. Let $\psi$ be a smooth function supported on $\partial D$ which verifies $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on $B(x_0, r) \cap \partial\Omega$ and $\psi = 0$ on $\partial D \setminus (B(x_0, 2r) \cap \partial\Omega)$. It can be checked that the problem

$$\begin{cases} \Delta z = z^{q(x)} & \text{in } D, \\ z = n\psi & \text{on } \partial D, \end{cases}$$

has a unique positive solution $z_n$ for every positive integer $n$ (see the proof of Theorem 2 for a similar argument). Moreover, if (1.3) has a positive solution, it follows by comparison that

$$u \geqslant z_n \quad \text{in } D, \tag{2.1}$$

since $u \geqslant n\psi$ on $\partial D$ for every $n$.

On the other hand, we have $z_n = nw_n$, where $w_n$ solves

$$\begin{cases} \Delta w = n^{q(x)-1} w^{q(x)} & \text{in } D, \\ w = \psi & \text{on } \partial D. \end{cases}$$

Now $0 \leqslant w_n \leqslant 1$ and $q(x) < 1$, so it is standard to conclude (for a subsequence if necessary) that $w_n \to w_0$ as $n \to \infty$, where $w_0$ is the harmonic function in $D$ which equals $\psi$ on $\partial D$. Since $w_0 > 0$ in $D$, we obtain that $z_n \to +\infty$ uniformly in compact subsets of $D \cup (B(x_0, r) \cap \partial\Omega)$. But then (2.1) implies $u = +\infty$ in $D \cup (B(x_0, r) \cap \partial\Omega)$, which is not possible. Hence no positive solution to (1.3) exists.

Finally, observe that the previous argument continues to be valid – with only minor changes – if $q \leqslant 1$ in a neighborhood of a point $x_0 \in \partial\Omega$. Thus the proof is concluded. □

Now we prove our existence result. The approach is the standard one: we construct solutions with finite datum on $\partial\Omega$ and then show that they are locally uniformly bounded.

**Proof of Theorem 2.** Let $n$ be a positive integer. Then the problem

$$\begin{cases} \Delta u = u^{q(x)} & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

has a unique positive solution. Indeed, $\underline{u} = 0$ is a subsolution and $\bar{u} = n$ is a supersolution, and then, by a well-known approach (see [1]) the existence of a classical solution $u \in C^{2,\eta}(\overline{\Omega})$ follows. To prove uniqueness, let $u$, $v$ be positive solutions and consider the set $\Omega_0 = \{x \in \Omega:\ u < v\}$. If $\Omega_0$ is nonempty, since $\Delta u \leqslant \Delta v$ in $\Omega_0$ and $u = v$ on $\partial\Omega_0$, it follows from the maximum principle that $u > v$ in $\Omega_0$, which is impossible. Thus $u \geqslant v$ and the symmetric argument shows $u = v$, giving uniqueness. The solution to (2.2) will be denoted by $u_n$.

Thanks to uniqueness, the solutions $u_n$ is increasing in $n$. Indeed, $u_{n+1}$ is a supersolution to (2.2) and by uniqueness $u_{n+1} \geqslant u_n$.

Let us prove next that $u_n$ is bounded in compact subsets of $\Omega$. Taking $\delta$ small, we can assume $u_n > 1$ in a strip $\mathcal{U}_\delta = \{x\colon \operatorname{dist}(x, \partial\Omega) < \delta\}$ for all $n$. Fix $\varepsilon$ with $0 < \varepsilon < \delta$ and a point $x_0$ such that $d(x_0) = \varepsilon/2$. Since $q > 1$ in $\mathcal{U}_\delta$, we have that $q \geqslant q_0 > 1$ in $B(x_0, \varepsilon/4)$ and thus $\Delta u_n \geqslant u_n^{q_0}$ in $B(x_0, \varepsilon/4)$. Hence $u_n \leqslant U$, the unique solution to

$$\begin{cases} \Delta U = U^{q_0} & \text{in } B(x_0, \varepsilon/4), \\ U = +\infty & \text{on } \partial B(x_0, \varepsilon/4). \end{cases}$$

This shows that $u_n$ is uniformly bounded in $B(x_0, \varepsilon/8)$. A compactness argument proves that $u_n$ is uniformly bounded in the set $\{x \in \Omega\colon d(x) = \varepsilon/2\}$, and since every $u_n$ is subharmonic, we obtain uniform bounds in the whole $\{x \in \Omega\colon d(x) > \varepsilon/2\}$. Since $\varepsilon$ was arbitrarily small, the sequence $\{u_n\}$ is locally uniformly bounded in $\Omega$.

Finally, it is standard to obtain that $\{u_n\}$ is precompact in $C^2_{\mathrm{loc}}(\Omega)$, and thus, passing to a subsequence, $u_n \to u$ in $C^2_{\mathrm{loc}}(\Omega)$, where $u$ verifies $\Delta u = u^{q(x)}$ in $\Omega$. Notice that, since $u_n$ is increasing in $n$, it also follows that the whole sequence converges to $u$. Moreover, $u = +\infty$ on $\partial\Omega$, and is thus a positive solution to (1.3). This finishes the proof. □

## 3. Boundary estimates

This section is devoted to prove the assertions concerning the boundary behavior of the solutions to (1.3). To prove Theorem 4 we use ideas from [6]. To this aim, it is important to obtain first a rough estimate for the solutions. This is the content of the next lemma.

**Lemma 8.** *Assume $x_0 \in \partial\Omega$ is such that $q(x_0) > 1$, and let $u$ be a positive solution to* (1.3). *Then there exist a neighborhood $\mathcal{V}$ of $x_0$* (*relative to $\overline{\Omega}$*) *and positive constants $C_1, C_2$ such that*

$$C_1 d(x)^{-\alpha(x)} \leqslant u(x) \leqslant C_2 d(x)^{-\alpha(x)} \quad \text{in } \mathcal{V}, \tag{3.1}$$

*where $\alpha(x) = 2/(q(x) - 1)$.*

**Proof.** Choose a neighborhood $\mathcal{V}'$ of $x_0$ such that $q>1$ in $\mathcal{V}'$ (we can take for instance a ball centered at $x_0$ intersected with $\Omega$). By diminishing the radius of $\mathcal{V}'$, we can select a smaller neighborhood $\mathcal{V}$ such that $B(x,d(x)/2)\subset\mathcal{V}'$ for $x\in\mathcal{V}$. Take $x\in\mathcal{V}$ and define the scaled function

$$v(y)=d(x)^{\alpha(x)}u\left(x+\frac{d(x)}{2}y\right),$$

for $y\in B:=B(0,1)$. It can be checked that the function $v$ solves the equation

$$\Delta v=\frac{1}{4}d(x)^{\alpha(x)\{q(x)-q(x+(d(x)/2)y)\}}v^{q(x+(d(x)/2)y)}\quad\text{in } B.$$

Since $q$ is $\eta$-Hölder, there exists a constant such that

$$\left|q(x)-q\left(x+\frac{d(x)}{2}y\right)\right|\leqslant Cd(x)^{\eta},$$

and hence

$$\Delta v\geqslant Cv^{q(x+(d(x)/2)y)}\quad\text{in } B$$

for some positive constant $C$ (we are using throughout the paper the letter $C$ to denote constants, that may change from one line to another but are independent of the relevant quantities). That is, $v$ is a subsolution to the equation $\Delta v=Cv^{q(x+(d(x)/2)y)}$ in $B$. Now we will construct a supersolution to the same equation which blows up on the boundary of $B$.

Let $\phi$ be the solution to $-\Delta\phi=1$ in $B$ with $\phi=0$ on $\partial B$. For a large positive $A_0$ and some $\beta>0$ to be chosen, we define $\bar v=A_0\phi^{-\beta}$. Then $\bar v$ will be a supersolution to $\Delta v=Cv^{q(x+(d(x)/2)y)}$ in $B$ provided that

$$\beta(\beta+1)|\nabla\phi|^2+\beta\phi\leqslant CA_0^{\{q(x+(d(x)/2)y)-1\}}\phi^{\{\beta+2-\beta q(x+(d(x)/2)y)\}}$$

for all $y\in B$. This inequality can be obtained choosing $\beta$ large in order to have

$$\beta+2-\beta q\left(x+\frac{d(x)}{2}\right)<0,$$

and then $A_0$ large enough. By comparison, we arrive at $v\leqslant\bar v$ in $B$, and setting $y=0$ we obtain

$$u(x)\leqslant A_0\phi(0)^{-\beta}d(x)^{-\alpha(x)}$$

for $x\in\mathcal{V}$. This shows the upper inequality in (3.1).

To prove the lower inequality we take a point $x\in\mathcal{V}'$ and denote by $\bar x$ the closest point to $x$ on $\partial\Omega$. Modulus an extra reduction of $\mathcal{V}'$ if necessary it can be assumed that $d(\bar x+d(x)\nu(\bar x))=d(x)$ for every $x\in\mathcal{V}'$ where $\nu$ stands for the outward unit normal and $d(x)$ designates the distance from $x$ to $\partial\Omega$. Denoting by $\mathcal{A}$ the annulus

$$\mathcal{A}=\{y\in\mathbb{R}^N:\ 1<|y|<2+\tau\},$$

where $\tau>0$, we introduce $\mathcal{A}_x=\bar x+d(x)\nu(\bar x)+d(x)\mathcal{A}$ and $Q_x=\mathcal{A}_x\cap\Omega$ (observe that $x\in Q_x$, while the annulus $\mathcal{A}_x$ is tangent to $\partial\Omega$ at $\bar x$). We remark that the outer radius can be any fixed number greater than 2, but for its later use in the proof of Theorem 7 we let it depend on the parameter $\tau$, which is of no importance in the present proof.

We can assume, by diminishing the radius of $\mathcal{V}$, that $Q_x\subset\mathcal{V}'$ for every $x\in\mathcal{V}$. Now define the normalized function

$$w(y)=d(x)^{\alpha(x)}u\big(\bar x+d(x)\nu(\bar x)+d(x)y\big),$$

for $y\in\widetilde Q_x$, where $\widetilde Q_x=\mathcal{A}\cap\{y\in\mathbb{R}^N:\ \bar x+d(x)\nu(\bar x)+d(x)y\in\Omega\}$. Then $w$ satisfies

$$\Delta w=d(x)^{\alpha(x)\{q(x)-q(\bar x+d(x)\nu(\bar x)+d(x)y)\}}w^{q(\bar x+d(x)\nu(\bar x)+d(x)y)}$$

in $\widetilde Q_x$. Thanks to the Hölder condition verified by $q$ it follows as before that

$$\Delta w\leqslant Cw^{q(\bar x+d(x)\nu(\bar x)+d(x)y)}\quad\text{in } \widetilde Q_x,$$

for a certain positive constant $C$.

On the other hand, it can be seen as before that the problem

$$\begin{cases} \Delta z = Cz^{q(\bar{x}+d(x)\nu(\bar{x})+d(x)y)} & \text{in } \mathcal{A}, \\ z = 1 & \text{on } |y| = 1, \\ z = 0 & \text{on } |y| = 2+\tau, \end{cases}$$

has a unique positive solution $z$. Since $w \geqslant z$ on $\partial \widetilde{Q}_x$, it follows by comparison that $w \geqslant z$ in $\widetilde{Q}_x$. Setting $y = -2\nu(\bar{x})$, we arrive at

$$u(x) \geqslant z\big(-2\nu(\bar{x})\big)\, d(x)^{-\alpha(x)},$$

and the proof of (3.1) concludes by noticing that since $z$ is bounded from below in $|y| = 2$ we obtain $z(-2\nu(\bar{x})) \geqslant C > 0$, where $C$ is independent of $x$. $\square$

**Proof of Theorem 4.** Choose an open neighborhood $\mathcal{W}$ of $x_0$ such that $\partial\Omega$ admits $C^{2,\eta}$ local coordinates $\xi = (\xi_1, \dots, \xi_N)$, $\xi : \mathcal{W} \to \mathbb{R}^N$, with $x \in \mathcal{W} \cap \Omega$ if and only if $\xi_1(x) > 0$. With no loss of generality we can assume $\xi(x_0) = 0$. Setting $u(x) = \bar{u}(\xi(x))$, $q(x) = \bar{q}(\xi(x))$, then $\bar{u}$ verifies an equation

$$\sum_{i,j=1}^{N} a_{ij}(\xi) \frac{\partial^2 \bar{u}}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_i(\xi) \frac{\partial \bar{u}}{\partial \xi_i} = \bar{u}^{\bar{q}(\xi)}$$

in $\xi(\mathcal{W} \cap \Omega)$ whose coefficients $a_{ij}$, $b_i$ are $C^\eta$ functions and $a_{ij}(0) = \delta_{ij}$. We can assume further that $\mathcal{W} \cap \Omega \subset \mathcal{V}$, where $\mathcal{V}$ is the neighborhood given by Lemma 8.

Let $\{x_n\}$ be an arbitrary sequence converging to $x_0$, and denote by $t_n$ the projection of $\xi(x_n)$ onto $\xi(\mathcal{W} \cap \partial\Omega)$ (a subset of the hyperplane $\xi_1 = 0$). We introduce the functions

$$v_n(y) = d_n^{\alpha_n} u(t_n + d_n y)$$

where $d_n = d(x_n)$, $\alpha_n = \alpha(x_n)$. Then $v_n$ verifies the equation

$$\sum_{i,j=1}^{N} a_{ij}(t_n + d_n y) \frac{\partial^2 v}{\partial y_i \partial y_j} + d_n \sum_{i=1}^{N} b_i(t_n + d_n y) \frac{\partial v}{\partial y_i} = d_n^{\{\alpha_n(\bar{q}(\xi(x_n)) - \bar{q}(t_n + d_n y))\}} v^{\bar{q}(t_n + d_n y)}.$$

We now use the estimates (3.1) provided by Lemma 8. They imply that for every compact set of the half-space $D := \{y \in \mathbb{R}^N \colon y_1 > 0\}$ there exist positive constants $C_1$, $C_2$ such that

$$C_1 d_n^{\{\alpha_n - \bar{\alpha}(t_n + d_n y)\}} y_1^{-\bar{\alpha}(t_n + d_n y)} \leqslant v_n(y) \leqslant C_2 d_n^{\{\alpha_n - \bar{\alpha}(t_n + d_n y)\}} y_1^{-\bar{\alpha}(t_n + d_n y)} \tag{3.2}$$

where $\alpha(x) = \bar{\alpha}(\xi(x))$. As in the proof of Lemma 8, we use the Hölder condition on $q$ to obtain that

$$d_n^{\alpha_n - \bar{\alpha}(t_n + d_n y)} \to 1$$

uniformly for $y$ in compacts of $D$ as $n \to \infty$. Thus (3.2) gives bounds for the sequence $\{v_n\}$, and it is now standard to obtain that for a subsequence we have $v_n \to v$ in $C^2_{\mathrm{loc}}(D)$, where $v$ verifies

$$\begin{cases} \Delta v = v^{q(x_0)} \\ C_1 y_1^{-\alpha(x_0)} \leqslant v(y) \leqslant C_2 y_1^{-\alpha(x_0)} \end{cases} \quad \text{in } D. \tag{3.3}$$

Theorem 3.4 and Remark 3.6 (b) in [6] imply that problem (3.3) has a unique solution, which can be checked to be

$$v(y) = \big\{\alpha(x_0)\big(\alpha(x_0) + 1\big)\big\}^{\frac{1}{q(x_0)-1}} y_1^{-\alpha(x_0)}.$$

Then (1.4) is proved just by setting $y = e_1$. $\square$

Now we prove Theorem 7. The proof is based on that of Lemma 8, but taking into account that the exponents there may be variable, and the involved constants have to be precisely estimated.

**Proof of Theorem 7.** Let $\varepsilon > 0$ and choose a neighborhood $\mathcal{W}$ of $x_0$ such that $q(y) \geqslant 1 + (Q - \varepsilon) d(y)^\gamma$ for $y \in \mathcal{W}$. For $x$ close to $x_0$, and $0 < \tau < 1$, we have $d(y) \geqslant (1 - \tau) d(x)$ if $y \in B(x, \tau d(x))$ and hence

$$q(y) \geqslant 1 + (Q - \varepsilon)(1 - \tau)^\gamma d(x)^\gamma$$

in $B(x,\tau d(x))$, provided $B(x,\tau d(x))\subset\mathcal{W}$, which is certainly true if $x$ is close enough to $x_0$. Denote for simplicity

$$\sigma=\sigma_{\varepsilon,\tau,x}=(Q-\varepsilon)(1-\tau)^\gamma d(x)^\gamma.$$

If $x$ is close enough to $x_0$ we may further assume that $u>1$ in $B(x,\tau d(x))$. Hence

$$\Delta u\geqslant u^{1+\sigma}\quad \text{in } B\big(x,\tau d(x)\big).$$

We now introduce the function

$$v(y)=\big(\tau d(x)\big)^{\frac{2}{\sigma}}u\big(x+\tau d(x)y\big)\quad y\in B=B(0,1),$$

which satisfies

$$\Delta v\geqslant v^{1+\sigma}$$

in $B$. On the other hand, we may look for a supersolution to the equation $\Delta v=v^{1+\sigma}$ of the form

$$\bar v=A\phi^{-\beta}$$

where $\beta=2/\sigma$, $A>0$ and $\phi$ is the solution to $-\Delta\phi=1$ in $B$ with $\phi=0$ on $\partial B$. Then $\bar v$ is a supersolution provided that

$$\frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)|\nabla\phi|^2+\frac{2}{\sigma}\phi\leqslant A^\sigma.$$

Since $\sigma=(Q-\varepsilon)(1-\tau)^\gamma d(x)^\gamma$, it is enough to take

$$A=\big\{Cd(x)\big\}^{\frac{-2\gamma}{(Q-\varepsilon)(1-\tau)^\gamma d(x)^\gamma}},$$

for some positive, large enough constant $C$. By comparison,

$$v(y)\leqslant\bar v(y)$$

if $y\in B$. Setting in particular $y=0$ we obtain

$$u(x)\leqslant\big\{Cd(x)\big\}^{\frac{-2\gamma}{(Q-\varepsilon)(1-\tau)^\gamma d(x)^\gamma}}\big\{\tau d(x)\phi(0)\big\}^{\frac{-2}{(Q-\varepsilon)(1-\tau)^\gamma d(x)^\gamma}}.$$

It follows from the last estimate that

$$\limsup_{x\to x_0}\frac{d(x)^\gamma\log u(x)}{-\log d(x)}\leqslant\frac{2\gamma+2}{(Q-\varepsilon)(1-\tau)^\gamma}.$$

Letting $\varepsilon\to 0$ and then $\tau\to 0$, we obtain

$$\limsup_{x\to x_0}\frac{d(x)^\gamma\log u(x)}{-\log d(x)}\leqslant\frac{2\gamma+2}{Q}. \tag{3.4}$$

Next we prove the lower estimate. As in the first part of the proof, we may assume a neighborhood $\mathcal{W}$ of $x_0$ has been chosen so that

$$q(y)\leqslant 1+(Q+\varepsilon)d(y)^\gamma$$

for $y\in\mathcal{W}$. For $x$ close to $x_0$, we consider the sets $\mathcal{A}$, $\mathcal{A}_x$, $Q_x$, $\widetilde{Q}_x$ introduced in the proof of Lemma 8. In $Q_x$ we have

$$q(y)\leqslant 1+(Q+\varepsilon)(1+\tau)^\gamma d(x)^\gamma$$

and then if $u>1$ we have

$$\Delta u\leqslant u^{1+\theta}\quad \text{in } Q_x,$$

where we now set $\theta=(Q+\varepsilon)(1+\tau)^\gamma d(x)^\gamma$. Introduce the function

$$w(y)=d(x)^{\frac{2}{\theta}}u\big(\bar x+d(x)\nu(\bar x)+d(x)y\big)$$

for $y \in \widetilde{Q}_x$. Then $\Delta w \leqslant w^{1+\theta}$ in $\widetilde{Q}_x$, and it follows by comparison that $w \geqslant U$ in $\widetilde{Q}_x$, where $U$ is the unique positive solution to

$$\begin{cases} \Delta U = U^{1+\theta} & \text{in } \mathcal{A}, \\ U = \infty & \text{on } |y| = 1, \\ U = 0 & \text{on } |y| = 2+\tau. \end{cases}$$

Thus our next aim will be to estimate from below the solution $U$ when $\theta \to 0$. Since $U$ is radial, it verifies $U = U(r)$, where $r = |y|$ and

$$\begin{cases} U'' + \frac{N-1}{r} U' = U^{1+\theta}, \quad 1 < r < 2+\tau, \\ U(1) = \infty, \\ U(2+\tau) = 0. \end{cases}$$

We introduce the change of variables

$$\rho = \begin{cases} \frac{1}{N}(1 - \frac{1}{r^N}), & \text{if } N \geqslant 3, \\ \log r, & \text{if } N = 2, \end{cases}$$

and denote $V(\rho) = U(r)$. Then $V$ verifies

$$\begin{cases} V'' = g(\rho) V^{1+\theta}, \quad 0 < \rho < L, \\ V(0) = \infty, \\ V(L) = 0, \end{cases}$$

with $g(\rho) = r^{2(N-1)}$ and $L$ is given by $L = 1/N(1 - 1/(2+\tau)^N)$ if $N \geqslant 3$, $L = \log(2+\tau)$ for $N = 2$.

Notice that $V$ is convex, and hence thanks to the mean value theorem:

$$V(\rho) = -V'(\xi)(L-\rho) \geqslant -V'(L)(L-\rho) \tag{3.5}$$

where $\xi \in (\rho, L)$ and $0 < \rho < L$. This shows that it is enough to obtain a lower estimate for $-V'(L)$.

Since $V' < 0$ and $g(\rho) \leqslant (2+\tau)^{2(N-1)} =: c$, we get

$$V'V'' \geqslant c V^{1+\theta} V'.$$

An integration in $(\rho, L)$ gives

$$\frac{-V'(\rho)}{\sqrt{V'(L)^2 + (2c/(2+\theta)) V(\rho)^{2+\theta}}} \leqslant 1.$$

Integrating with respect to $\rho$ in $(0, L)$ and setting $t = V(\rho)$, we obtain

$$\int_0^\infty \left( V'(L)^2 + \frac{2c}{2+\theta} t^{2+\theta} \right)^{-1/2} dt \leqslant L. \tag{3.6}$$

We take $t = ((2+\theta)V'(L)^2/2c)^{\frac{1}{2+\theta}} \sigma$ and denote

$$I(\theta) = \int_0^\infty \frac{d\sigma}{\sqrt{1+\sigma^{2+\theta}}}.$$

Then, it follows from (3.6) that

$$-V'(L) \geqslant \left( \frac{2+\theta}{2c} \right)^{\frac{1}{\theta}} \left( \frac{1}{L} I(\theta) \right)^{\frac{2+\theta}{\theta}}. \tag{3.7}$$

On the other hand, if we perform in the integral defining $I$ the change of variable $1 + \sigma^{2+\theta} = t^{-1}$, we obtain

$$I(\theta) = \frac{1}{2+\theta} \int_0^1 (1-t)^{\frac{1}{2+\theta}-1} t^{-\frac{1}{2}-\frac{1}{2+\theta}} \, dt$$
$$= \frac{1}{2+\theta} B\left(\frac{1}{2+\theta}, \frac{1}{2} - \frac{1}{2+\theta}\right) = \frac{1}{2+\theta} \frac{\Gamma(\frac{1}{2+\theta})\Gamma(\frac{1}{2} - \frac{1}{2+\theta})}{\Gamma(\frac{1}{2})},$$

where $B$ and $\Gamma$ stand for Euler Beta and Gamma functions, respectively. Since $\Gamma(z) \sim 1/z$ as $z \to 0$, it follows that $I(\theta) \sim 2/\theta$ as $\theta \to 0$, and hence $I(\theta) \geqslant 1/\theta$ for small $\theta$. This implies, thanks to (3.7), that

$$-V'(L) \geqslant \left(\frac{2+\theta}{2c}\right)^{\frac{1}{\theta}} \left(\frac{1}{L\theta}\right)^{\frac{2+\theta}{\theta}},$$

and then (3.5) gives

$$\log V(\rho) \geqslant \frac{1}{\theta} \log\left(\frac{2+\theta}{2c}\right) + \frac{2+\theta}{\theta} \log\left(\frac{1}{L\theta}\right) + \log(L-\rho).$$

Going back to the original variables, we arrive at

$$\log U(y) \geqslant \frac{1}{\theta} \log\left(\frac{2+\theta}{2c}\right) + \frac{2+\theta}{\theta} \log\left(\frac{1}{L\theta}\right) + H\big(|y|\big),$$

where $H$ is a function which does not depend on $\theta$. Taking into account that $w(y) \geqslant U(y)$, and setting $y = -2\nu(\bar{x})$, we get

$$\log\big(d(x)^{\frac{2}{\theta}} u(x)\big) \geqslant \frac{1}{\theta} \log\left(\frac{2+\theta}{2c}\right) + \frac{2+\theta}{\theta} \log\left(\frac{1}{L\theta}\right) + H(2),$$

and then, since $\theta = (Q+\varepsilon)(1+\tau)^\gamma d(x)$, it follows that

$$\liminf_{x \to x_0} \frac{d(x)^\gamma \log u(x)}{-\log d(x)} \geqslant \frac{2\gamma+2}{(Q+\varepsilon)(1+\tau)^\gamma}.$$

Finally, letting $\varepsilon \to 0$ and $\tau \to 0$ we have

$$\liminf_{x \to x_0} \frac{d(x)^\gamma \log u(x)}{-\log d(x)} \geqslant \frac{2\gamma+2}{Q}$$

which together with (3.4) proves (1.7). $\square$

## 4. Uniqueness

This section will be devoted to obtain the uniqueness results Theorems 3 and 5. We begin with the case in which $q \geqslant 1$ in $\Omega$ and $q > 1$ on $\partial\Omega$.

**Proof of Theorem 3.** Let $u$, $v$ be positive solutions to (1.3). Since $q > 1$ on $\partial\Omega$, we have, thanks to Theorem 4, that

$$\lim_{x \to x_0} \frac{u(x)}{v(x)} = 1$$

for every $x_0 \in \partial\Omega$. By the compactness of $\overline{\Omega}$, this limit holds uniformly, and so for small enough $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1-\varepsilon)v \leqslant u \leqslant (1+\varepsilon)v$$

for all $x \in \Omega$ such that $d(x) \leqslant \delta$. Consider the problem

$$\begin{cases} \Delta z = z^{q(x)} & \text{in } \Omega_\delta, \\ z = u & \text{on } \partial\Omega_\delta, \end{cases} \tag{4.1}$$

with $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$. Problem (4.1) has a unique positive solution, which is precisely $u$. Now it can be checked that $(1-\varepsilon)v$ and $(1+\varepsilon)v$ are a sub and a supersolution respectively to (4.1), since $q \geqslant 1$ in $\Omega$. It follows from the uniqueness of $u$ that $(1-\varepsilon)v \leqslant u \leqslant (1+\varepsilon)v$ in $\Omega_\delta$. Thus this inequality is valid throughout $\Omega$, and letting $\varepsilon \to 0$ we arrive at $u = v$, which shows the desired result. $\square$

Now we consider the case where $q$ may be less or equal than one somewhere in $\Omega$, but it is strictly greater than 3 on $\partial\Omega$ and smooth in a neighborhood of $\partial\Omega$.

**Proof of Theorem 5.** We first show that (1.5) holds for all $\beta \in (0, \frac{q_0-3}{q_0-1})$. For this aim we construct sub and supersolutions near the boundary. We claim that for $\beta \in (0, \frac{q_0-3}{q_0-1})$ and large enough $B$, the function

$$\bar{u} = A(x)d(x)^{-\alpha(x)} + Bd(x)^\beta$$

is a supersolution in $\mathcal{U}_\rho := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}$ if $\delta > 0$ is small enough, where

$$A = \big(\alpha(\alpha+1)\big)^{\frac{1}{q-1}}$$

and

$$\alpha = 2/(q-1).$$

We choose $\delta$ small to have $d \in C^2(\mathcal{U}_\delta)$ and $q > 3$ in $\mathcal{U}_\delta$. Notice that in the present situation $\alpha, A \in C^2(\mathcal{U}_\delta)$. Thus a direct computation gives:

$$\begin{aligned}\Delta\bar{u} &= d^{-\alpha}\Delta A - 2\alpha d^{-\alpha-1}\nabla A\nabla d - 2d^{-\alpha}\log d\nabla A\nabla\alpha - 2Ad^{-\alpha-1}\nabla\alpha\nabla d + A\alpha(\alpha+1)d^{-\alpha-2}\\ &\quad + 2A\alpha d^{-\alpha-1}\log d\nabla\alpha\nabla d - A\alpha d^{-\alpha-1}\Delta d - Ad^{-\alpha}\log d\Delta\alpha + Ad^{-\alpha}(\log d)^2|\nabla\alpha|^2\\ &\quad + B\beta(\beta-1)d^{\beta-2} + B\beta d^{\beta-1}\Delta d,\end{aligned}$$

where the fact that the distance $d(x)$ verifies $|\nabla d| = 1$ in $\mathcal{U}_\delta$ has been used. Some further computations show that $\bar{u}$ is a supersolution provided that the following inequality holds,

$$\begin{aligned}&d^2\Delta A - 2\alpha d\nabla A\nabla d - 2d^2\log d\nabla A\nabla\alpha - 2Ad\nabla\alpha\nabla d + 2A\alpha d\log d\nabla\alpha\nabla d - A\alpha d\Delta d - Ad^2\log d\Delta\alpha\\ &\quad + Ad^2(\log d)^2|\nabla\alpha|^2 + B\beta(\beta-1)d^{\alpha+\beta} + B\beta d^{\alpha+\beta+1}\Delta d\\ &\quad \leqslant \big(A + Bd^{\alpha+\beta}\big)^{q(x)} - A^{q(x)},\end{aligned}$$

where we have used that $A^{q(x)-1} = \alpha(\alpha+1)$.

On the other hand, we also have by convexity that $(x+y)^q \geqslant x^q + qx^{q-1}y$ for all real positive numbers $x$, $y$ and thus $\bar{u}$ will be a supersolution if

$$\begin{aligned}&d^2\Delta A - 2\alpha d\nabla A\nabla d - 2d^2\log d\nabla A\nabla\alpha - 2Ad\nabla\alpha\nabla d + 2A\alpha d\log d\nabla\alpha\nabla d - A\alpha d\Delta d\\ &\quad - Ad^2\log d\Delta\alpha + Ad^2(\log d)^2|\nabla\alpha|^2 + B\beta(\beta-1)d^{\alpha+\beta} + B\beta d^{\alpha+\beta+1}\Delta d\\ &\quad \leqslant qBA^{q-1}d^{\alpha+\beta}.\end{aligned} \tag{4.2}$$

Now, since $0 < \beta < \frac{q_0-3}{q_0-1}$, we have $0 < \beta < 1 - \alpha$ on $\partial\Omega$, so that we can diminish $\delta$ further to have this inequality in $\mathcal{U}_\delta$. Thus (4.2) can be written as

$$-B\beta d^{\alpha+\beta}\big((1-\beta) - d\Delta d\big) + \mathrm{o}\big(d^{\alpha+\beta}\big) \leqslant qA^{q-1}Bd^{\alpha+\beta} \tag{4.3}$$

in $\mathcal{U}_\delta$, where the $o$-term does not depend on $B$. Notice that the first term in the left-hand side of (4.3) is positive for small $\delta$, and thus if $B > 1$ (4.3) is implied by the inequality

$$-\beta\big((1-\beta) - d\Delta d\big) + \mathrm{o}\big(d^{\alpha+\beta}\big) \leqslant qA^{q-1}$$

in $\mathcal{U}_\delta$, which can be achieved by taking $\delta$ smaller if necessary, since $\beta < 1$. The election of $\delta$ is thus independent of $B$ as long as $B > 1$, and we have shown that $\bar{u}$ is a supersolution in $\mathcal{U}_\delta$ if $B > 1$.

Analogously it can be proved that

$$w = A(x)d(x)^{-\alpha(x)} - Bd(x)^{\beta}$$

is a subsolution in the subset of $\mathcal{U}_\delta$ where it is positive, and hence $\underline{u} = \max\{w, 0\}$ is a subsolution in the whole $\Omega$.

Now let $u$ be any solution to (1.3). We choose $B$ large so that $\underline{u} \leqslant u \leqslant \bar{u}$ in $d = \delta$, and then it follows by comparison and Theorem 3 that $\underline{u} \leqslant u \leqslant \bar{u}$ in $\mathcal{U}_\delta$. This proves (1.5).

Finally, we show uniqueness. Let $u$, $v$ be solutions to (1.3). Then, according to (1.5):

$$u(x) - v(x) = \mathrm{O}\big(d(x)^{\beta}\big)$$

for every $\beta \in (0, \frac{q_0-3}{q_0-1})$. Let $\Omega_0 = \{x \in \Omega\colon\ u(x) < v(x)\}$. If $\Omega_0 \neq \emptyset$, we would have $u = v$ on $\partial\Omega_0$ (since $u - v = 0$ on $\partial\Omega$) and $\Delta u \leqslant \Delta v$ in $\Omega_0$. The maximum principle would imply $u > v$ in $\Omega_0$, which is impossible. Thus $\Omega_0 = \emptyset$, that is, $u \leqslant v$. The symmetric argument gives $u = v$, and uniqueness is shown. This concludes the proof. □

## 5. Dead core formation

In this final section we analyze the existence of dead cores for problem (1.3).

**Proof of Theorem 6.** The proof rests on the construction of a suitable weak supersolution to (1.3). For this aim, consider $\widetilde{\Omega} = \{x \in \Omega\colon\ d(x) > \delta\}$ for a fixed small $\delta$ (we only require on $\delta$ that $q > 1$ in a neighborhood of $\partial\widetilde{\Omega}$). Set $\widetilde{\Omega}_\lambda = \lambda\widetilde{\Omega}$. We will construct a supersolution $u = \bar{u}_\lambda \in C(\widetilde{\Omega}_\lambda) \cap H^1_{\mathrm{loc}}(\widetilde{\Omega}_\lambda)$ exhibiting the following features: $\bar{u}_\lambda = \infty$ on $\partial\widetilde{\Omega}_\lambda$, $\bar{u}_\lambda$ possesses a dead core $\mathcal{O}'_\lambda$ which uniformly fills $Q_\lambda$ as $\lambda \to \infty$. Thus, once $\bar{u}_\lambda$ has been obtained, we will obtain by comparison that $u \leqslant \bar{u}_\lambda$ in $\widetilde{\Omega}_\lambda$ for every positive solution $u$ to (1.3), since $u < +\infty$ on $\partial\widetilde{\Omega}_\lambda$ while $\bar{u}_\lambda = +\infty$ on $\partial\widetilde{\Omega}_\lambda$, and the assertions of the theorem will follow.

The supersolution $\bar{u}_\lambda$ will be constructed separately in the sets $Q_\lambda$ and $\widetilde{\Omega}_\lambda \setminus \overline{Q}_\lambda$. Let us proceed first in $Q_\lambda$ and for a fixed number $m_0 > 0$ let $u = \tilde{u}_\lambda \in C^{2,\eta}(Q_\lambda) \cap C(\overline{Q}_\lambda)$ be the solution to

$$\begin{cases} \Delta u = u^{q_\lambda(x)} & \text{in } Q_\lambda,\\ u = m_0 & \text{on } \partial Q_\lambda. \end{cases}$$

Then, $v = v_\lambda(x) \in C^{2,\eta}(Q) \cap C(\overline{Q})$ defined as

$$v_\lambda(x) = \tilde{u}_\lambda(\lambda x),$$

solves

$$\begin{cases} \Delta v = \lambda^2 v^{q(x)} & \text{in } Q,\\ v = m_0 & \text{on } \partial Q. \end{cases}$$

For large $\lambda$, $v_\lambda$ develops a dead core $\widetilde{\mathcal{O}}_\lambda = \{x \in Q\colon\ v_\lambda(x) = 0\}$ such that $\widetilde{\mathcal{O}}_\lambda \supset \{x \in Q\colon\ \mathrm{dist}(x, \partial Q) \geqslant d(\lambda)\}$ with $d(\lambda) \to 0$ as $\lambda \to \infty$. In fact, choose $d_0 > 0$ small and set $Q_{d_0} = \{x \in Q\colon\ \mathrm{dist}(x, \partial Q) > d_0\}$. Then $\partial Q_{d_0}$ can be covered with a finite number of balls $B \subset \overline{B} \subset Q$ with the same radius, $d_0/2$. In each of such balls $B$ consider the auxiliary problem

$$\begin{cases} \Delta w = \lambda^2 f(w), & x \in B,\\ w = m_0, & x \in \partial B, \end{cases} \tag{5.1}$$

where for $u \geqslant 0$, $f(u) = \min\{u^{q_0}, u^{q_1}\}$, $0 < q_0 \leqslant q_1 < 1$ being, respectively, the minimum and the maximum of $q$ extended to the region $\{x \in Q\colon\ \mathrm{dist}(x, \partial Q_{d_0}) \leqslant d_0/2\}$.

Due to the fact that $v_\lambda$ is subharmonic in $Q$ we can assert that

$$v_\lambda(x) \leqslant w_\lambda(x), \quad x \in B,$$

where $w_\lambda$ stands for the unique positive solution to (5.1). We now claim the existence of a critical $\lambda_c$, only depending on $d_0$, such that for $\lambda \geqslant \lambda_c$ there exists a positive $r(\lambda)$ so that $w_\lambda = 0$ in the ball with the same center as $B$ and radius $r(\lambda)$. From this fact and the subharmonicity of $v_\lambda$ it follows that $v_\lambda = 0$ in $Q_{d_0}$ for all $\lambda \geqslant \lambda_c$. Therefore, the dead cores properties stated above hold true (both for $v_\lambda$ and $\tilde{u}_\lambda$).

Let us now proceed with the construction of $\bar{u}_\lambda$ in $\widetilde{\Omega}_\lambda \setminus Q_\lambda$ and to this goal set $u = \hat{u}_\lambda \in C^{2,\eta}((\widetilde{\Omega}_\lambda \setminus Q_\lambda) \cup \partial Q_\lambda)$ the unique solution to

$$\begin{cases} \Delta u = u^{q_\lambda(x)}, & x \in \widetilde{\Omega}_\lambda \setminus Q_\lambda, \\ u = \infty, & x \in \partial\widetilde{\Omega}_\lambda, \\ u = m_0, & x \in \partial Q_\lambda. \end{cases} \tag{5.2}$$

Define:

$$\bar{u}_\lambda(x) = \begin{cases} \tilde{u}_\lambda(x), & x \in \overline{Q}_\lambda, \\ \hat{u}_\lambda(x), & x \in \widetilde{\Omega}_\lambda \setminus Q_\lambda. \end{cases}$$

Then $\bar{u}_\lambda$ is a weak supersolution to (1.3) provided $m_0 > 0$ is large enough. In fact it is enough to show that

$$\frac{\partial \hat{u}_\lambda}{\partial \nu} > 0 \tag{5.3}$$

on $\partial Q_\lambda$ where $\nu$ stands for the outer unit normal to $\widetilde{\Omega}_\lambda \setminus Q_\lambda$ on the component $\partial Q_\lambda$ of its boundary. To prove that fact, let

$$W = \{x \in \widetilde{\Omega}_\lambda \setminus Q_\lambda\colon \operatorname{dist}(x, \partial Q_\lambda) < d_1\}$$

for certain small positive $d_1$. Observe that $\hat{u}_\lambda \to \hat{u}$ in $C^{2,\eta}(\widetilde{\Omega}_\lambda \setminus Q_\lambda)$ as $m_0 \to \infty$ where $\hat{u}$ is the minimal solution to (5.2) with $m_0 = \infty$. This means that $\hat{u}_\lambda$ remains finite on $\partial W \setminus \partial Q_\lambda$ while it increases with $m_0$ on $\partial Q_\lambda$. By subharmonicity, $\hat{u}_\lambda < m_0$ in $W$ provided $m_0$ is large and (5.3) follows from the maximum principle. This finishes the construction of the supersolution $\bar{u}_\lambda$ with the desired properties.

To complete the proof let us next show the claim concerning problem (5.1). Consider the normalized case of the ball $B = B(0, R)$. In order to demonstrate the dead core features of (5.1) it is enough to handle the slight variation of the problem in the annulus $\mathcal{A} = \{x\colon \varepsilon_0 < |x| < R\} \subset B$, $0 < \varepsilon_0 < R$, consisting in setting $w = 0$ on $|x| = \varepsilon_0$. By radial symmetry such problem in $\mathcal{A}$ can be written as

$$\begin{cases} w'' = \lambda^2 g(\rho) f(w), & 0 < \rho < L, \\ w(0) = 0, \\ w(L) = m_0, \end{cases} \tag{5.4}$$

where for $\varepsilon_0 < r < R$ the suitable modification of the change of variables $\rho = \rho(r)$ introduced in the proof of Theorem 7 has been performed. As in that proof, $g(\rho) = r^{2(N-1)}$ while now

$$L = \log(R/\varepsilon_0), \quad \text{and} \quad L = \frac{1}{N}\left(\frac{1}{\varepsilon_0^N} - \frac{1}{R^N}\right),$$

in the cases $N = 2$ and $N \geqslant 3$, respectively.

In virtue of the uniqueness in the solvability of (5.4) it follows that its solution $w_\lambda$ satisfies

$$w_\lambda(\rho) \leqslant \tilde{v}_\lambda(\rho)$$

$0 < \rho < L$, where $v = \tilde{v}_\lambda(\rho)$ is the unique solution to

$$\begin{cases} v'' = \lambda^2 \varepsilon_0^{2(N-1)} f(v), & 0 < \rho < L, \\ v(0) = 0, \\ w(L) = m_0. \end{cases} \tag{5.5}$$

Thus the proof of the claim reduces to perform a dead core analysis in problem (5.5). In such case, direct integration shows that for $\lambda$ greater than some critical $\lambda_c$, $\tilde{v}_\lambda(\rho)$ vanishes in the interval $(0, \rho(\lambda))$ where $\rho(\lambda)$ is expressed as

$$\rho(\lambda) = L - \frac{1}{\varepsilon_0^{N-1}\lambda} \int_0^{m_0} \frac{ds}{\sqrt{F(s)}}, \tag{5.6}$$

where $F(u) = \int_0^u f(s)\,ds$. More precisely, $\lambda_c$ is given by the unique value of $\lambda > 0$ which makes zero the difference in (5.6). This concludes the proof of Theorem 6. $\square$

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