Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 25 (2008) 1221–1251

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

High-electric-field limit for the Vlasov–Maxwell–Fokker–Planck system

Mihai Bostan [a], Thierry Goudon [b,c,*]

[a] *Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16, route de Gray, 25030 Besançon cedex, France*
[b] *Project-Team SIMPAF, INRIA Lille Nord Europe Research Centre, Park Plazza, 40, avenue Halley, F-59650 Villeneuve d'Ascq cedex, France*
[c] *Laboratoire Paul Painlevé, UMR CNRS 8524-USTL, France*

Received 19 September 2005; accepted 28 July 2008

Available online 9 September 2008

Abstract

In this paper we derive the high-electric-field limit of the three-dimensional Vlasov–Maxwell–Fokker–Planck system. We use the relative entropy method which requires the smoothness of the solution of the limit problem. We obtain convergences of the electro-magnetic field, charge and current densities.

MSC: 35Q99; 35B40

Keywords: High-field limit; Vlasov–Maxwell–Fokker–Planck system; Relative entropy

1. Introduction

We consider a plasma in which the dilute charged particles interact both through collisions and through the action of their self-consistent electro-magnetic field. Actually, we are concerned with the evolution of the negative particles which are described in terms of a distribution function in phase space while the charge and current of the positive particles are given functions. Up to a dimensional analysis (postponed to Appendix A) the evolution of the plasma is governed by the following equations

$$\varepsilon(\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon) - \big(E_\varepsilon + \alpha\varepsilon(v\wedge B_\varepsilon)\big)\cdot\nabla_v f_\varepsilon = \mathrm{div}_v(vf_\varepsilon + \nabla_v f_\varepsilon), \tag{1}$$

for $(t,x,v)\in]0,T[\times\mathbb{R}^3\times\mathbb{R}^3$ and

$$\partial_t E_\varepsilon - \mathrm{curl}_x B_\varepsilon = -(J - j_\varepsilon), \tag{2}$$

$$\alpha\varepsilon\partial_t B_\varepsilon + \mathrm{curl}_x E_\varepsilon = 0, \tag{3}$$

$$\mathrm{div}_x E_\varepsilon = D(t,x) - \rho_\varepsilon(t,x) \quad \text{and} \quad \mathrm{div}_x B_\varepsilon = 0, \tag{4}$$

* Corresponding author.
E-mail addresses: mbostan@univ-fcomte.fr (M. Bostan), Thierry.Goudon@inria.fr (T. Goudon).

0294-1449/$ – see front matter © 2008 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.anihpc.2008.07.004

for $(t,x) \in]0,T[\times \mathbb{R}^3$ and where we have set

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,dv, \qquad j_\varepsilon(t,x) = \int_{\mathbb{R}^3} v f_\varepsilon(t,x,v)\,dv.$$

The system (1)–(4) is referred to as the Vlasov–Maxwell–Fokker–Planck (VMFP) system. Here $f_\varepsilon(t,x,v) \geqslant 0$ is the distribution function of the negative particles, $E_\varepsilon, B_\varepsilon$ stand for the electric and magnetic fields respectively while $D(t,x)$, $J(t,x)$ are the (given) charge and current densities of positive particles. They are supposed to satisfy the conservation law

$$\partial_t D + \operatorname{div}_x J = 0, \quad (t,x) \in]0,T[\times \mathbb{R}^3. \tag{5}$$

The system is completed by prescribing initial conditions for the distribution function f_ε and the electro-magnetic field $(E_\varepsilon, B_\varepsilon)$

$$f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{6}$$

$$E_\varepsilon(0,x) = E_\varepsilon^0(x), \qquad B_\varepsilon(0,x) = B_\varepsilon^0(x), \quad x \in \mathbb{R}^3. \tag{7}$$

We suppose that initially the plasma is globally neutral, i.e.,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0(x,v)\,dv\,dx = \int_{\mathbb{R}^3} D(0,x)\,dx, \tag{8}$$

and also that the initial conditions satisfy

$$\operatorname{div}_x E_\varepsilon^0 = D(0,x) - \int_{\mathbb{R}^3} f_\varepsilon^0(x,v)\,dv, \quad \operatorname{div}_x B_\varepsilon^0 = 0, \quad x \in \mathbb{R}^3. \tag{9}$$

After integration of (1) with respect to $v \in \mathbb{R}^3$ we deduce that the charge density ρ_ε and the current density j_ε of the negative particles verify the conservation law

$$\partial_t \rho_\varepsilon + \operatorname{div}_x j_\varepsilon = 0, \quad (t,x) \in]0,T[\times \mathbb{R}^3. \tag{10}$$

By using (5), (10) and by taking the divergence with respect to x of Eqs. (2), (3) we deduce that (4) are consequences of (9).

The problem is motivated from plasma physics, as for instance in the theory of semiconductors, the evolution of laser-produced plasmas or the description of tokamaks. The coupling between the kinetic equation (1) and the Maxwell system (2)–(4) describes how the local concentration and movements of charges create electric fields and currents which, in turn, influence the motion of the electrons in the whole domain. The Fokker–Planck operator in the right-hand side of (1) accounts for the collisions of the electrons with the background. These collisions produce both a friction and a diffusion effect; we refer to [18] for the introduction of such an operator based on the principles of Brownian motion and to e.g. [6] for specific applications to plasma physics. The dimensional analysis is detailed in Appendix A. Let us only say that the dimensionless parameter $\varepsilon = (\frac{l}{\Lambda})^2$ is the square of the ratio between the mean free path and the Debye length and $\alpha = (\frac{\Lambda}{\tau \cdot c_0})^2$ is the square of the ratio between the Debye length and the distance traveled by the light during the relaxation time due to collisions. We are interested in the asymptotic regime

$$0 < \varepsilon \ll 1, \quad \alpha \text{ bounded.}$$

(The parameter α might depend on ε in our analysis and tend either to 0 or a positive constant.) It can be convenient to detail this regime by means of the characteristic time scales of the evolution of the plasma: $T_c =$ 1/cyclotronic frequency, $T_p =$ 1/plasma frequency (definitions are recalled in Appendix A) and τ the relaxation time associated to the collisions which have to be compared to the time scale of light propagation T_0 and the time scale of observation T. Then, the asymptotic regime we are interested in means that

$$\tau \ll T_p \ll T$$

while the other time scales are governed by the behavior of $\alpha = T/T_c = \frac{1}{\varepsilon}(T_0/T)^2$. This kind of asymptotic problem is crucial for applications such as the modeling of Inertial Confinement Fusion devices or in some delimited regions of tokamaks where there is a strong interplay between the collisional effects and the electro-magnetic effects, see for instance [23,30].

The mathematical difficulty is related to the nonlinear term $E_\varepsilon \cdot \nabla_v f_\varepsilon$ which appears in (1) with the same order of magnitude than the diffusion Fokker–Planck term (it is due to the hypothesis that the mean free path l is much smaller than the Debye length Λ). We call this asymptotic regime for $\varepsilon \searrow 0$ the high-electric-field limit. Now, note that the Fokker–Planck operator can be written as follows

$$L_{\mathrm{FP}}(f) := \mathrm{div}_v(vf + \nabla_v f) = \mathrm{div}_v\big(e^{-\frac{|v|^2}{2}} \nabla_v\big(f e^{\frac{|v|^2}{2}}\big)\big), \tag{11}$$

and therefore the kinetic equation (1) becomes

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - \alpha(v \wedge B_\varepsilon) \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} \mathrm{div}_v\big(e^{-\frac{|v+E_\varepsilon(t,x)|^2}{2}} \nabla_v\big(f_\varepsilon e^{\frac{|v+E_\varepsilon(t,x)|^2}{2}}\big)\big). \tag{12}$$

From (12) we can expect that when $\varepsilon \searrow 0$, the distribution function f_ε converges to

$$f_\varepsilon \approx \rho(t,x) M_{E(t,x)}(v), \quad M_E(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v+E|^2}{2}}, \tag{13}$$

and therefore we can guess that

$$j_\varepsilon(t,x) = \int_{\mathbb{R}^3} v f_\varepsilon \, dv \approx -\rho(t,x) E(t,x).$$

Using the charge conservation law (10) together with (2)–(4), we are thus formally led to the following limit system

$$\begin{cases} \partial_t \rho - \mathrm{div}_x(\rho E) = 0, & & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \mathrm{div}_x E = D(t,x) - \rho(t,x), & \mathrm{curl}_x E = 0, & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \partial_t E - \mathrm{curl}_x B = -J(t,x) - \rho(t,x)E(t,x), & \mathrm{div}_x B = 0, & (t,x) \in]0,T[\times \mathbb{R}^3. \end{cases} \tag{14}$$

We wish to justify rigorously this asymptotic behavior.

High-field asymptotics have been first analyzed in the kinetic theory of semiconductors in [37], see also [17]. Then, further extensions and mathematical results for different physical models have been obtained in [1], with a discussion based on numerical simulations, [5] for a derivation of so-called SHE models for charge transport in semiconductors and [21] for a derivation of energy-transport models, [34] for application to quantum hydrodynamics model.... The problem combines the difficulty of hydrodynamic regimes with the treatment of the nonlinear acceleration term $E_\varepsilon \cdot \nabla_v f_\varepsilon$. The problem slightly simplifies in the electrostatic case where the electric field is simply defined through the Poisson equation (complete (1) by $E_\varepsilon = -\nabla_x \Phi_\varepsilon$, $\Delta_x \Phi_\varepsilon = \rho_\varepsilon - D$ and $B_\varepsilon = 0$). This actually means that the electric field E_ε is defined by a convolution with $\rho_\varepsilon - D$. The resulting Vlasov–Poisson–Fokker–Planck (VPFP) system can be seen, at least formally, as an asymptotic limit of the VMFP model in a physical regime where the light speed is large compared to the thermal velocity, the other physical parameters being fixed. The high-field limit of the VPFP system can be addressed by appealing to usual compactness methods; however, constraints on the space dimension appear, due to the singularity of the convolution kernel. It turns out that the strategy works in dimension 1 [35] and dimension 2 [26]. Another approach uses relative entropy (or modulated energy) methods, as introduced in [44]. With such an approach, we try to evaluate how far the solution is from the expected limit. This method has been used to treat various asymptotic questions in colisionless plasma physics, in particular the derivation of quasineutral regimes [13,14,25,39], and for hydrodynamic limits in gas dynamics [40,7], or for fluid-particles interaction models [28].... Further references and examples of applications of the method can be found with many deep comments in the review [41]. Concerning the VPFP system, it allows to justify the L^2 strong convergence for the electric field and we can pass to the limit for any space dimension [26]. However, this method requires some smoothness on the solutions of the limit system. Eventually, we point out that a low-field regime, where diffusion dominates the transport terms, can also be considered: for the VPFP system, we refer to [38,27] and for an attempt with the VMFP system to [8].

The aim of this paper is therefore to analyze the high-electric-field limit of the three-dimensional VMFP system by using the relative entropy method. This extension is interesting both from the viewpoint of physics: we are dealing with a more realistic and complete model; and those of mathematics: replacing the Poisson equation by the Maxwell system we cannot expect too much regularizing effects from the coupling, and this also shows how robust the relative entropy method is. By the way, the mathematical theory of the solutions of the VMFP is far from being completely known. By contrast, the theory for VPFP is well established: existence of weak solutions can be found in [42] with refinements in [15], while for existence and uniqueness results of strong solutions we refer to [20,36] and the complete results of [9,10]. The coupling with the Maxwell equations leads to a much more difficult analysis. The collisionless case has been further investigated and global existence of classical solutions relies on the behavior of the tip of the support of the solution, as shown by different approaches in [24,11,31], while the local well-posedness of smooth solutions is due to [43]. It is also worth mentioning the recent result [16] concerning a reduced version of the Vlasov–Maxwell equation. For the VMFP model but neglecting the friction forces, the global existence of renormalized solutions has been obtained in [22]. Recently, the global existence and uniqueness of smooth solutions have been obtained for the relativistic version of the VMFP system, in the specific one and one half dimensional framework [33]. Considering the full Vlasov–Maxwell–Boltzmann system with data close to equilibrium stunning progress appear in [29], with an extension to the Landau operator in [45]. However, it is still an open question to investigate if the Fokker–Planck operator introduces some regularizing effects which would lead to the well-posedness of the full VMFP system in a general framework or if we should definitely deal with renormalized solutions verifying the conservation laws up to defect measures. These questions are clearly beyond the scope of this paper where we focus on the asymptotic questions, considering essentially smooth solutions of the VMFP system. Our main result states as follows. We establish this result for smooth solutions but we will see that the same conclusions hold true in the framework of renormalized solutions (cf. Appendix C).

Theorem 1. *Let $\rho^0 \geqslant 0$ and $D \geqslant 0$ such that $\rho^0 \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$, with $\nabla_x \ln(\rho^0) \in L^\infty(\mathbb{R}^3)$ and $D \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap W^{1,\infty}(]0,T[\times\mathbb{R}^3)$, with $\partial_t D \in L^\infty(]0,T[; L^1(\mathbb{R}^3))$. Let $J \in L^\infty(]0,T[; L^2(\mathbb{R}^3))^3 \cap L^\infty(]0,T[; L^q(\mathbb{R}^3))^3$, with furthermore $\partial_t J \in L^\infty(]0,T[; H^{-1}(\mathbb{R}^3))^3 \cap L^\infty(]0,T[; W^{-1,q}(\mathbb{R}^3))^3$ for some $q \in]3,+\infty[$ and $\partial_t^2 J \in L^\infty(]0,T[; H^{-2}(\mathbb{R}^3))^3$ such that $\partial_t D + \operatorname{div}_x J = 0$. Consider (ρ, E, B) the unique solution of* (14) *with the initial condition ρ^0. Let $f_\varepsilon^0 \geqslant 0$, $E_\varepsilon^0, B_\varepsilon^0$ be a sequence of smooth distribution functions and electromagnetic fields verifying*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx = \int_{\mathbb{R}^3} D(0,x)\, dx, \quad \sup_{\varepsilon>0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon^0 \, dv\, dx < +\infty, \tag{15}$$

$$\lim_{\varepsilon \searrow 0} \frac{1}{\alpha\varepsilon}\left\{ \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \ln \frac{f_\varepsilon^0}{\rho M_E}\, dv\, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left|E_\varepsilon^0 - E^0\right|^2 + \alpha\varepsilon \left|B_\varepsilon^0 - B^0\right|^2 \right) dx \right\} = 0 \tag{16}$$

where E^0 is the solution of $\operatorname{div}_x E^0 = D(0,x) - \rho^0(x)$, $\operatorname{curl}_x E^0 = 0$, $x \in \mathbb{R}^3$ and $\varepsilon/\alpha \to 0$. We assume that $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ are strong solutions of the VMFP system (1)–(7). *Then $(E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ converges to (E, B) in $L^\infty(]0,T[; L^2(\mathbb{R}^3))^6$, whereas $(\rho_\varepsilon, j_\varepsilon)_{\varepsilon>0}$ converges to $(\rho, -\rho E)$ in $L^\infty(]0,T[, L^1(\mathbb{R}^3))^4$.*

Let us make a couple of comments on the results and mention a few open questions. First of all, the scaling assumption on α in Theorem 1 is maybe not the most relevant on the physical viewpoint; but our analysis covers much more general cases, as it will be detailed later on (see Theorem 15). Second, as already said the existence theory of the full VMFP system is not complete; nevertheless we assume we have at hand a sequence of solutions of the system, smooth enough to justify the manipulations below. Dealing with a more general class of solutions, involving defect measures in the macroscopic conservation laws is possible by adapting the reasoning in [39]. We will give some hints in this directions in Appendix C. The assumption on the initial data is necessary with the method we use, which also requires the smoothness of the solution of the limit equations. In some sense this assumption means that the initial state is already close to the shape of the limit, a shifted Maxwellian function. Of course it would be very interesting to design a proof involving only compactness arguments. We also completely neglect (like most of the papers on the topics which usually restrict to the whole space problem or the periodic framework) the difficulties coming from boundary conditions, that induce delicate boundary layer analysis. Finally, a very important question consists in dealing with the full system involving kinetic equations for both positive and negative particles. This leads to a tough analysis and again most of the results in the literature are not able to deal with the two species model.

The paper is organized as follows. In Section 2 we establish some a priori estimates satisfied by smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ of the three-dimensional VMFP system. In the next section, we introduce the relative entropy and calculate its time evolution. There, we also analyze the well-posedness of the limit equation. In Section 4 we detail the passage to the limit. The dimensional analysis of the equations, the physical meaning of the different parameters and the extension to the renormalized solutions are detailed in Appendix C.

2. A priori estimates

In this section we establish a priori estimates for the smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ of VMFP, uniformly with respect to $\varepsilon > 0$. These estimates are deduced from the natural conservation properties of the system and from the dissipation mechanism due to the collisions.

Proposition 2. *Let $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ be a smooth solution of the problem* (1)–(4), (6), (7) *where the initial conditions satisfy $f_\varepsilon^0 \geqslant 0$, and*

$$M_\varepsilon^0 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx < +\infty,$$

$$W_\varepsilon^0 := \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon^0 \, dv\, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|E_\varepsilon^0\right|^2 dx + \frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3} \left|B_\varepsilon^0\right|^2 dx < +\infty,$$

$$L_\varepsilon^0 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon^0 \, dv\, dx < +\infty,$$

$$H_\varepsilon^0 := \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \left|\ln f_\varepsilon^0\right| dv\, dx < +\infty.$$

We assume also that $J \in L^1(]0,T[; L^2(\mathbb{R}^3))^3$. Then, we have for any $0 < t < T < \infty$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\, dv\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx < +\infty,$$

$$\sup_{0\leqslant t\leqslant T} \left\{ \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon(t,x,v)\, dv\, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|E_\varepsilon(t,x)\right|^2 dx + \frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3} \left|B_\varepsilon(t,x)\right|^2 dx \right\} + \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \, dv\, dx\, dt$$
$$\leqslant \left(\left(2W_\varepsilon^0 + 6TM_\varepsilon^0\right)^{\frac{1}{2}} + \sqrt{2}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))} \right)^2,$$

$$\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon(t,x,v)\, dv\, dx \leqslant C_T\left(M_\varepsilon^0 + W_\varepsilon^0 + L_\varepsilon^0 + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right),$$

$$\sup_{0\leqslant t\leqslant T} \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon |\ln f_\varepsilon|(t,x,v)\, dv\, dx + \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|\nabla_v \sqrt{f_\varepsilon}\right|^2 dv\, dx\, dt$$
$$\leqslant C_T\left(\varepsilon + M_\varepsilon^0 + W_\varepsilon^0 + \varepsilon L_\varepsilon^0 + H_\varepsilon^0 + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right).$$

Remark 2.1. As said above M_ε^0 (and its evolution counterpart) stands for the total negative charge, and the result only states that it is conserved: indeed there is no production nor loss of electrons within the model. The quantity W_ε^0 collects, taking into account the scaling, the kinetic energy of the particles and the energy of the electro-magnetic fields. The quantity $\varepsilon \iint f_\varepsilon \ln f_\varepsilon \, dv\, dx$ represents the (scaled) entropy associated to the particles, and the collisions induce a dissipation of this quantity. For technical purposes, we will be interested instead in the positive quantity H_ε^0. Finally L_ε^0 can be thought of as a measure of how particles spread in space.

Before starting our computations let us state the following lemma, based on classical arguments due to Carleman.

Lemma 3. *Assume that $f = f(x, v)$ satisfies $f \geqslant 0$, $(|x| + |v|^2 + |\ln f|) f \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. Then for all $k > 0$ we have*

$$f|\ln f| \leqslant f \ln f + 2k\big(|x| + |v|^2\big) f + 2Ce^{-\frac{k}{2}(|x|+|v|^2)}, \quad \textit{with } C = \sup_{0<y<1} \big\{-\sqrt{y}\ln y\big\},$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f|\ln f|\,dv\,dx \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\ln f\,dv\,dx + 2k\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \big(|x| + |v|^2\big) f\,dv\,dx + C_k,$$

with $C_k = 2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{k}{2}(|x|+|v|^2)}\,dv\,dx$.

Proof. Since $f|\ln f| = f\ln f + 2f(\ln f)_-$, it is sufficient to estimate $f(\ln f)_-$. Take $k > 0$ and let $C = \sup_{0<y<1}\{-\sqrt{y}\ln y\} < +\infty$. We have

$$\begin{aligned} f(\ln f)_- &= -f\ln f \cdot \mathbf{1}_{\{0<f<e^{-k(|x|+|v|^2)}\}} - f\ln f \cdot \mathbf{1}_{\{e^{-k(|x|+|v|^2)}\leqslant f<1\}} \\ &\leqslant Ce^{-\frac{k}{2}(|x|+|v|^2)} + k\big(|x| + |v|^2\big) f, \quad \forall (x, v) \in \mathbb{R}^N \times \mathbb{R}^N. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\ln f)_-\,dv\,dx \leqslant k\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \big(|x| + |v|^2\big) f\,dv\,dx + C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{k}{2}(|x|+|v|^2)}\,dv\,dx,$$

and the conclusion follows easily. □

Proof of Proposition 2. Integrating (1) with respect to $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ yields

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(t, x, v)\,dv\,dx = 0, \quad t \in]0, T[,$$

which implies that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(t, x, v)\,dv\,dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0\,dv\,dx = M_\varepsilon^0, \quad t \in]0, T[. \tag{17}$$

Note that integrating (5) with respect to x implies $\frac{d}{dt}\int_{\mathbb{R}^3} D(t, x)\,dx = 0$ and therefore we deduce that if initially the plasma is globally neutral, i.e., $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0\,dv\,dx = \int_{\mathbb{R}^3} D(0, x)\,dx$, then it remains globally neutral for all $t \in]0, T[$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(t, x, v)\,dv\,dx = \int_{\mathbb{R}^3} D(t, x)\,dx.$$

Multiplying (1) by $\frac{|v|^2}{2}$ and integrating with respect to (x, v) implies

$$\varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon\,dv\,dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} E_\varepsilon \cdot v f_\varepsilon\,dv\,dx = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon\,dv\,dx + 3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\,dv\,dx. \tag{18}$$

Multiplying (2), (3) by E_ε, respectively B_ε and integrating with respect to x yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \big(|E_\varepsilon|^2 + \alpha\varepsilon|B_\varepsilon|^2\big)\,dx = -\int_{\mathbb{R}^3} E_\varepsilon \cdot (J - j_\varepsilon)\,dx. \tag{19}$$

By combining (18), (19) we obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \,dv\,dx + \frac{d}{dt}\left(\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \,dv\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(|E_\varepsilon|^2 + \alpha\varepsilon |B_\varepsilon|^2\right) dx\right)$$
$$= -\int_{\mathbb{R}^3} E_\varepsilon \cdot J\,dx + 3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \,dv\,dx,$$

and therefore we have

$$\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \,dv\,dx\,ds + \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \,dv\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(|E_\varepsilon|^2 + \alpha\varepsilon |B_\varepsilon|^2\right) dx$$
$$\leqslant \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon^0 \,dv\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left|E_\varepsilon^0\right|^2 + \alpha\varepsilon \left|B_\varepsilon^0\right|^2\right) dx$$
$$+ \int_0^t \left(\int_{\mathbb{R}^3} \left|E_\varepsilon(s,x)\right|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^3} \left|J(s,x)\right|^2 dx\right)^{\frac{1}{2}} ds + 3t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \,dv\,dx.$$

By using Bellman's lemma (see Appendix B) we obtain for all $0 \leqslant t \leqslant T$

$$\left(\int_{\mathbb{R}^3} \left|E_\varepsilon(t,x)\right|^2 dx\right)^{\frac{1}{2}} \leqslant \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(\varepsilon|v|^2 + 6T\right) f_\varepsilon^0 \,dv\,dx + \int_{\mathbb{R}^3} \left(\left|E_\varepsilon^0\right|^2 + \alpha\varepsilon\left|B_\varepsilon^0\right|^2\right) dx\right)^{\frac{1}{2}}$$
$$+ \int_0^T \left(\int_{\mathbb{R}^3} \left|J(s,x)\right|^2 dx\right)^{\frac{1}{2}} ds$$
$$= \left(2W_\varepsilon^0 + 6TM_\varepsilon^0\right)^{\frac{1}{2}} + \|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))}.$$

Finally we get

$$\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \,dv\,dx\,dt + \sup_{0\leqslant t\leqslant T}\left\{\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \,dv\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(|E_\varepsilon|^2 + \alpha\varepsilon |B_\varepsilon|^2\right) dx\right\}$$
$$\leqslant \left(\left(2W_\varepsilon^0 + 6TM_\varepsilon^0\right)^{\frac{1}{2}} + \sqrt{2}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right)^2. \tag{20}$$

We multiply now (1) by $|x|$ and we obtain after integration with respect to (x,v)

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon \,dv\,dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(v\cdot x)}{|x|} f_\varepsilon \,dv\,dx = 0.$$

We deduce that

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon \,dv\,dx \leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon^0 \,dv\,dx + \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v| f_\varepsilon \,dv\,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon^0 \,dv\,dx + \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2}\left(|v|^2 + 1\right) f_\varepsilon \,dv\,dx\,dt$$
$$\leqslant C_T\left(M_\varepsilon^0 + W_\varepsilon^0 + L_\varepsilon^0 + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right). \tag{21}$$

We multiply now (1) by $(1 + \ln f_\varepsilon)$ and after integration with respect to (x,v) we get

$$\varepsilon \frac{d}{dt} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln f_\varepsilon \, dv\, dx = - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (v f_\varepsilon + \nabla_v f_\varepsilon) \frac{\nabla_v f_\varepsilon}{f_\varepsilon} \, dv\, dx$$
$$= 3 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \, dv\, dx - 4 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|\nabla_v \sqrt{f_\varepsilon}\right|^2 dv\, dx.$$

Finally we deduce that

$$\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln f_\varepsilon \, dv\, dx + 4 \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|\nabla_v \sqrt{f_\varepsilon}\right|^2 dv\, dx\, ds = \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \ln f_\varepsilon^0 \, dv\, dx + 3t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx. \tag{22}$$

Combining (20)–(22) and Lemma 3 with $k = 1$ yields

$$\sup_{0 \leqslant t \leqslant T} \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon |\ln f_\varepsilon| \, dv\, dx + \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|\nabla_v \sqrt{f_\varepsilon}\right|^2 dv\, dx\, dt$$
$$\leqslant C_T \left(\varepsilon + M_\varepsilon^0 + W_\varepsilon^0 + \varepsilon L_\varepsilon^0 + H_\varepsilon^0 + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right). \qquad \square$$

3. The relative entropy method

In this section we introduce the relative entropy, according to the seminal works [13,44]: it will allows us to establish convergence results, uniformly on any finite time interval $[0, T]$. The proof requires some regularity properties of the limit solutions (ρ, E, B) of (14) as well as the convergence of the initial data like in particular for the electromagnetic field

$$\lim_{\varepsilon \searrow 0} \left(\int_{\mathbb{R}^3} \left|E_\varepsilon^0(x) - E(0,x)\right|^2 dx + \alpha\varepsilon \int_{\mathbb{R}^3} \left|B_\varepsilon^0(x) - B(0,x)\right|^2 dx \right) = 0.$$

We introduce the Maxwellian

$$\rho M_E(v) = \frac{\rho}{(2\pi)^{3/2}} \exp\left(-\frac{|v+E|^2}{2}\right)$$

parametrized by ρ, E so that $\rho = \int_{\mathbb{R}^3} \rho M_E \, dv$, and $\rho E = -\int_{\mathbb{R}^3} v \rho M_E \, dv$. Given two nonnegative functions f, g defined on $\mathbb{R}^3 \times \mathbb{R}^3$, we define the nonnegative quantity

$$H(f|g) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[\frac{f}{g} \ln\left(\frac{f}{g}\right) - \frac{f}{g} + 1\right] g \, dv\, dx$$

which is a way to evaluate how far f is from g. We are interested in the evolution of

$$\mathcal{H}_\varepsilon(t) = \varepsilon H(f_\varepsilon | \rho M_E) + \frac{1}{2} \int_{\mathbb{R}^3} \left(|E_\varepsilon - E|^2 + \alpha\varepsilon |B_\varepsilon - B|^2\right) dx$$

where $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon > 0}$ are smooth solutions of (1)–(7) and (ρ, E, B) is a smooth solution of (14). This quantity splits into the standard (rescaled) L^2 norm of the electro-magnetic field plus the relative entropy between the solution $f_\varepsilon(t, x, v)$ and the leading term $\rho(t, x) M_{E(t,x)}(v)$.

3.1. Analysis of the limit system

We start with the analysis of the system (14). Note that this system can be split into two problems. First solve for (ρ, E)

$$\begin{cases} \partial_t \rho - \operatorname{div}_x(\rho E) = 0, & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \operatorname{div}_x E = D(t,x) - \rho(t,x), & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \operatorname{curl}_x E = 0, & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^3, \end{cases} \tag{23}$$

and secondly find B solution of

$$\begin{cases} \partial_t E - \operatorname{curl}_x B = -J(t,x) - \rho(t,x)E(t,x), & (t,x) \in]0,T[\times \mathbb{R}^3, \\ \operatorname{div}_x B = 0, & (t,x) \in]0,T[\times \mathbb{R}^3, \end{cases} \tag{24}$$

where the charge and current densities D, J are given functions satisfying $\partial_t D + \operatorname{div}_x J = 0$. We give here an existence result for (23) which is a direct consequence of the existence result obtained in [35], see also [26].

Proposition 4. *Let $\rho^0 \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$, $D \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap W^{1,\infty}(]0,T[\times \mathbb{R}^3)$, $\partial_t D \in L^\infty(]0,T[; L^1(\mathbb{R}^3))$. Then there is a unique solution for* (23) *satisfying*

$$\rho \in W^{1,\infty}\big(]0,T[\times \mathbb{R}^3\big), \qquad E \in W^{1,\infty}\big(]0,T[\times \mathbb{R}^3\big)^3.$$

Proof. We introduce the exterior electric field E_0 given by $\operatorname{div}_x E_0 = D$, $\operatorname{curl}_x E_0 = 0$, so that $\operatorname{div}_x(E - E_0) = -\rho$, $\operatorname{curl}_x(E - E_0) = 0$. The hypotheses imply that $E_0 \in W^{1,\infty}(]0,T[\times \mathbb{R}^3)^3$. Following the arguments of Theorem 3 and Lemma 8 of [35] we deduce that there is a unique strong solution (ρ, E) for (23) verifying $\rho \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$, $D - \rho \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$, $E \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))^3$. By differentiating the first equation of (23) with respect to x we check that $\nabla_x \rho \in L^\infty(]0,T[; L^1(\mathbb{R}^3))^3$ and since $\partial_t \rho = E \cdot \nabla_x \rho + \rho(D - \rho) \in L^\infty(]0,T[; L^1(\mathbb{R}^3)) \cap L^\infty(]0,T[; L^\infty(\mathbb{R}^3))$ and $\partial_t D \in L^\infty(]0,T[; L^1(\mathbb{R}^3)) \cap L^\infty(]0,T[\times \mathbb{R}^3)$ we deduce that $\partial_t E \in L^\infty(]0,T[; W^{1,p}(\mathbb{R}^3))^3 \subset L^\infty(]0,T[; L^\infty(\mathbb{R}^3))^3$ for all $p > 3$. Finally we obtain that $\rho \in W^{1,\infty}(]0,T[\times \mathbb{R}^3)$ and $E \in W^{1,\infty}(]0,T[\times \mathbb{R}^3)^3$. In fact, since $D - \rho \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$ we have $E \in L^\infty(]0,T[; W^{2,p}(\mathbb{R}^3))^3$ for all $1 < p < +\infty$. On the other hand since $\partial_t D - \partial_t \rho \in L^\infty(]0,T[; L^1(\mathbb{R}^3)) \cap L^\infty(]0,T[\times \mathbb{R}^3)$ we have $\partial_t E \in L^\infty(]0,T[; W^{1,p}(\mathbb{R}^3))^3$ for all $1 < p < +\infty$. In particular we obtain that $E, \partial_t E \in L^\infty(]0,T[; L^2(\mathbb{R}^3))^3$.

Having disposed of this existence result, we can also show by looking at the system satisfied by the $\partial_{x_i} \ln(\rho)$'s that these quantities are bounded on $(0,T) \times \mathbb{R}^3$ when $\nabla_x \rho^0$ belongs to $L^\infty(\mathbb{R}^3)^3$. □

Once we find (ρ, E) it is easy to solve (24).

Proposition 5. *Under the hypotheses of Proposition* 4 *assume also that*

$$J \in L^\infty\big(]0,T[; L^2\big(\mathbb{R}^3\big)\big)^3 \cap L^\infty\big(]0,T[; L^q\big(\mathbb{R}^3\big)\big)^3,$$
$$\partial_t J \in L^\infty\big(]0,T[; H^{-1}\big(\mathbb{R}^3\big)\big)^3 \cap L^\infty\big(]0,T[; W^{-1,q}\big(\mathbb{R}^3\big)\big)^3,$$
$$\partial_t^2 J \in L^\infty\big(]0,T[; H^{-2}\big(\mathbb{R}^3\big)\big)^3,$$

for some $q \in]3,+\infty[$ and $\partial_t D + \operatorname{div}_x J = 0$ in $\mathcal{D}'(]0,T[\times \mathbb{R}^3)$. Then there is a unique solution B for (24) *verifying $B \in L^\infty(]0,T[; H^1(\mathbb{R}^3))^3 \cap L^\infty(]0,T[; W^{1,q}(\mathbb{R}^3))^3$, $\partial_t B \in L^\infty(]0,T[; L^2(\mathbb{R}^3))^3 \cap L^\infty(]0,T[; L^q(\mathbb{R}^3))^3$, $\partial_t^2 B \in L^\infty(]0,T[; H^{-1}(\mathbb{R}^3))^3$. In particular $B \in L^\infty(]0,T[\times \mathbb{R}^3)^3$.*

Proof. Observe that we have $\operatorname{div}_x(\partial_t E + \rho E + J) = 0$ and that $\partial_t E + \rho E + J \in L^\infty(]0,T[; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3))^3$. Therefore there is a unique $B \in L^\infty(]0,T[; H^1(\mathbb{R}^3) \cap W^{1,q}(\mathbb{R}^3))^3$ such that $\partial_t E + \rho E + J = \operatorname{curl}_x B$, $\operatorname{div}_x B = 0$. In order to estimate $\partial_t B$ in $L^\infty(]0,T[; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3))^3$ it is sufficient to estimate $\partial_t(\partial_t E + \rho E + J)$ in $L^\infty(]0,T[; H^{-1}(\mathbb{R}^3) \cap W^{-1,q}(\mathbb{R}^3))^3$. We have

$$\operatorname{div}_x\big(\partial_t^2 E\big) = \partial_t^2(D - \rho) = \partial_t\big(-\operatorname{div}_x J - \operatorname{div}_x(\rho E)\big),$$

and thus

$$\begin{aligned} \big\|\partial_t^2 E\big\|_{L^\infty(]0,T[; H^{-1}(\mathbb{R}^3))} &\leqslant C\big\|\partial_t(J + \rho E)\big\|_{L^\infty(]0,T[; H^{-1}(\mathbb{R}^3))} \\ &\leqslant C\big\{\|\partial_t J\|_{L^\infty(]0,T[; H^{-1}(\mathbb{R}^3))} + \|\rho\|_{L^\infty} \cdot \|\partial_t E\|_{L^\infty(]0,T[; L^2(\mathbb{R}^3))} \\ &\quad + \|\partial_t \rho\|_{L^\infty} \cdot \|E\|_{L^\infty(]0,T[; L^2(\mathbb{R}^3))}\big\}. \end{aligned}$$

By the previous proof we already know that $E \in L^\infty(]0,T[; W^{2,p}(\mathbb{R}^3))^3$, $\partial_t E \in L^\infty(]0,T[; W^{1,p}(\mathbb{R}^3))^3$ for all $1 < p < +\infty$ and we obtain similarly

$$\begin{aligned}\|\partial_t^2 E\|_{L^\infty(]0,T[;W^{-1,q}(\mathbb{R}^3))} &\leqslant C\|\partial_t(J+\rho E)\|_{L^\infty(]0,T[;W^{-1,q}(\mathbb{R}^3))}\\ &\leqslant C\{\|\partial_t J\|_{L^\infty(]0,T[;W^{-1,q}(\mathbb{R}^3))} + \|\rho\|_{L^\infty}\|\partial_t E\|_{L^\infty(]0,T[;L^q(\mathbb{R}^3))}\\ &\quad + \|\partial_t \rho\|_{L^\infty}\|E\|_{L^\infty(]0,T[;L^q(\mathbb{R}^3))}\}.\end{aligned}$$

It remains to estimate $\partial_t^2 B$ in $L^\infty(]0,T[;H^{-1}(\mathbb{R}^3))^3$. As before we have

$$\begin{aligned}\|\partial_t^3 E\|_{L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))} &\leqslant C\|\partial_t^2(J+\rho E)\|_{L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))}\\ &\leqslant C\{\|\partial_t^2 J\|_{L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))} + \|\rho\|_{W^{1,\infty}}\|\partial_t^2 E\|_{L^\infty(]0,T[;H^{-1}(\mathbb{R}^3))}\\ &\quad + \|\partial_t \rho\|_{L^\infty}\|\partial_t E\|_{L^\infty(]0,T[;L^2(\mathbb{R}^3))} + \|\partial_t^2 \rho E\|_{L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))}\}.\end{aligned}$$

And we are done if we bound the norm of $\partial_t^2 \rho E$ in $L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))^3$. By the continuity equation we have

$$\partial_t^2 D + \mathrm{div}_x\, \partial_t J = 0,$$

implying that $\partial_t^2 D \in L^\infty(]0,T[;W^{-2,q}(\mathbb{R}^3))$. Therefore we deduce that

$$\partial_t^2 \rho = \partial_t^2 D - \mathrm{div}_x\, \partial_t^2 E \in L^\infty(]0,T[;W^{-2,q}(\mathbb{R}^3)).$$

And finally taking p_0 such that $1/q' = 1/p_0 + 1/2$ with $1/q' + 1/q = 1$ and by observing that

$$\|E(t)\varphi\|_{W^{2,q'}(\mathbb{R}^3)} \leqslant C\|E(t)\|_{W^{2,p_0}(\mathbb{R}^3)}\|\varphi\|_{H^2(\mathbb{R}^3)},$$

we obtain

$$\|\partial_t^2 \rho E\|_{L^\infty(]0,T[;H^{-2}(\mathbb{R}^3))} \leqslant C\|\partial_t^2 \rho\|_{L^\infty(]0,T[;W^{-2,q}(\mathbb{R}^3))}\|E\|_{L^\infty(]0,T[;W^{2,p_0}(\mathbb{R}^3))} < +\infty. \qquad \square$$

For further computations it is worth introducing the vector potential U such that

$$B = \mathrm{curl}_x\, U \quad \text{and} \quad \mathrm{div}_x\, U = 0.$$

Since $\mathrm{curl}_x\, B = \mathrm{curl}_x\, \mathrm{curl}_x\, U = -\Delta_x U$ the vector potential U has the regularity

$$\begin{aligned}&U \in L^\infty(]0,T[;H^2(\mathbb{R}^3))^3,\\ &\partial_t U \in L^\infty(]0,T[;H^1(\mathbb{R}^3) \cap W^{1,q}(\mathbb{R}^3))^3,\\ &\partial_t^2 U \in L^\infty(]0,T[;L^2(\mathbb{R}^3))^3.\end{aligned}$$

In particular, since $q > 3$, we have $\partial_t U \in L^\infty(]0,T[\times\mathbb{R}^3)^3$.

3.2. Evolution of the relative entropy

This section is devoted to the study of the evolution of the relative entropy, deduced from

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_\varepsilon = {}&\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f_\varepsilon\left(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2}\right)dv\,dx + \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \partial_t E\cdot(v+E)\,dv\,dx\\ &- \frac{d}{dt}\left(\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln(\rho)\,dv\,dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3} |E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2\,dx\right),\end{aligned} \tag{25}$$

where we used the charge conservation (17).

Proposition 6. *Let $f_\varepsilon^0 \geqslant 0$ verify the assumptions of Proposition 2. Let $D \geqslant 0$, $D \in L^\infty(]0,T[;L^1(\mathbb{R}^3))$ and $J \in L^\infty(]0,T[;L^1(\mathbb{R}^3))^3$ verify $\partial_t D + \mathrm{div}_x\, J = 0$. Let $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ be a smooth solution of the VMFP system (1)–(4) with the initial conditions $f_\varepsilon^0, E_\varepsilon^0, B_\varepsilon^0$ satisfying (9). Assume that the solution (ρ, E, B) of (14) verifies $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, $B, \partial_t U \in L^\infty(]0,T[\times\mathbb{R}^3)^3$, $E, B, \partial_t E, \partial_t B \in L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$, $\partial_t^2 B \in L^\infty(]0,T[;H^{-1}(\mathbb{R}^3))^3$. Then the balance of the relative entropy is given by*

$$\frac{d}{dt}\left(\mathcal{H}_\varepsilon(t)+\alpha\varepsilon\int_{\mathbb{R}^3}\big((E_\varepsilon-E)\wedge(B_\varepsilon-B)\big)\cdot E\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}\partial_t U\cdot(E_\varepsilon-E)\,dx\right)+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q_\varepsilon|^2\,dv\,dx$$
$$=\int_{\mathbb{R}^3}\mathcal{A}(E_\varepsilon-E,E_\varepsilon-E)\cdot E\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}\mathcal{A}(B_\varepsilon-B,B_\varepsilon-B)\cdot E\,dx$$
$$+\alpha\varepsilon\int_{\mathbb{R}^3}\big((E_\varepsilon-E)\wedge(B_\varepsilon-B)\big)\cdot\partial_t E\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}\big\{\partial_t^2U+E\wedge\partial_t B-\nabla_x(\partial_t U\cdot E)\big\}\cdot(E_\varepsilon-E)\,dx$$
$$+\varepsilon\int_{\mathbb{R}^3}\left\{\alpha(\partial_t U+E\wedge B)+\partial_t E-(\mathrm{D}_x E)E-\frac{\nabla_x\rho}{\rho}\right\}\cdot q_\varepsilon\sqrt{f_\varepsilon}\,dx$$
$$+\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}q_\varepsilon\cdot(\mathrm{D}_x E)(v+E)\sqrt{f_\varepsilon}\,dv\,dx, \tag{26}$$

where, for a given $u:\mathbb{R}^3\to\mathbb{R}^3$, $\mathcal{A}(u,u)$ *denotes the vector* $u\,\mathrm{div}_x\,u-u\wedge\mathrm{curl}_x\,u$ *and* $q_\varepsilon=\sqrt{f_\varepsilon}(v+E)+2\nabla_v\sqrt{f_\varepsilon}$.

We wish to establish from identity (26) an estimate like

$$\mathcal{H}_\varepsilon(t)\leqslant\mathcal{H}_\varepsilon(0)+\omega(\varepsilon)+C_T\int_0^t\mathcal{H}_\varepsilon(s)\,ds$$

for any $0\leqslant t\leqslant T<+\infty$ where the constant C_T depends on T and on various bounds on the data and the solution of the limit problem while $\omega(\varepsilon)$, which also depends on $0<T<\infty$, tends to 0 as ε goes to 0. Having such an estimate implies convergence properties by a simple application of the Gronwall lemma.

To start the proof of these statements, it is convenient to rewrite (1) as follows

$$\varepsilon(\partial_t f_\varepsilon+v\cdot\nabla_x f_\varepsilon)-\mathrm{div}_v\big(f_\varepsilon(v+E)+\nabla_v f_\varepsilon\big)-\mathrm{div}_v\big\{\big(E_\varepsilon-E+\alpha\varepsilon(v\wedge B_\varepsilon)\big)f_\varepsilon\big\}=0. \tag{27}$$

Let (ρ,E,B) be a solution of (14), and let us multiply (1) by

$$1+\ln f_\varepsilon+\frac{|v+E|^2}{2}$$

so that we will recognize the first term in the right-hand side of (25). It thus makes the following quantities appear

$$Q_1(t)=\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\partial_t f_\varepsilon+v\cdot\nabla_x f_\varepsilon)\cdot\left(1+\ln f_\varepsilon+\frac{|v+E|^2}{2}\right)dv\,dx,$$
$$Q_2(t)=-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{div}_v\big(f_\varepsilon(v+E)+\nabla_v f_\varepsilon\big)\left(1+\ln f_\varepsilon+\frac{|v+E|^2}{2}\right)dv\,dx,$$
$$Q_3(t)=-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{div}_v\big(\big(E_\varepsilon-E+\alpha\varepsilon(v\wedge B_\varepsilon)\big)f_\varepsilon\big)\left(1+\ln f_\varepsilon+\frac{|v+E|^2}{2}\right)dv\,dx.$$

We split the evaluation of these quantities into three lemma.

Lemma 7. *Assume that* $E\in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$. *Then we have*

$$Q_1(t)=\varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon\left(\ln f_\varepsilon+\frac{|v+E|^2}{2}\right)dv\,dx-\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon(v+E)\cdot\big(\partial_t E+(\mathrm{D}_x E)v\big)\,dv\,dx,$$

where $\mathrm{D}_x E$ *stands for the Jacobian matrix of* E.

Proof. We can write

$$(\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon)\cdot\left(1+\ln f_\varepsilon+\frac{|v+E|^2}{2}\right) = \partial_t(f_\varepsilon\ln f_\varepsilon)+v\cdot\nabla_x(f_\varepsilon\ln f_\varepsilon)+\partial_t\left(f_\varepsilon\frac{|v+E|^2}{2}\right) + v\cdot\nabla_x\left(f_\varepsilon\frac{|v+E|^2}{2}\right) - f_\varepsilon(v+E)\cdot\big(\partial_t E+(\mathrm{D}_x E)v\big),$$

where $\mathrm{D}_x E = (\frac{\partial E_i}{\partial x_j})_{1\leqslant i,j\leqslant 3}$. After integration with respect to (x,v) we get

$$Q_1(t) = \varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\left(\ln f_\varepsilon+\frac{|v+E|^2}{2}\right)dv\,dx - \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(v+E)\cdot\big(\partial_t E+(\mathrm{D}_x E)v\big)\,dv\,dx. \qquad \square$$

Lemma 8. *We have*

$$Q_2(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\sqrt{f_\varepsilon}(v+E)+2\nabla_v\sqrt{f_\varepsilon}\right|^2 dv\,dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q_\varepsilon|^2\,dv\,dx.$$

Proof. By using the formula

$$\mathrm{div}_v\big(f_\varepsilon(v+E)+\nabla_v f_\varepsilon\big) = \mathrm{div}_v\big\{e^{-\frac{|v+E|^2}{2}}\nabla_v\big(f_\varepsilon e^{\frac{|v+E|^2}{2}}\big)\big\},$$

we deduce that

$$\begin{aligned} Q_2(t) &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{div}_v\big\{e^{-\frac{|v+E|^2}{2}}\nabla_v\big(f_\varepsilon e^{\frac{|v+E|^2}{2}}\big)\big\}\ln\big(f_\varepsilon e^{\frac{|v+E|^2}{2}}\big)\,dv\,dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-|v+E|^2}}{f_\varepsilon}\left|\nabla_v\big(f_\varepsilon e^{\frac{|v+E|^2}{2}}\big)\right|^2 dv\,dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\sqrt{f_\varepsilon}(v+E)+2\nabla_v\sqrt{f_\varepsilon}\right|^2 dv\,dx. \qquad \square \end{aligned}$$

Lemma 9. *Let* (ρ,E,B) *be a solution of* (14) *satisfying* $E\in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, $B,\partial_t U\in L^\infty(]0,T[\times\mathbb{R}^3)^3$, $E,B,\partial_t E,\partial_t B\in L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$. *Then, we have*

$$\begin{aligned} Q_3(t) &= \frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^3}|E_\varepsilon-E|^2\,dx+\frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3}|B_\varepsilon-B|^2\,dx\right\} \\ &\quad -\int_{\mathbb{R}^3}\big(\mathcal{A}(E_\varepsilon-E,E_\varepsilon-E)+\alpha\varepsilon\mathcal{A}(B_\varepsilon-B,B_\varepsilon-B)\big)\cdot E\,dx \\ &\quad -\alpha\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\partial_t U+E\wedge B)\cdot(v+E)f_\varepsilon\,dv\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}\big(\nabla_x(\partial_t U\cdot E)-E\wedge\partial_t B\big)\cdot(E_\varepsilon-E)\,dx \\ &\quad +\alpha\varepsilon\int_{\mathbb{R}^3}\partial_t U\cdot\partial_t(E_\varepsilon-E)\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}\partial_t\big((E_\varepsilon-E)\wedge(B_\varepsilon-B)\big)\cdot E\,dx. \end{aligned}$$

Proof. We can write

$$\mathrm{div}_v\big(f_\varepsilon\big(E_\varepsilon-E+\alpha\varepsilon(v\wedge B_\varepsilon)\big)\big)\left(1+\ln f_\varepsilon+\frac{|v+E|^2}{2}\right) = \mathrm{div}_v\big(f_\varepsilon\ln f_\varepsilon\big(E_\varepsilon-E+\alpha\varepsilon(v\wedge B_\varepsilon)\big)\big) + \mathrm{div}_v\big(f_\varepsilon\big(E_\varepsilon-E+\alpha\varepsilon(v\wedge B_\varepsilon)\big)\big)\frac{|v+E|^2}{2}.$$

After integration with respect to (x, v) we get

$$\begin{aligned}
Q_3(t) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\big(E_\varepsilon - E + \alpha\varepsilon(v \wedge B_\varepsilon)\big)\cdot(v+E)\,dv\,dx \\
&= \int_{\mathbb{R}^3} (E_\varepsilon - E)\cdot j_\varepsilon\,dx + \int_{\mathbb{R}^3} \rho_\varepsilon(E_\varepsilon - E)\cdot E\,dx + \alpha\varepsilon\int_{\mathbb{R}^3} (j_\varepsilon \wedge B_\varepsilon)\cdot E\,dx \\
&= \int_{\mathbb{R}^3} (E_\varepsilon - E)\cdot(j_\varepsilon + \rho E)\,dx + \int_{\mathbb{R}^3} (\rho_\varepsilon - \rho)(E_\varepsilon - E)\cdot E\,dx \\
&\quad + \alpha\varepsilon\int_{\mathbb{R}^3} \big((j_\varepsilon + \rho E)\wedge(B_\varepsilon - B)\big)\cdot E\,dx + \alpha\varepsilon\int_{\mathbb{R}^3} \big((j_\varepsilon + \rho_\varepsilon E)\wedge B\big)\cdot E\,dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

From (2), (3) and (14) we have

$$\partial_t(E_\varepsilon - E) - \operatorname{curl}_x(B_\varepsilon - B) = j_\varepsilon + \rho E, \tag{28}$$

$$\alpha\varepsilon\partial_t(B_\varepsilon - B) + \operatorname{curl}_x(E_\varepsilon - E) = -\alpha\varepsilon\partial_t B. \tag{29}$$

By multiplying (28), (29) by $E_\varepsilon - E$, and $B_\varepsilon - B$ respectively, we find after integration with respect to x

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big(|E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2\big)\,dx &= \int_{\mathbb{R}^3}(j_\varepsilon + \rho E)\cdot(E_\varepsilon - E)\,dx - \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon - B)\,dx \\
&= I_1 - \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon - B)\,dx.
\end{aligned} \tag{30}$$

By using (28) and the vector potential U the last term in the above right-hand side can be written

$$\begin{aligned}
-\alpha\varepsilon\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon - B)\,dx &= -\alpha\varepsilon\int_{\mathbb{R}^3}\partial_t U\cdot\operatorname{curl}_x(B_\varepsilon - B)\,dx \\
&= \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t U\cdot\big\{j_\varepsilon + \rho E - \partial_t(E_\varepsilon - E)\big\}\,dx \\
&= \alpha\varepsilon\int_{\mathbb{R}^3}\big\{\partial_t U\cdot(j_\varepsilon + \rho_\varepsilon E) + (\rho - \rho_\varepsilon)\partial_t U\cdot E\big\}\,dx \\
&\quad - \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t U\cdot\partial_t(E_\varepsilon - E)\,dx.
\end{aligned} \tag{31}$$

From (4), (14) we have $\operatorname{div}_x(E_\varepsilon - E) = -(\rho_\varepsilon - \rho)$ and thus

$$I_2 = \int_{\mathbb{R}^3}(\rho_\varepsilon - \rho)(E_\varepsilon - E)\cdot E\,dx = -\int_{\mathbb{R}^3}\operatorname{div}_x(E_\varepsilon - E)(E_\varepsilon - E)\cdot E\,dx. \tag{32}$$

Now by using (2), (3) we deduce

$$\begin{aligned}
I_3 &= \alpha\varepsilon\int_{\mathbb{R}^3}\big((j_\varepsilon + \rho E)\wedge(B_\varepsilon - B)\big)\cdot E\,dx \\
&= \alpha\varepsilon\int_{\mathbb{R}^3}\big(\big(\partial_t(E_\varepsilon - E) - \operatorname{curl}_x(B_\varepsilon - B)\big)\wedge(B_\varepsilon - B)\big)\cdot E\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\varepsilon \int_{\mathbb{R}^3} \big((B_\varepsilon - B) \wedge \operatorname{curl}_x (B_\varepsilon - B)\big) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E \, dx \\
&\quad - \alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge \partial_t (B_\varepsilon - B)\big) \cdot E \, dx \\
&= \alpha\varepsilon \int_{\mathbb{R}^3} \big((B_\varepsilon - B) \wedge \operatorname{curl}_x (B_\varepsilon - B)\big) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge \partial_t B\big) \cdot E \, dx \\
&\quad + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E \, dx + \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge \operatorname{curl}_x (E_\varepsilon - E)\big) \cdot E \, dx.
\end{aligned} \tag{33}
$$

Equalities (32) and (33) imply

$$
\begin{aligned}
I_2 + I_3 &= -\int_{\mathbb{R}^3} \mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) \cdot E \, dx - \alpha\varepsilon \int_{\mathbb{R}^3} \mathcal{A}(B_\varepsilon - B, B_\varepsilon - B) \cdot E \, dx \\
&\quad + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge \partial_t B\big) \cdot E \, dx.
\end{aligned}
$$

Finally we arrive at the formula for Q_3 stated in Lemma 9. □

From (27) we know that $Q_1(t) + Q_2(t) + Q_3(t) = 0$ for all $0 \leqslant t \leqslant T$. Therefore, combining Lemmas 7–9 yields

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \left(\ln f_\varepsilon + \frac{1}{2}|v + E|^2\right) dv \, dx + \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \big(|E_\varepsilon - E|^2 + \alpha\varepsilon |B_\varepsilon - B|^2\big) \, dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2 \, dv \, dx \\
&= \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon (v + E) \cdot \big(\partial_t E + (\mathrm{D}_x E) v\big) \, dv \, dx + \alpha\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\partial_t U + E \wedge B) \cdot (v + E) f_\varepsilon \, dv \, dx \\
&\quad + \int_{\mathbb{R}^3} \mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \mathcal{A}(B_\varepsilon - B, B_\varepsilon - B) \cdot E \, dx \\
&\quad - \alpha\varepsilon \int_{\mathbb{R}^3} \big(\nabla_x (\partial_t U \cdot E) - E \wedge \partial_t B\big) \cdot (E_\varepsilon - E) \, dx - \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E \, dx \\
&\quad - \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t U \cdot \partial_t (E_\varepsilon - E) \, dx.
\end{aligned} \tag{34}
$$

The two last terms can be recast as

$$
\begin{aligned}
&-\alpha\varepsilon \frac{d}{dt} \left(\int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge (B_\varepsilon - B) \cdot E + \partial_t U \cdot (E_\varepsilon - E)\big) \, dx \right), \\
&+\alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge (B_\varepsilon - B) \cdot \partial_t E + \partial_t^2 U \cdot (E_\varepsilon - E)\big) \, dx.
\end{aligned}
$$

For computing the time derivative of the relative entropy we also need the expression of the third term in the right-hand side of (25).

Lemma 10. *We have*

$$
\frac{d}{dt} \left(\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln(\rho) \, dv \, dx \right) = \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \big((v + E) \cdot \nabla_x \ln(\rho) + \operatorname{div}_x E\big) \, dv \, dx.
$$

Proof. Using the equations satisfied by f_ε and ρ, we obtain when integrating by parts

$$\frac{d}{dt}\bigg(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln(\rho)\,dv\,dx\bigg) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon v\cdot\nabla_x \ln(\rho)\,dv\,dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_\varepsilon}{\rho}\operatorname{div}_x(\rho E)\,dv\,dx.$$

We conclude by expanding $\frac{1}{\rho}\operatorname{div}_x(\rho E) = \operatorname{div}_x E + E\cdot\nabla_x \ln(\rho)$. □

Combining (34) and Lemma 10 characterizes the evolution of the relative entropy and proves Proposition 6 with the following observations (based on integration by parts with respect to v):

– On the one hand for any (vector valued) function Ψ depending only on (t,x), we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Psi(t,x)\cdot(v+E) f_\varepsilon\,dv\,dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Psi(t,x)\cdot q_\varepsilon\sqrt{f}_\varepsilon\,dv\,dx.$$

– On the other hand, we have

$$\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\mathrm{D}_x E)(v+E)\cdot(v+E) f_\varepsilon\,dv\,dx - \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \operatorname{div}_x E\,dv\,dx = \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q_\varepsilon\cdot(\mathrm{D}_x E)(v+E)\sqrt{f_\varepsilon}\,dv\,dx,$$

where we recognize one of the integrals produced in Lemma 10.

We intend to show that the terms in the right-hand side of (26) are dominated by the relative entropy and the entropy production term $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q_\varepsilon|^2\,dv\,dx$ up to a reminder term of order ε^2. This will allow us to conclude by the Gronwall lemma.

Corollary 3.1. *Under the hypotheses of Proposition* 6 *we have for any* $\varepsilon\in\,]0,1]$

$$\begin{cases} \sup_{t\in[0,T]} \mathcal{H}_\varepsilon(t) \leqslant C_T\big(\mathcal{H}_\varepsilon(0)+\varepsilon^2(1+\alpha^2)\big),\\ \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2\,dv\,dx\,dt \leqslant C_T\big(\mathcal{H}_\varepsilon(0)+\varepsilon^2(1+\alpha^2)\big). \end{cases}$$

Proof. The estimates are standard, except that of the term

$$\varepsilon\bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q_\varepsilon\cdot(\mathrm{D}_x E)(v+E)\sqrt{f_\varepsilon}\,dv\,dx\bigg| \leqslant \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q_\varepsilon|^2\,dv\,dx + C\varepsilon^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v+E|^2 f_\varepsilon\,dv\,dx,$$

which actually needs a sharp estimate of $\varepsilon^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v+E|^2 f_\varepsilon\,dv\,dx$. This can be done by using the properties of the entropic convergence, introduced in [4]. For the sake of the completeness we recall here the arguments. Let $h :]-1,+\infty[\,\to\mathbb{R}$ be the strictly convex function

$$h(z) = (1+z)\ln(1+z) - z,\quad z>-1,$$

which enters into the definition of the relative entropy. Indeed, let us denote by g_ε the ε-fluctuations of f_ε with respect to the equilibrium ρM_E (and normalized by ρM_E)

$$\frac{f_\varepsilon}{\rho M_E} = 1+\varepsilon g_\varepsilon.$$

Then, we get

$$H(f_\varepsilon|\rho M_E) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho M_E h(\varepsilon g_\varepsilon)\,dv\,dx.$$

We shall make use of the following properties of the function h:

- Its Legendre transform is explicitly given by
$$h^\star(y) = e^y - 1 - y, \quad y \in \mathbb{R}.$$
- Reflection inequality
$$h\big(|z|\big) \leqslant h(z), \quad z > -1.$$
- Super-quadratic homogeneity
$$h^\star(\lambda y) \leqslant \lambda^2 h^\star(y), \quad y > 0,\ 0 \leqslant \lambda \leqslant 1.$$
- Young inequality
$$yz \leqslant h(z) + h^\star(y).$$

Indeed, apply the Young inequality with

$$y = \frac{\varepsilon}{4a}\big(1 + |v + E|^2\big), \qquad z = \varepsilon|g_\varepsilon|.$$

Using the reflection inequality and the super-quadratic homogeneity yields for $0 < \varepsilon \leqslant a$

$$\frac{\varepsilon}{4a}\big(1 + |v + E|^2\big)\varepsilon|g_\varepsilon| \leqslant h(\varepsilon g_\varepsilon) + \frac{\varepsilon^2}{a^2} h^\star\left(\frac{1}{4}\big(1 + |v + E|^2\big)\right).$$

Multiplying by $a\varepsilon\rho M_E$ and integrating with respect to $(x, v) \in \mathbb{R}^6$ we deduce that

$$\begin{aligned}\frac{\varepsilon^2}{4}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\big(1 + |v + E|^2\big)|f_\varepsilon - \rho M_E|\,dv\,dx &\leqslant a\varepsilon\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\rho M_E h(\varepsilon g_\varepsilon)\,dv\,dx + C\frac{\varepsilon^3}{a}\\ &\leqslant a\mathcal{H}_\varepsilon(t) + C\frac{\varepsilon^3}{a}.\end{aligned} \tag{35}$$

Consequently, choosing $a = 1$, we obtain

$$\varepsilon^2\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{1}{2}|v + E|^2 f_\varepsilon\,dv\,dx \leqslant C\big(\mathcal{H}_\varepsilon(t) + \varepsilon^2\big), \quad 0 < \varepsilon \leqslant 1. \tag{36}$$

(Another choice of a will be useful later on.)

It remains to estimate the other terms in the right-hand side of (26). Using the formula

$$\mathcal{A}(u, u) = \mathrm{div}_x\left(u \otimes u - \frac{1}{2}|u|^2 I_3\right),$$

we get easily after integration by parts that

$$\left|\int\limits_{\mathbb{R}^3}\mathcal{A}(E_\varepsilon - E, E_\varepsilon - E)\cdot E\,dx\right| \leqslant C\int\limits_{\mathbb{R}^3}|E_\varepsilon - E|^2\,dx \leqslant C\mathcal{H}_\varepsilon(t),$$

and similarly

$$\alpha\varepsilon\left|\int\limits_{\mathbb{R}^3}\mathcal{A}(B_\varepsilon - B, B_\varepsilon - B)\cdot E\,dx\right| \leqslant C\alpha\varepsilon\int\limits_{\mathbb{R}^3}|B_\varepsilon - B|^2\,dx \leqslant C\mathcal{H}_\varepsilon(t),$$

where C depends on $\|E\|_{W^{1,\infty}}$. Next, we use the trivial inequalities

$$\begin{aligned}\alpha\varepsilon\left|\int\limits_{\mathbb{R}^3}\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big)\cdot\partial_t E\,dx\right| &\leqslant C\sqrt{\alpha\varepsilon}\int\limits_{\mathbb{R}^3}\big(|E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2\big)\,dx\\ &\leqslant C\sqrt{\alpha\varepsilon}\,\mathcal{H}_\varepsilon(t).\end{aligned}$$

By virtue of our assumptions on the limit solution, $\partial_t^2 U + E \wedge \partial_t B - \nabla_x(\partial_t U \cdot E)$ belongs to $L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$ and therefore

$$\alpha\varepsilon \left| \int_{\mathbb{R}^3} \left\{ \partial_t^2 U + E \wedge \partial_t B - \nabla_x(\partial_t U \cdot E) \right\} \cdot (E_\varepsilon - E)\,dx \right| \leqslant C\alpha^2\varepsilon^2 + \frac{1}{2}\int_{\mathbb{R}^3} |E_\varepsilon - E|^2\,dx$$
$$\leqslant C\alpha^2\varepsilon^2 + \mathcal{H}_\varepsilon(t).$$

Similarly, by using the charge conservation we get

$$\varepsilon \left| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left\{ \alpha(\partial_t U + E \wedge B) + \partial_t E - (\mathrm{D}_x E)E - \nabla_x \ln\rho \right\} \cdot q_\varepsilon \sqrt{f_\varepsilon}\,dv\,dx \right|$$
$$\leqslant \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2\,dv\,dx + C\varepsilon^2\big(1+\alpha^2\big). \tag{37}$$

Plugging all these estimates in (26) we obtain

$$\frac{d}{dt}\left\{ \mathcal{H}_\varepsilon(t) + \alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E\,dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t U \cdot (E_\varepsilon - E)\,dx \right\} + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2\,dv\,dx$$
$$\leqslant C\big(\mathcal{H}_\varepsilon(t) + \varepsilon^2\big(1+\alpha^2\big)\big).$$

Eventually the conclusion follows by integrating with respect to the time and by observing that

$$\left| \alpha\varepsilon \int_{\mathbb{R}^3} \big((E_\varepsilon - E) \wedge (B_\varepsilon - B)\big) \cdot E\,dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t U \cdot (E_\varepsilon - E)\,dx \right| \leqslant C\big(1+\sqrt{\alpha\varepsilon}\big)\mathcal{H}_\varepsilon(t) + C\alpha^2\varepsilon^2. \qquad \square \tag{38}$$

4. Asymptotics

In this section we analyze the asymptotic behavior of smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ of the VMFP system (1)–(7) when the parameter $\varepsilon \searrow 0$ and we establish rigorously the connection to the system (14). We start with the following consequence of Corollary 3.1.

Proposition 11. *Under the hypotheses of Proposition* 6*, we assume moreover that*

(i) $\lim_{\varepsilon\searrow 0} \mathcal{H}_\varepsilon(0) = 0$. *Then, we have*

$$\begin{cases} E_\varepsilon \underset{\varepsilon\searrow 0}{\to} E \quad \text{strongly in } L^\infty\big(]0,T[;L^2\big(\mathbb{R}^3\big)\big)^3, \\ \displaystyle\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2\,dv\,dx\,dt \underset{\varepsilon\searrow 0}{\to} 0, \\ f_\varepsilon - \rho_\varepsilon M_E \underset{\varepsilon\searrow 0}{\to} 0 \quad \text{strongly in } L^1\big(]0,T[\times\mathbb{R}^3\times\mathbb{R}^3\big). \end{cases}$$

(ii) $\lim_{\varepsilon\searrow 0} \mathcal{H}_\varepsilon(0)/\varepsilon = 0$. *Then, we have furthermore,*

$$\begin{cases} f_\varepsilon \underset{\varepsilon\searrow 0}{\to} \rho M_E \quad \text{strongly in } L^\infty\big(]0,T[;L^1\big(\mathbb{R}^3\times\mathbb{R}^3\big)\big), \\ \rho_\varepsilon \underset{\varepsilon\searrow 0}{\to} \rho \quad \text{and} \quad j_\varepsilon \underset{\varepsilon\searrow 0}{\to} -\rho E \quad \text{strongly in } L^\infty\big(]0,T[;L^1\big(\mathbb{R}^3\big)\big). \end{cases}$$

(iii) $\lim_{\varepsilon\searrow 0} \mathcal{H}_\varepsilon(0)/(\alpha\varepsilon) = 0$, *with* $\lim_{\varepsilon\searrow 0} \varepsilon/\alpha = 0$. *Then, we have furthermore,*

$$B_\varepsilon \underset{\varepsilon\searrow 0}{\to} B \quad \text{strongly in } L^\infty\big(]0,T[;L^2\big(\mathbb{R}^3\big)\big)^3.$$

Proof. The two first statements in (i) are obvious consequences of Corollary 3.1 since $H(f_\varepsilon|\rho M_E) \geqslant 0$. Next, we appeal to the logarithmic Sobolev inequality, see e.g. [3,2], which yields

$$0 \leqslant \int_{\mathbb{R}^3} \left\{ \frac{f_\varepsilon}{\rho_\varepsilon M_E} \ln\left(\frac{f_\varepsilon}{\rho_\varepsilon M_E}\right) - \frac{f_\varepsilon}{\rho_\varepsilon M_E} + 1 \right\} \rho_\varepsilon M_E \, dv = \int_{\mathbb{R}^3} f_\varepsilon \ln\left(\frac{f_\varepsilon}{\rho_\varepsilon M_E}\right) dv$$
$$\leqslant \lambda \int_{\mathbb{R}^3} \left|\nabla_v \sqrt{f_\varepsilon / M_E}\,\right|^2 M_E \, dv = \frac{\lambda}{4} \int_{\mathbb{R}^3} |q_\varepsilon|^2 \, dv,$$

for some $\lambda > 0$. Hence $\int_0^T H(f_\varepsilon|\rho_\varepsilon M_E)\,dt$ tends to 0 as $\varepsilon \searrow 0$. Eventually, we conclude by using the Csiszar–Kullback–Pinsker inequality, see [19,32], which implies that

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f_\varepsilon - \rho_\varepsilon M_E| \, dv \, dx \right)^2 \leqslant \mu \int_{\mathbb{R}^3} f_\varepsilon \ln\left(\frac{f_\varepsilon}{\rho_\varepsilon M_E}\right) dv \, dx,$$

with $\mu > 0$.

With the additional assumption in (ii), we strengthen also the behavior of the relative entropy:

$$\lim_{\varepsilon \searrow 0} \sup_{0 \leqslant t \leqslant T} \frac{\mathcal{H}_\varepsilon(t)}{\varepsilon} = 0.$$

Now, let us go back to (35). Optimizing with respect to a, we arrive at

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(1 + |v + E|^2\right) |f_\varepsilon - \rho M_E| \, dv \, dx \leqslant C_T \sqrt{\frac{\mathcal{H}_\varepsilon(t)}{\varepsilon}}$$

which tends to 0 uniformly with respect to $0 \leqslant t \leqslant T$ as $\varepsilon \searrow 0$. Therefore we readily check that

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f_\varepsilon - \rho M_E| \, dv \, dx \underset{\varepsilon \searrow 0}{\to} 0,$$
$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} |\rho_\varepsilon - \rho| \, dx = \sup_{t \in [0,T]} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} (f_\varepsilon - \rho M_E) \, dv \right| dx \underset{\varepsilon \searrow 0}{\to} 0,$$
$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} |j_\varepsilon + \rho E| \, dx = \sup_{t \in [0,T]} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} v(f_\varepsilon - \rho M_E) \, dv \right| dx \underset{\varepsilon \searrow 0}{\to} 0.$$

The control on the magnetic field under the strengthened assumption in (iii) follows from the simple remark

$$\|B_\varepsilon - B\|(t)_{L^2(\mathbb{R}^3)} \leqslant \frac{\mathcal{H}_\varepsilon(t)}{\alpha\varepsilon} \leqslant C_T \frac{\mathcal{H}_\varepsilon(0) + \varepsilon^2(1 + \alpha^2)}{\alpha\varepsilon}. \qquad \square$$

Clearly, Proposition 11(iii) ends the proof of Theorem 1. However, we can still investigate the asymptotic behavior of the solutions under the weaker hypothesis of (i). The difficulty comes from the fact that the relative entropy does not provide useful information on $H(f_\varepsilon|\rho M_E)$ and $\|B_\varepsilon - B\|^2_{L^2(\mathbb{R}^3)}$ due to the ε and $\alpha\varepsilon$ in front of these terms in the definition of the relative entropy. Nevertheless, we will be able to establish convergences in a weaker sense. For instance, since $\rho_\varepsilon - \rho = \mathrm{div}_x(E - E_\varepsilon)$ we obtain that $\lim_{\varepsilon \searrow 0} \rho_\varepsilon = \rho$ in $\mathcal{D}'(\mathbb{R}^3)$, uniformly for $t \in [0, T]$. Actually we can prove that the previous convergence holds in the space of bounded measures. Throughout the paper we denote by $\mathcal{M}^1(\mathbb{R}^3)$ the set of bounded Radon measures on $\mathbb{R}^3$, while $\mathcal{M}^1_+(\mathbb{R}^3)$ stands for its positive cone. We recall some definitions and compactness properties in measure spaces (see [12] for more details).

Definition 4.1. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1(\mathbb{R}^3)$. We say that

(1) $(\rho_n)_{n\in\mathbb{N}}$ converges vaguely to ρ iff

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\varphi\,d\rho_n=\int_{\mathbb{R}^3}\varphi\,d\rho, \tag{39}$$

for any continuous function with compact support $\varphi\in C_c^0(\mathbb{R}^3)$ (actually the convergence holds for any continuous function φ vanishing at infinity, i.e., $\lim_{|x|\to+\infty}\varphi(x)=0$);

(2) $(\rho_n)_{n\in\mathbb{N}}$ converges tightly to ρ iff (39) holds for any continuous and bounded function $\varphi\in C^0(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$.

We have the following classical results.

Proposition 12.

(1) *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1_+(\mathbb{R}^3)$ which converges vaguely to ρ. Assume also that $\lim_{n\to+\infty}\rho_n(\mathbb{R}^3)=\rho(\mathbb{R}^3)$. Then $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ tightly.*

(2) *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1(\mathbb{R}^3)$ verifying $\sup_n|\rho_n|(\mathbb{R}^3)<+\infty$ and such that for any $\eta>0$ there exists a compact set $K_\eta\subset\mathbb{R}^3$ satisfying $\sup_n|\rho_n|(\mathbb{R}^3-K_\eta)\leqslant\eta$. Then $(\rho_n)_{n\in\mathbb{N}}$ is relatively compact for the tight topology.*

We recall also the following compactness result, cf. [26].

Proposition 13. *Assume that $(\rho_\varepsilon)_{\varepsilon>0}$, $(j_\varepsilon)_{\varepsilon>0}$ satisfy $\rho_\varepsilon\geqslant 0$, $\partial_t\rho_\varepsilon+\operatorname{div}_x j_\varepsilon=0$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$, $\forall\varepsilon>0$ and*

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\rho_\varepsilon(t,x)\big(1+|x|\big)\,dx<+\infty,$$

$$\sup_{\varepsilon>0}\int_0^T\bigg(\int_{\mathbb{R}^3}\big|j_\varepsilon(t,x)\big|\,dx\bigg)^2dt<+\infty,$$

$$\sup_{\varepsilon>0}\int_0^T\int_{\mathbb{R}^3}\big(1+\sqrt{|x|}\big)\big|j_\varepsilon(t,x)\big|\,dx\,dt<+\infty.$$

Then $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];\mathcal{M}^1_+(\mathbb{R}^3)\text{-tight})$ and $(j_\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight.

Our goal is to complete Proposition 11 as follows.

Lemma 14. *Let the assumptions of Proposition* 11(i) *be fulfilled. Then, we also have the following convergence properties*:

(a) *ρ_ε converges to ρ in $C^0([0,T];\mathcal{M}^1_+(\mathbb{R}^3)\text{-tight})$,*
(b) *j_ε converges to $-\rho E$ in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)$ tightly,*
(c) *B_ε converges to B in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$.*

Proof. We observe that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v+E|^2f_\varepsilon\,dv\,dx\,dt&\leqslant\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(|v+E|^2f_\varepsilon+4\big|\nabla_v\sqrt{f_\varepsilon}\,\big|^2\Big)\,dv\,dx\,dt\\&=\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(|q_\varepsilon|^2-2(v+E)\cdot\nabla_vf_\varepsilon\big)\,dv\,dx\,dt\end{aligned}$$

$$= \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(|q_\varepsilon|^2 + 6 f_\varepsilon\right) dv\, dx\, dt.$$

Hence, the charge conservation together with Corollary 3.1 imply that

$$\sup_{0<\varepsilon\leqslant 1} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 f_\varepsilon \, dv\, dx\, dt < +\infty.$$

Next, reasoning as in the proof of Proposition 2 (see (21)) we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(1+|x|\big) f_\varepsilon \, dv\, dx &\leqslant \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(1+|x|\big) f_\varepsilon^0 \, dv\, dx + \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v| f_\varepsilon \, dv\, dx\, dt \\
&\leqslant \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(1+|x|\big) f_\varepsilon^0 \, dv\, dx + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(1+|v|^2\big) f_\varepsilon \, dv\, dx\, dt,
\end{aligned}$$

by using (15) and therefore

$$\sup_{0<\varepsilon\leqslant 1} \sup_{t\in[0,T]} \int_{\mathbb{R}^3} \big(1+|x|\big) \rho_\varepsilon(t,x)\, dx < +\infty.$$

Moreover we have the inequalities

$$\begin{aligned}
\int_0^T \big\| j_\varepsilon(t) \big\|_{L^1(\mathbb{R}^3)}^2 dt &\leqslant \int_0^T \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v| f_\varepsilon \, dv\, dx \right)^2 dt \\
&\leqslant \int_0^T \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 f_\varepsilon \, dv\, dx \right) \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon \, dv\, dx \right) dt \\
&\leqslant \left(\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 f_\varepsilon \, dv\, dx\, dt \right) \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx \right) \\
&\leqslant C, \quad 0<\varepsilon\leqslant 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \big(1+\sqrt{|x|}\big) \big| j_\varepsilon(t,x) \big| \, dx\, dt &\leqslant \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(1+\sqrt{|x|}\big) |v| f_\varepsilon \, dv\, dx\, dt \\
&\leqslant \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big\{ \big(1+|v|^2\big) + \big(|x|+|v|^2\big) \big\} f_\varepsilon \, dv\, dx\, dt \\
&\leqslant C, \quad 0<\varepsilon\leqslant 1.
\end{aligned}$$

Therefore, by using Proposition 13 we deduce that $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T]; \mathcal{M}_+^1(\mathbb{R}^3)\text{-tight})$ and $(j_\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight. Since $\operatorname{div}_x(E_\varepsilon - E) = -(\rho_\varepsilon - \rho)$ we obtain that ρ_ε converges to ρ in $C^0([0,T]; \mathcal{M}_+^1(\mathbb{R}^3)\text{-tight})$. Next, we remark that

$$\int_0^T\int_{\mathbb{R}^3} |j_\varepsilon + \rho_\varepsilon E|\,dx\,dt \leqslant \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sqrt{f_\varepsilon}|q_\varepsilon|\,dv\,dx\,dt$$
$$\leqslant \left(\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\,dv\,dx\,dt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2\,dv\,dx\,dt\right)^{\frac{1}{2}}, \tag{40}$$

and thus we have for all continuous bounded function θ

$$\left|\int_0^T\int_{\mathbb{R}^3} (j_\varepsilon + \rho E)\theta\,dx\,dt\right| \leqslant \left|\int_0^T\int_{\mathbb{R}^3} (j_\varepsilon + \rho_\varepsilon E)\theta\,dx\,dt\right| + \left|\int_0^T\int_{\mathbb{R}^3} (\rho - \rho_\varepsilon)E\theta\,dx\,dt\right|$$
$$\leqslant \|\theta\|_{L^\infty}\int_0^T\int_{\mathbb{R}^3} |j_\varepsilon + \rho_\varepsilon E|\,dx\,dt + \left|\int_0^T\int_{\mathbb{R}^3} (\rho_\varepsilon - \rho)E\theta\,dx\,dt\right|. \tag{41}$$

Since $E\theta$ is bounded and continuous we have $\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(\rho_\varepsilon - \rho)E\theta\,dx\,dt = 0$ and thus Proposition 11 implies that j_ε converges to $-\rho E$ in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight.

It remains to deal with the magnetic field: we aim at showing that

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3} (B_\varepsilon - B)\cdot\varphi\,dx\,dt = 0,$$

for all function $\varphi \in C_c^2(]0,T[\times\mathbb{R}^3)^3$. Pick φ such a function and observe that in particular we have $\varphi, \partial_t\varphi \in L^2(]0,T[;H^1(\mathbb{R}^3))^3$. By using the decomposition

$$\varphi = \nabla_x\varphi_1 + \mathrm{curl}_x\,\varphi_2,$$

with $\varphi_1, \partial_t\varphi_1 \in L^2(]0,T[;H^2(\mathbb{R}^3))$ and $\varphi_2, \partial_t\varphi_2 \in L^2(]0,T[;H^2(\mathbb{R}^3))^3$, it is sufficient to prove that

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3} (B_\varepsilon - B)\cdot\nabla_x\varphi_1\,dx\,dt = 0, \tag{42}$$

and

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3} (B_\varepsilon - B)\cdot\mathrm{curl}_x\,\varphi_2\,dx\,dt = 0. \tag{43}$$

The convergence (42) is trivial since $\mathrm{div}_x\,B_\varepsilon = \mathrm{div}_x\,B = 0$. To justify (43) we use the equations

$$\partial_t E_\varepsilon - \mathrm{curl}_x\,B_\varepsilon = j_\varepsilon - J, \qquad \partial_t E - \mathrm{curl}_x\,B = -\rho E - J.$$

After multiplication by the test function φ_2 we find

$$-\int_0^T\int_{\mathbb{R}^3} (E_\varepsilon - E)\cdot\partial_t\varphi_2\,dx\,dt - \int_0^T\int_{\mathbb{R}^3} (B_\varepsilon - B)\cdot\mathrm{curl}_x\,\varphi_2\,dx\,dt = \int_0^T\int_{\mathbb{R}^3} (j_\varepsilon + \rho E)\cdot\varphi_2\,dx\,dt.$$

Since $\partial_t\varphi_2 \in L^2(]0,T[;L^2(\mathbb{R}^3))^3$ and $\lim_{\varepsilon\searrow 0} E_\varepsilon = E$ in $L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$ the first integral in the left-hand side vanishes as $\varepsilon \searrow 0$. To deal with the right-hand side, observe that φ_2 is a continuous bounded function since $\varphi_2, \partial_t\varphi_2 \in L^2(]0,T[;H^2(\mathbb{R}^3))^3$ imply $\varphi_2 \in C^0([0,T];H^2(\mathbb{R}^3))^3 \subset C^0([0,T]\times\mathbb{R}^3)^3 \cap L^\infty(]0,T[\times\mathbb{R}^3)^3$. By using the convergence $\lim_{\varepsilon\searrow 0} j_\varepsilon = -\rho E$ in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight we deduce

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(j_\varepsilon+\rho E)\cdot\varphi_2\,dx\,dt=0,$$

and therefore (43) holds.

Let us end with the following remark, which makes the formal result (13) clear:

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\big(f_\varepsilon(t,x,v)-\rho(t,x)M_E(t,x,v)\big)\varphi(x)\,dx\right|dv\,dt=0,$$

holds for any test function $\varphi\in C^0(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. At first, we expand

$$f_\varepsilon-\rho M_E=(f_\varepsilon-\rho_\varepsilon M_E)+(\rho_\varepsilon-\rho)M_E.$$

Consider $\varphi\in C^0(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Since $E\in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, for all $(t,v)\in[0,T]\times\mathbb{R}^3$ the function $x\to M_E(t,x,v)\varphi(x)$ is continuous and bounded. We have already shown that $\rho_\varepsilon-\rho$ tends to 0 in $C^0([0,T];\mathcal{M}^1(\mathbb{R}^3)\text{-tight})$ and thus we have

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3}\big(\rho_\varepsilon(t,x)-\rho(t,x)\big)M_E(t,x,v)\varphi(x)\,dx=0,\quad \forall(t,v)\in[0,T]\times\mathbb{R}^3.$$

Moreover we have the inequality $|v+E(t,x)|^2\geqslant\frac{1}{2}|v|^2-|E(t,x)|^2\geqslant\frac{1}{2}|v|^2-\|E\|^2_{L^\infty}$ and thus $M_E(t,x,v)\leqslant C(\|E\|_{L^\infty})e^{-|v|^2/4}$, $\forall(t,x,v)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$. We deduce that

$$\left|\int_{\mathbb{R}^3}\big(\rho_\varepsilon(t,x)-\rho(t,x)\big)M_E(t,x,v)\varphi(x)\,dx\right|\leqslant 2C\big(\|E\|_{L^\infty}\big)\|\varphi\|_{L^\infty}\int_{\mathbb{R}^3}D(0,x)\,dx\,e^{-|v|^2/4},$$

and by using the dominated convergence theorem we obtain

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\big(\rho_\varepsilon(t,x)-\rho(t,x)\big)M_E(t,x,v)\varphi(x)\,dx\right|dv\,dt=0.$$

The behavior of $f_\varepsilon-\rho_\varepsilon M_E$ has been already discussed in Proposition 11. □

To conclude, we are led to the following statement.

Theorem 15. *We assume that the assumptions of Theorem* 1 *are fulfilled, but we replace hypothesis* (16) *on the initial relative entropy by* $\lim_{\varepsilon\searrow 0}\mathcal{H}_\varepsilon(0)=0$. *Then* $(E_\varepsilon)_{\varepsilon>0}$ *converges to* E *in* $L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$, $(\sqrt{\varepsilon}B_\varepsilon)_{\varepsilon>0}$ *converges to* 0 *in* $L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$, $(B_\varepsilon)_{\varepsilon>0}$ *converges to* B *in* $\mathcal{D}'(]0,T[\times\mathbb{R}^3)^3$, $(\rho_\varepsilon)_{\varepsilon>0}$ *converges to* ρ *in* $C^0([0,T];\mathcal{M}^1_+(\mathbb{R}^3)$*-tight) and* $(j_\varepsilon)_{\varepsilon>0}$ *converges to* $-\rho E$ *in* $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$*-tight.*

Acknowledgement

We thank the referees for their valuable suggestions, which have allowed us to obtain a better result than in the first version of the paper.

Appendix A. Dimensional analysis

We detail here the dimensional analysis of the equations and the physical meaning of the different parameters introduced previously. Let us write the equations in physical variables. We distinguish the following physical constants:

- ε_0 the vacuum permittivity;
- μ_0 the vacuum permeability;
- c_0 the vacuum light speed given by $\varepsilon_0\mu_0c_0^2=1$;

- q the charge of (negative) particles;
- m the mass of particles;
- τ the relaxation time which characterizes the interactions of the particles with the thermal bath;
- K_B the Boltzmann constant;
- T_{th} the temperature of the thermal bath.

Let $f(t,x,v)$ denote the particle distribution function, which depends on the time $t>0$, space coordinates $x\in\mathbb{R}^3$ and velocity coordinates $v\in\mathbb{R}^3$. The evolution of f is described by the Fokker–Planck equation

$$\partial_t f + v\cdot\nabla_x f + \frac{F}{m}\cdot\nabla_v f = L_{\mathrm{FP}}(f), \quad (t,x,v)\in]0,+\infty[\,\times\mathbb{R}^3\times\mathbb{R}^3,$$

where the Fokker–Planck collision operator is given by

$$L_{\mathrm{FP}}(f) = \frac{1}{\tau}\operatorname{div}_v\left(vf + \frac{K_B T_{th}}{m}\nabla_v f\right),$$

and $F(t,x,v) = q(E(t,x) + v\wedge B(t,x))$ represents the Lorentz force. The evolution of the electro-magnetic field (E,B) is given by the Maxwell equations

$$\partial_t E - c_0^2 \operatorname{curl}_x B = -\frac{j(t,x)}{\varepsilon_0}, \qquad \partial_t B + \operatorname{curl}_x E = 0, \quad (t,x)\in]0,+\infty[\,\times\mathbb{R}^3,$$
$$\operatorname{div}_x E = \frac{\rho(t,x)}{\varepsilon_0}, \qquad \operatorname{div}_x B = 0, \quad (t,x)\in]0,+\infty[\,\times\mathbb{R}^3,$$

where $\rho = q\int_{\mathbb{R}^3} f\,dv$ and $j = q\int_{\mathbb{R}^3} vf\,dv$ are respectively the charge and current densities. The plasma is characterized by the mean free path $l = \sqrt{\frac{K_B T_{th}}{m}}\cdot\tau$, which is the average distance traveled by a particle between two successive collisions, and the Debye length $\Lambda = \sqrt{\frac{\varepsilon_0 K_B T_{th} L^3}{q^2\mathcal{N}}}$, which is the typical length of perturbations of a quasi-neutral plasma. Here $\mathcal{N}$ stands for a typical value for the number of particles in the plasma. In this paper we focus our attention on asymptotic regimes where the mean free path is much smaller than the Debye length, i.e., $l \ll \Lambda$. We set

$$\varepsilon = \left(\frac{l}{\Lambda}\right)^2$$

which is a small parameter. We introduce time, length and velocity units

$$T = \frac{\tau}{\varepsilon}, \qquad L = \frac{l}{\varepsilon}, \qquad V = \sqrt{\frac{K_B T_{th}}{m}}.$$

Observe also that we have $L = TV$ and $\Lambda = \sqrt{\varepsilon}L$. We define dimensionless variables and unknowns by the relations

$$t = Tt', \qquad x = Lx', \qquad v = Vv',$$
$$f(t,x,v) = \frac{\mathcal{N}}{L^3V^3} f'\left(\frac{t}{T},\frac{x}{L},\frac{v}{V}\right), \qquad E(t,x) = \frac{U_{th}}{L\varepsilon}E'\left(\frac{t}{T},\frac{x}{L}\right), \qquad B(t,x) = \frac{VU_{th}}{c_0^2 L\varepsilon}B'\left(\frac{t}{T},\frac{x}{L}\right),$$
$$\rho(t,x) = \frac{q\mathcal{N}}{L^3}\rho'\left(\frac{t}{T},\frac{x}{L}\right), \qquad j(t,x) = \frac{qV\mathcal{N}}{L^3}j'\left(\frac{t}{T},\frac{x}{L}\right),$$

where $U_{th} = \frac{K_B T_{th}}{q}$ is the thermal potential. After changing variables and unknowns, we obtain dropping the primes

$$\varepsilon(\partial_t f + v\cdot\nabla_x f) - \left(E(t,x) + \frac{V^2}{c_0^2} v\wedge B(t,x)\right)\cdot\nabla_v f = \operatorname{div}_v(vf + \nabla_v f),$$
$$\partial_t E - \operatorname{curl}_x B = -j(t,x), \qquad \frac{V^2}{c_0^2}\partial_t B + \operatorname{curl}_x E = 0,$$
$$\operatorname{div}_x E = \rho(t,x), \qquad \operatorname{div}_x B = 0,$$

where $\rho(t,x)=\int_{\mathbb{R}^3} f(t,x,v)\,dv$, $j(t,x)=\int_{\mathbb{R}^3} vf(t,x,v)\,dv$. Notice that we have

$$\frac{V^2}{c_0^2}=\left(\frac{l}{\tau}\right)^2\frac{1}{c_0^2}=\frac{\Lambda^2\varepsilon}{\tau^2c_0^2}=\alpha\varepsilon,$$

where $\alpha=(\frac{\Lambda}{\tau c_0})^2$, so that we are interested in a regime where the light speed remains large compared to the velocity unit of observation.

Let us define

the collision frequency	$\frac{1}{\tau}=\frac{1}{\sqrt{\varepsilon}T_p}$,
the plasma frequency	$\frac{1}{T_p}=\sqrt{\frac{\mathcal{N}q^2}{mL^3\varepsilon_0}}$,
the scale of light propagation	$T_0=\frac{L}{c_0}$,
the cyclotronic frequency	$\frac{1}{T_c}=\frac{K_BT_{th}}{mc_0^2T}\frac{1}{\varepsilon}$

(the last definition takes into account the scaling of the magnetic field) while $1/\tau$ is the collision frequency. With the previous scaling assumptions, we arrive at $T_p/T=\sqrt{\varepsilon}$, $T_0/T=\sqrt{\alpha\varepsilon}$, $T_c/T=1/\alpha$. The assumption $\varepsilon/\alpha\to 0$ is physically questionable since it means that T_0 which is the time necessary for light to travel the distance L is large compared to the time τ between two collisions events. This remark justifies the analysis of the general situation.

Appendix B. Bellman's lemma

In the proof of Proposition 2 we have used Bellman's lemma. We recall here the statement

Lemma 16. *Assume that* $x:[0,T]\to\mathbb{R}$ *and* $a:[0,T]\to\mathbb{R}_+$ *are given functions satisfying*

$$\frac{1}{2}\left|x(t)\right|^2\leqslant\frac{1}{2}|x_0|^2+\int_0^t a(s)x(s)\,ds,\quad t\in[0,T].$$

Then we have the inequality

$$\left|x(t)\right|\leqslant|x_0|+\int_0^t a(s)\,ds,\quad t\in[0,T].$$

Appendix C. Renormalized solutions

We have investigated the high-electric-field limit of the VMFP system provided that the solutions $(f_\varepsilon,E_\varepsilon,B_\varepsilon)_{\varepsilon>0}$ are smooth. The global existence of smooth solution for the VMFP system is a largely open problem in general situations. Therefore, following the ideas in [39] we intend to establish similar results in the framework of weaker solutions for the VMFP equations. In order to simplify our computations we work in the space periodic setting: we consider the space domain $\mathbb{T}^3=(\mathbb{R}/\mathbb{Z})^3$. A weak solution is a triplet

$$\begin{aligned}(f_\varepsilon,E_\varepsilon,B_\varepsilon)\in{}& L^\infty\big(]0,T[;L^2(\mathbb{T}^3\times\mathbb{R}^3)\big)\times L^\infty\big(]0,T[;L^2(\mathbb{T}^3)\big)^6\\ &\cap C\big([0,T];\text{w-}L^2(\mathbb{T}^3\times\mathbb{R}^3)\big)\times C\big([0,T];\text{w-}L^2(\mathbb{T}^3)\big)^6,\end{aligned}$$

which satisfies (1)–(4), (6), (7) in the sense of distributions and verifies

$$\varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\ln f_\varepsilon + \frac{1}{2}|v|^2\right) f_\varepsilon \, dv \, dx + \frac{1}{2}\int_{\mathbb{T}^3} \left(|E_\varepsilon|^2 + \alpha\varepsilon |B_\varepsilon|^2\right) dx$$
$$+ \int_0^t \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left|v\sqrt{f_\varepsilon} + 2\nabla_v \sqrt{f_\varepsilon}\right|^2 dv \, dx \, ds + \int_0^t \int_{\mathbb{T}^3} J \cdot E_\varepsilon \, dx \, ds$$
$$\leqslant \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\ln f_\varepsilon^0 + \frac{1}{2}|v|^2\right) f_\varepsilon^0 \, dv \, dx + \frac{1}{2}\int_{\mathbb{T}^3} \left(\left|E_\varepsilon^0\right|^2 + \alpha\varepsilon \left|B_\varepsilon^0\right|^2\right) dx, \quad t \in [0,T].$$

It is well known that such a solution satisfies the local conservation law of the charge

$$\partial_t \int_{\mathbb{R}^3} f_\varepsilon \, dv + \operatorname{div}_x \int_{\mathbb{R}^3} v f_\varepsilon \, dv = 0, \tag{C.1}$$

but whether the local conservation law of the momentum holds in the sense of distributions is still an open problem. We consider a particular type of weak solutions, i.e., the renormalized solutions for the VMFP equations, as introduced by DiPerna and Lions [22]. Their construction yields a solution which satisfies in addition a conservation law of momentum and a global energy equality with defect measures. The idea is to consider approximate solutions $(f_\varepsilon^n, E_\varepsilon^n, B_\varepsilon^n)_n$ (here ε is kept fixed) and to extract subsequences (still indexed by n) such that

$$f_\varepsilon^n \rightharpoonup f_\varepsilon, \quad \mathrm{w}\star{-}L^\infty\left(]0,T[; L^2\left(\mathbb{T}^3 \times \mathbb{R}^3\right)\right),$$
$$\left(E_\varepsilon^n, B_\varepsilon^n\right) \rightharpoonup (E_\varepsilon, B_\varepsilon), \quad \mathrm{w}\star{-}L^\infty\left(]0,T[; L^2\left(\mathbb{T}^3\right)\right)^6.$$

Therefore (after extraction eventually) there are symmetric nonnegative matrix-valued defect measures $\mu_E^\varepsilon, \mu_B^\varepsilon \in L^\infty(]0,T[; \mathcal{M}^1(\mathbb{T}^3))^9$, $\mu_{EB} \in L^\infty(]0,T[; \mathcal{M}^1(\mathbb{T}^3))^3$ such that for any $\varphi \in C^0([0,T] \times \mathbb{T}^3)$

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{T}^3} \left(E_\varepsilon^n \otimes E_\varepsilon^n\right)\varphi(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} (E_\varepsilon \otimes E_\varepsilon)\varphi(t,x) \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} \varphi(t,x) \, d\mu_E^\varepsilon,$$
$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{T}^3} \left(B_\varepsilon^n \otimes B_\varepsilon^n\right)\varphi(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} (B_\varepsilon \otimes B_\varepsilon)\varphi(t,x) \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} \varphi(t,x) \, d\mu_B^\varepsilon,$$
$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{T}^3} \left(E_\varepsilon^n \wedge B_\varepsilon^n\right)\varphi(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} (E_\varepsilon \wedge B_\varepsilon)\varphi(t,x) \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} \varphi(t,x) \, d\mu_{EB}^\varepsilon.$$

Observe also that the sequence $(\int_0^\infty r^4 f_\varepsilon^n(t,x,\sigma r)\, dr)_n$ is bounded in $L^\infty(]0,T[; \mathcal{M}_+^1(\mathbb{T}^3 \times \mathbb{S}^2))$ and therefore (after extraction eventually) there is a nonnegative measure $m_\varepsilon \in L^\infty(]0,T[; \mathcal{M}_+^1(\mathbb{T}^3 \times \mathbb{S}^2))$ such that for any $\psi \in C^0([0,T] \times \mathbb{T}^3 \times \mathbb{S}^2)$ we have

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon^n \psi\left(t,x,\frac{v}{|v|}\right) dv \, dx \, dt$$
$$= \int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \psi\left(t,x,\frac{v}{|v|}\right) dv \, dx \, dt + \int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{S}^2} \psi(t,x,\sigma) \, dm_\varepsilon. \tag{C.2}$$

Taking $\psi(t,x,\sigma) = \theta(t,x)(\sigma \otimes \sigma)$ we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} (v \otimes v) f_\varepsilon^n \, dv = \int_{\mathbb{R}^3} (v \otimes v) f_\varepsilon \, dv + \int_{\mathbb{S}^2} (\sigma \otimes \sigma) \, dm_\varepsilon,$$

in $\mathcal{D}'(]0,T[\times\mathbb{T}^3)$. Using the formula

$$\rho_\varepsilon^n E_\varepsilon^n + \alpha\varepsilon j_\varepsilon^n \wedge B_\varepsilon^n = DE_\varepsilon^n + \alpha\varepsilon J \wedge B_\varepsilon^n - \mathcal{A}\big(E_\varepsilon^n, E_\varepsilon^n\big) - \alpha\varepsilon\mathcal{A}\big(B_\varepsilon^n, B_\varepsilon^n\big) + \alpha\varepsilon\partial_t\big(E_\varepsilon^n \wedge B_\varepsilon^n\big),$$

we deduce that the limit solution $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ satisfies the local conservation law of momentum in the sense of distributions on $[0,T[\times\mathbb{T}^3$

$$\begin{aligned}
&\varepsilon\partial_t \int_{\mathbb{R}^3} v f_\varepsilon\,dv + \varepsilon\operatorname{div}_x\bigg(\int_{\mathbb{R}^3}(v\otimes v) f_\varepsilon\,dv + \int_{\mathbb{S}^2}(\sigma\otimes\sigma)\,dm_\varepsilon\bigg) + DE_\varepsilon + \alpha\varepsilon J\wedge B_\varepsilon \\
&\quad - \mathcal{A}(E_\varepsilon, E_\varepsilon) - \alpha\varepsilon\mathcal{A}(B_\varepsilon, B_\varepsilon) + \alpha\varepsilon\partial_t(E_\varepsilon\wedge B_\varepsilon) \\
&\quad - \operatorname{div}_x\bigg(\mu_E^\varepsilon - \frac{1}{2}\operatorname{tr}\big(\mu_E^\varepsilon\big)I_3\bigg) - \alpha\varepsilon\operatorname{div}_x\bigg(\mu_B^\varepsilon - \frac{1}{2}\operatorname{tr}\big(\mu_B^\varepsilon\big)I_3\bigg) + \alpha\varepsilon\partial_t\mu_{EB} \\
&= -\int_{\mathbb{R}^3} v f_\varepsilon\,dv.
\end{aligned} \tag{C.3}$$

In the above formula the terms $\mathcal{A}(E_\varepsilon, E_\varepsilon)$, $\mathcal{A}(B_\varepsilon, B_\varepsilon)$ must be understood in the sense of distributions accordingly to the formula

$$\mathcal{A}(E_\varepsilon, E_\varepsilon) = \operatorname{div}_x\bigg(E_\varepsilon\otimes E_\varepsilon - \frac{1}{2}|E_\varepsilon|^2 I_3\bigg), \qquad \mathcal{A}(B_\varepsilon, B_\varepsilon) = \operatorname{div}_x\bigg(B_\varepsilon\otimes B_\varepsilon - \frac{1}{2}|B_\varepsilon|^2 I_3\bigg).$$

For further computations it is convenient to transform (C.3) using the identities in $\mathcal{D}'([0,T[\times\mathbb{T}^3)$

$$\begin{aligned}
\alpha\varepsilon\partial_t(E_\varepsilon\wedge B_\varepsilon) &= \alpha\varepsilon\partial_t\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big) + \alpha\varepsilon\partial_t(E\wedge B) \\
&\quad + \alpha\varepsilon\big(j_\varepsilon + \rho E + \operatorname{curl}_x(B_\varepsilon - B)\big)\wedge B + \alpha\varepsilon(E_\varepsilon - E)\wedge\partial_t B \\
&\quad + \alpha\varepsilon\partial_t E\wedge(B_\varepsilon - B) + E\wedge\big(-\alpha\varepsilon\partial_t B - \operatorname{curl}_x(E_\varepsilon - E)\big),
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
\mathcal{A}(E_\varepsilon, E_\varepsilon) &= \mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) + \mathcal{A}(E, E) + (E_\varepsilon - E)\operatorname{div}_x E \\
&\quad + E\operatorname{div}_x(E_\varepsilon - E) - E\wedge\operatorname{curl}_x(E_\varepsilon - E),
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
\alpha\varepsilon\mathcal{A}(B_\varepsilon, B_\varepsilon) &= \alpha\varepsilon\mathcal{A}(B_\varepsilon - B, B_\varepsilon - B) + \alpha\varepsilon\mathcal{A}(B, B) - \alpha\varepsilon(B_\varepsilon - B)\wedge\operatorname{curl}_x B \\
&\quad - \alpha\varepsilon B\wedge\operatorname{curl}_x(B_\varepsilon - B),
\end{aligned} \tag{C.6}$$

$$\alpha\varepsilon\partial_t(E\wedge B) - \mathcal{A}(E, E) - \alpha\varepsilon\mathcal{A}(B, B) = E(\rho - D) - \alpha\varepsilon(J + \rho E)\wedge B + \alpha\varepsilon E\wedge\partial_t B. \tag{C.7}$$

Combining (C.4)–(C.7), (28), (29) we obtain

$$\begin{aligned}
&\alpha\varepsilon\partial_t(E_\varepsilon\wedge B_\varepsilon) - \mathcal{A}(E_\varepsilon, E_\varepsilon) - \alpha\varepsilon\mathcal{A}(B_\varepsilon, B_\varepsilon) \\
&= \alpha\varepsilon\partial_t\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big) - \mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) - \alpha\varepsilon\mathcal{A}(B_\varepsilon - B, B_\varepsilon - B) \\
&\quad - (D - \rho)E_\varepsilon - \alpha\varepsilon(J + \rho E)\wedge B_\varepsilon + \alpha\varepsilon(E_\varepsilon - E)\wedge\partial_t B - E\operatorname{div}_x(E_\varepsilon - E) + \alpha\varepsilon(j_\varepsilon + \rho E)\wedge B.
\end{aligned}$$

Therefore the conservation law of momentum (C.3) can be written

$$\begin{aligned}
&\varepsilon\partial_t \int_{\mathbb{R}^3} v f_\varepsilon\,dv + \varepsilon\operatorname{div}_x\bigg(\int_{\mathbb{R}^3}(v\otimes v) f_\varepsilon\,dv + \int_{\mathbb{S}^2}(\sigma\otimes\sigma)\,dm_\varepsilon\bigg) \\
&\quad + \alpha\varepsilon\partial_t\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big) - \mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) - \alpha\varepsilon\mathcal{A}(B_\varepsilon - B, B_\varepsilon - B) \\
&\quad + \rho E_\varepsilon - \alpha\varepsilon\rho E\wedge B_\varepsilon + \alpha\varepsilon(E_\varepsilon - E)\wedge\partial_t B - E(\rho - \rho_\varepsilon) + \alpha\varepsilon(j_\varepsilon + \rho E)\wedge B \\
&\quad - \operatorname{div}_x\bigg(\mu_E^\varepsilon - \frac{1}{2}\operatorname{tr}\big(\mu_E^\varepsilon\big)I_3\bigg) - \alpha\varepsilon\operatorname{div}_x\bigg(\mu_B^\varepsilon - \frac{1}{2}\operatorname{tr}\big(\mu_B^\varepsilon\big)I_3\bigg) + \alpha\varepsilon\partial_t\mu_{EB} \\
&= -\int_{\mathbb{R}^3} v f_\varepsilon\,dv.
\end{aligned} \tag{C.8}$$

Similarly the limit solution $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ satisfies the free-energy decay

$$\begin{aligned}
&\varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\ln f_\varepsilon + \frac{1}{2}|v|^2\right) f_\varepsilon \, dv\, dx + \frac{1}{2}\int_{\mathbb{T}^3} \left(|E_\varepsilon|^2 + \alpha\varepsilon |B_\varepsilon|^2\right) dx + \int_0^t \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left|v\sqrt{f_\varepsilon} + 2\nabla_v f_\varepsilon\right|^2 dv\, dx\, ds \\
&\quad + \frac{\varepsilon}{2}\int_{\mathbb{T}^3}\int_{\mathbb{S}^2} dm_\varepsilon + \frac{1}{2}\int_{\mathbb{T}^3} d\left(\operatorname{tr}\left(\mu_E^\varepsilon\right) + \alpha\varepsilon \operatorname{tr}\left(\mu_B^\varepsilon\right)\right) + \int_0^t\int_{\mathbb{T}^3} J\cdot E_\varepsilon \, dx\, ds \\
&\quad \leqslant \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\ln f_\varepsilon^0 + \frac{1}{2}|v|^2\right) f_\varepsilon^0 \, dv\, dx + \frac{1}{2}\int_{\mathbb{T}^3} \left(\left|E_\varepsilon^0\right|^2 + \alpha\varepsilon \left|B_\varepsilon^0\right|^2\right) dx, \quad t\in[0,T]. \qquad \text{(C.9)}
\end{aligned}$$

We call renormalized solution of the VMFP system a weak solution satisfying (C.3) and (C.9). The above arguments allow to construct a renormalized solution $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ on any time interval $[0, T]$ and for any $\varepsilon > 0$. In order to study the asymptotic behavior of these solutions as ε goes to zero, we introduce the relative entropy with defect measures.

$$\begin{aligned}
\tilde{\mathcal{H}}_\varepsilon(t) &= \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln \frac{f_\varepsilon}{\rho M_E} \, dv\, dx + \frac{1}{2}\int_{\mathbb{T}^3} \left(|E_\varepsilon - E|^2 + \alpha\varepsilon |B_\varepsilon - B|^2\right) dx \\
&\quad + \frac{\varepsilon}{2}\int_{\mathbb{T}^3}\int_{\mathbb{S}^2} dm_\varepsilon + \frac{1}{2}\int_{\mathbb{T}^3} d\left(\operatorname{tr}\left(\mu_E^\varepsilon\right) + \alpha\varepsilon \operatorname{tr}\left(\mu_B^\varepsilon\right)\right) \\
&= \mathcal{H}_\varepsilon(t) + \frac{\varepsilon}{2}\int_{\mathbb{T}^3}\int_{\mathbb{S}^2} dm_\varepsilon + \frac{1}{2}\int_{\mathbb{T}^3} d\left(\operatorname{tr}\left(\mu_E^\varepsilon\right) + \alpha\varepsilon \operatorname{tr}\left(\mu_B^\varepsilon\right)\right).
\end{aligned}$$

We have

$$f_\varepsilon \ln \frac{f_\varepsilon}{\rho M_E} = f_\varepsilon \left(\ln f_\varepsilon + \frac{1}{2}|v|^2\right) + f_\varepsilon \left(v\cdot E + \frac{1}{2}|E|^2\right) - f_\varepsilon \ln \frac{\rho}{(2\pi)^{3/2}},$$

and thus in order to evaluate $\tilde{\mathcal{H}}_\varepsilon$ we need to compute $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |E|^2 f_\varepsilon \, dv\, dx$, $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} E\cdot v f_\varepsilon \, dv\, dx$, $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln\rho \, dv\, dx$ and to combine with (C.9). This will be done by using the conservation laws of charge (C.1) and momentum (C.8). We obtain the following equalities in $\mathcal{D}'([0, T[)$

$$\varepsilon \frac{d}{dt}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{1}{2}|E|^2 f_\varepsilon \, dv\, dx - \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} E\cdot\left(\partial_t E + (\mathrm{D}_x E)v\right) f_\varepsilon \, dv\, dx = 0, \qquad \text{(C.10)}$$

$$\begin{aligned}
&\varepsilon \frac{d}{dt}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} v\cdot E f_\varepsilon \, dv\, dx - \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\partial_t E + (\mathrm{D}_x E)v\right)\cdot v f_\varepsilon \, dv\, dx - \varepsilon \int_{\mathbb{T}^3}\int_{\mathbb{S}^2} \sigma\cdot(\mathrm{D}_x E)\sigma \, dm_\varepsilon \\
&\quad + \alpha\varepsilon \frac{d}{dt}\int_{\mathbb{T}^3} \left((E_\varepsilon - E)\wedge(B_\varepsilon - B)\right)\cdot E \, dx - \alpha\varepsilon \int_{\mathbb{T}^3} \left((E_\varepsilon - E)\wedge(B_\varepsilon - B)\right)\cdot \partial_t E \, dx \\
&\quad - \int_{\mathbb{T}^3} \left(\mathcal{A}(E_\varepsilon - E, E_\varepsilon - E) + \alpha\varepsilon \mathcal{A}(B_\varepsilon - B, B_\varepsilon - B)\right)\cdot E \, dx \\
&\quad + \int_{\mathbb{T}^3} \mathrm{D}_x E : \left(\mu_E^\varepsilon - \frac{1}{2}\operatorname{tr}\left(\mu_E^\varepsilon\right) I_3\right) dx + \alpha\varepsilon \int_{\mathbb{T}^3} \mathrm{D}_x E : \left(\mu_B^\varepsilon - \frac{1}{2}\operatorname{tr}\left(\mu_B^\varepsilon\right) I_3\right) dx \\
&\quad + \int_{\mathbb{T}^3} \rho E_\varepsilon \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{T}^3} \left((E_\varepsilon - E)\wedge \partial_t B\right)\cdot E \, dx - \int_{\mathbb{T}^3} |E|^2(\rho - \rho_\varepsilon) \, dx \\
&\quad + \alpha\varepsilon \int_{\mathbb{T}^3} (j_\varepsilon \wedge B)\cdot E \, dx + \alpha\varepsilon \frac{d}{dt}\int_{\mathbb{T}^3} E\cdot d\mu_{EB}^\varepsilon - \alpha\varepsilon \int_{\mathbb{T}^3} \partial_t E\cdot d\mu_{EB}^\varepsilon
\end{aligned}$$

$$= -\int_{\mathbb{T}^3} j_\varepsilon \cdot E\,dx. \tag{C.11}$$

By standard computations using the Maxwell equations we obtain in $\mathcal{D}'([0,T[)$

$$\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}|E|^2 - E_\varepsilon\cdot E + \frac{\alpha\varepsilon}{2}|B|^2 - \alpha\varepsilon B_\varepsilon\cdot B\right)dx = \int_{\mathbb{T}^3}\big((j-j_\varepsilon)\cdot E - \alpha\varepsilon\partial_t B\cdot(B_\varepsilon - B)\big)\,dx - \int_{\mathbb{T}^3}(j-J)\cdot E_\varepsilon\,dx. \tag{C.12}$$

Summing up (C.9)–(C.11), (C.12) (the last three equalities being integrated over $[0,t]$) yields after elementary manipulations

$$\begin{aligned}
&\varepsilon\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\left(\ln f_\varepsilon + \frac{1}{2}|v+E|^2\right)f_\varepsilon\,dv\,dx + \frac{1}{2}\int_{\mathbb{T}^3}\big(|E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2\big)\,dx\\
&\quad + \int_0^t\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}|q_\varepsilon|^2\,dv\,dx\,ds + \frac{\varepsilon}{2}\int_{\mathbb{T}^3}\int_{\mathbb{S}^2}dm_\varepsilon + \frac{1}{2}\int_{\mathbb{T}^3}d\big(\mathrm{tr}(\mu_E^\varepsilon) + \alpha\varepsilon\,\mathrm{tr}(\mu_B^\varepsilon)\big)\\
&\quad + \alpha\varepsilon\int_{\mathbb{T}^3}\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big)\cdot E\,dx + \alpha\varepsilon\int_{\mathbb{T}^3}E\cdot d\mu_{EB}^\varepsilon\,dx\\
&\leqslant \varepsilon\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\left(\ln f_\varepsilon^0 + \frac{1}{2}|v+E|^2\right)f_\varepsilon^0\,dv\,dx + \frac{1}{2}\int_{\mathbb{T}^3}\big(\big|E_\varepsilon^0 - E^0\big|^2 + \alpha\varepsilon\big|B_\varepsilon^0 - B^0\big|^2\big)\,dx\\
&\quad + \alpha\varepsilon\int_{\mathbb{T}^3}\big(\big(E_\varepsilon^0 - E^0\big)\wedge\big(B_\varepsilon^0 - B^0\big)\big)\cdot E^0\,dx + \varepsilon\int_0^t\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\big(\partial_t E + (\mathrm{D}_x E)v\big)\cdot(v+E)f_\varepsilon\,dv\,dx\,ds\\
&\quad - \int_0^t\int_{\mathbb{T}^3}\left((E_\varepsilon - E)\otimes(E_\varepsilon - E) - \frac{1}{2}|E_\varepsilon - E|^2 I_3\right):\mathrm{D}_x E\,dx\,ds\\
&\quad - \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\left((B_\varepsilon - B)\otimes(B_\varepsilon - B) - \frac{1}{2}|B_\varepsilon - B|^2 I_3\right):\mathrm{D}_x E\,dx\,ds\\
&\quad + \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\big((E_\varepsilon - E)\wedge(B_\varepsilon - B)\big)\cdot\partial_t E\,dx\,ds + \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\partial_t E\cdot d\mu_{EB}^\varepsilon\,dx\,ds\\
&\quad - \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\partial_t B\cdot(B_\varepsilon - B)\,dx\,ds - \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\big((E_\varepsilon - E)\wedge\partial_t B\big)\cdot E\,dx\,ds\\
&\quad - \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}(j_\varepsilon\wedge B)\cdot E\,dx\,ds + \varepsilon\int_0^t\int_{\mathbb{T}^3}\int_{\mathbb{S}^2}\sigma\cdot(\mathrm{D}_x E)\sigma\,dm_\varepsilon\\
&\quad - \int_0^t\int_{\mathbb{T}^3}\mathrm{D}_x E:\left(\mu_E^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_E^\varepsilon\big)I_3\right) - \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\mathrm{D}_x E:\left(\mu_B^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_B^\varepsilon\big)I_3\right).
\end{aligned} \tag{C.13}$$

It is easily seen, by introducing the vector potential U that

$$-\int_{\mathbb{T}^3} \partial_t B\cdot(B_\varepsilon - B)\,dx - \int_0^t\int_{\mathbb{T}^3}\big((E_\varepsilon - E)\wedge\partial_t B\big)\cdot E\,dx\,ds - \int_0^t\int_{\mathbb{T}^3}(j_\varepsilon\wedge B)\cdot E\,dx\,ds$$
$$= \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}(\partial_t U + E\wedge B)\cdot(v+E) f_\varepsilon\,dv\,dx - \int_{\mathbb{T}^3}\big(\nabla_x(\partial_t U\cdot E) - E\wedge\partial_t B\big)\cdot(E_\varepsilon - E)\,dx$$
$$- \frac{d}{dt}\int_{\mathbb{T}^3}\partial_t U\cdot(E_\varepsilon - E)\,dx + \int_{\mathbb{T}^3}\partial_t^2 U\cdot(E_\varepsilon - E)\,dx. \tag{C.14}$$

Combining (C.13), (C.14) yields the analogous version of (34) (integrated over $[0,t]$) with defect measure terms. It remains to add the contribution of $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln((2\pi)^{-3/2}\rho)\,dv\,dx$ which is obtained by using one more time the charge conservation law

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon \ln\frac{\rho}{(2\pi)^{3/2}}\,dv\,dx = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \ln\frac{\rho^0}{(2\pi)^{3/2}}\,dv\,dx + \int_0^t\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f_\varepsilon(\partial_t + v\cdot\nabla_x)\ln\rho\,dv\,dx\,ds.$$

Performing now the same computations as in the proof of Proposition 6 leads to a relative entropy balance similar to (26) with the following additional defect terms in the left-hand side

$$\frac{\varepsilon}{2}\int_{\mathbb{T}^3}\int_{\mathbb{S}^2} dm_\varepsilon + \frac{1}{2}\int_{\mathbb{T}^3} d\big(\mathrm{tr}(\mu_E^\varepsilon) + \alpha\varepsilon\,\mathrm{tr}(\mu_B^\varepsilon)\big) + \alpha\varepsilon\int_{\mathbb{T}^3} E(t,x)\cdot d\mu_{EB}^\varepsilon, \tag{C.15}$$

and the additional defect terms in the right-hand side

$$\alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\partial_t E\cdot d\mu_{EB}^\varepsilon + \varepsilon\int_0^t\int_{\mathbb{T}^3}\int_{\mathbb{S}^2}\sigma\cdot(\mathrm{D}_x E)\sigma\,dm_\varepsilon - \int_0^t\int_{\mathbb{T}^3}\mathrm{D}_x E : \left(\mu_E^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_E^\varepsilon\big) I_3\right)$$
$$- \alpha\varepsilon\int_0^t\int_{\mathbb{T}^3}\mathrm{D}_x E : \left(\mu_B^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_B^\varepsilon\big) I_3\right). \tag{C.16}$$

From now on we can use the same arguments as in the case of smooth solutions. The only new thing to do is to observe that the above defect terms appearing under the time integration sign in (C.16) are dominated by the defect terms in $\tilde{\mathcal{H}}_\varepsilon$ (see (C.15)). Taking into account that for any matrix $A \in C^0(\mathbb{T}^3)^9$ we have the inequalities

$$\left|\int_{\mathbb{T}^3} A(x) : d\mu_E^\varepsilon\right| \leqslant \int_{\mathbb{T}^3}\big|A(x)\big|\,d\,\mathrm{tr}\big(\mu_E^\varepsilon\big), \qquad \left|\int_{\mathbb{T}^3} A(x) : d\mu_B^\varepsilon\right| \leqslant \int_{\mathbb{T}^3}\big|A(x)\big|\,d\,\mathrm{tr}\big(\mu_B^\varepsilon\big),$$

we deduce that

$$\left|\int_{\mathbb{T}^3}\mathrm{D}_x E : \left(\mu_E^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_E^\varepsilon\big) I_3\right) + \alpha\varepsilon\int_{\mathbb{T}^3}\mathrm{D}_x E : \left(\mu_B^\varepsilon - \frac{1}{2}\mathrm{tr}\big(\mu_B^\varepsilon\big) I_3\right)\right| \leqslant C\left(\int_{\mathbb{T}^3} d\big(\mathrm{tr}\big(\mu_E^\varepsilon\big) + \alpha\varepsilon\,\mathrm{tr}\big(\mu_B^\varepsilon\big)\big)\right)$$
$$\leqslant C\tilde{\mathcal{H}}_\varepsilon(t).$$

Observe now that for any vector $a \in C^0(\mathbb{T}^3)^3$ we have

$$\sqrt{\alpha\varepsilon}\left|\int_{\mathbb{T}^3} a(x)\cdot d\mu_{EB}^\varepsilon\right| \leqslant \frac{1}{2}\int_{\mathbb{T}^3}\big|a(x)\big|\,d\,\mathrm{tr}\big(\mu_E^\varepsilon\big) + \frac{1}{2}\int_{\mathbb{T}^3}\big|a(x)\big|\alpha\varepsilon\,d\,\mathrm{tr}\big(\mu_B^\varepsilon\big),$$

implying that

$$\alpha\varepsilon\left|\int_{\mathbb{T}^3}\partial_t E\cdot d\mu_{EB}^\varepsilon\right| \leqslant C\sqrt{\alpha\varepsilon}\int_{\mathbb{T}^3} d\big(\mathrm{tr}\big(\mu_E^\varepsilon\big) + \alpha\varepsilon\,\mathrm{tr}\big(\mu_B^\varepsilon\big)\big).$$

Eventually observe that

$$\left|\int_{\mathbb{T}^3}\int_{\mathbb{S}^2} \sigma \cdot (\mathrm{D}_x E)\sigma \, dm_\varepsilon\right| \leqslant C \int_{\mathbb{T}^3}\int_{\mathbb{S}^2} dm_\varepsilon.$$

Finally one gets the inequality

$$\tilde{\mathcal{H}}_\varepsilon(t) + \frac{1}{2}\int_0^t \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |q_\varepsilon|^2 \, dv \, dx \, ds \leqslant C\big(\tilde{\mathcal{H}}_\varepsilon(0) + \varepsilon^2\big) + C\int_0^t \tilde{\mathcal{H}}_\varepsilon(s)\, ds,$$

and we conclude by the Gronwall lemma.

References

[1] A. Arnold, J.-A. Carrillo, I. Gamba, C.-W. Shu, Low and high field scaling limits for the Vlasov– and Wigner–Poisson–Fokker–Planck system, Transport Theory Statist. Phys. 30 (2–3) (2001) 121–153.
[2] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differential Equations 26 (1–2) (2001) 43–100.
[3] D. Bakry, M. Emery, Hypercontractivité de semi-groupes de diffusion, C. R. Acad. Sci. Paris Sér. I Math. 299 (15) (1984) 775–778.
[4] C. Bardos, F. Golse, C.D. Levermore, Fluid dynamic limits of kinetic equations II. Convergence proofs for the Boltzmann equation, Comm. Pure Appl. Math. XLVI (1993) 667–753.
[5] N. Ben Abdallah, P. Degond, P. Markowich, C. Schmeiser, High field approximation of the spherical harmonics expansion model for semiconductors, Z. Angew. Math. Phys. 52 (2) (2001) 201–230.
[6] A. Bers, J.-L. Delcroix, Physique des plasmas, EDP Sciences, 2000.
[7] F. Berthelin, A. Vasseur, From kinetic equations to multidimensional isentropic gas dynamics before shocks, SIAM J. Math. Anal. 36 (6) (2005) 1807–1835.
[8] M. Bostan, Th. Goudon, Low field regime for the relativistic Vlasov–Maxwell–Fokker–Planck system; the one and one half dimensional case, Kinetic Related Models 1 (1) (2008) 139–169.
[9] F. Bouchut, Existence and uniqueness of a global smooth solution for the Vlasov–Poisson–Fokker–Planck system in three dimensions, J. Funct. Anal. 111 (1993) 239–258.
[10] F. Bouchut, Smoothing effect for the nonlinear Vlasov–Poisson–Fokker–Planck system, J. Differential Equations 122 (1995) 225–238.
[11] F. Bouchut, F. Golse, C. Pallard, Classical solutions and the Glassey–Strauss theorem for the 3D Vlasov–Maxwell system, Arch. Ration. Mech. Anal. 170 (1) (2003) 1–15.
[12] N. Bourbaki, Éléments de Mathématiques, Fascicule XXXV, Livre VI, Chapitre IX, Intégration, Hermann, Paris, 1969.
[13] Y. Brenier, Convergence of the Vlasov–Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 25 (2000) 737–754.
[14] Y. Brenier, N. Mauser, M. Puel, Incompressible Euler and e-MHD as scaling limits of the Vlasov–Maxwell system, Commun. Math. Sci. 1 (3) (2003) 437–447.
[15] J.-A. Carrillo, J. Soler, On the initial value problem for the Vlasov–Poisson–Fokker–Planck system with initial data in L^p spaces, Math. Methods Appl. Sci. 18 (10) (1995) 825–839.
[16] J.-A. Carrillo, S. Labrunie, Global solutions for the one-dimensional Vlasov–Maxwell system for laser-plasma interaction, Math. Models Methods Appl. Sci. 16 (1) (2006) 19–57.
[17] C. Cercignani, I.M. Gamba, C.D. Levermore, High field approximations to a Boltzmann–Poisson system and boundary conditions in a semiconductor, Appl. Math. Lett. 10 (4) (1997) 111–117.
[18] S. Chandrasekhar, Brownian motion, dynamical friction and stellar dynamics, Rev. Mod. Phys. 21 (1949) 383–388.
[19] I. Csiszar, Information-type measures of difference of probability distributions and indirect observations, Studia Sci. Math. Hungar. 2 (1967) 299–318.
[20] P. Degond, Global existence of smooth solutions for the Vlasov–Fokker–Planck equations in 1 and 2 space dimensions, Ann. Scient. Ecole Normale Sup. 19 (1986) 519–542.
[21] P. Degond, A. Jungel, High field approximation of the energy-transport model for semiconductors with non-parabolic band structure, Z. Angew. Math. Phys. 52 (6) (2001) 1053–1070.
[22] R. DiPerna, P.-L. Lions, Global weak solutions of Vlasov–Maxwell systems, Comm. Pure Appl. Math. 42 (6) (1989) 729–757.
[23] B. Dubroca, R. Duclous, F. Filbet, V. Tikhonchuk, High order resolution of the Maxwell–Fokker–Planck–Landau model intended for ICF/Fast ignition applications, CELIA-Université Bordeaux 1, in preparation.
[24] R. Glassey, W. Strauss, Singularity formation in a collisionless plasma could occur only at high velocities, Arch. Ration. Mech. Anal. 9 (1) (1986) 59–90.
[25] F. Golse, L. Saint-Raymond, The Vlasov–Poisson system with strong magnetic field in quasineutral regime, Math. Models Methods Appl. Sci. 13 (5) (2003) 661–714.
[26] T. Goudon, J. Nieto, F. Poupaud, J. Soler, Multidimensional high-field limit of the electrostatic Vlasov–Poisson–Fokker–Planck system, J. Differential Equations 213 (2) (2005) 418–442.

[27] T. Goudon, Hydrodynamic limit for the Vlasov–Poisson–Fokker–Planck system: Analysis of the two-dimensional case, Math. Models Methods Appl. Sci. 15 (5) (2005) 737–752.
[28] T. Goudon, P.-E. Jabin, A. Vasseur, Hydrodynamic limits for the Vlasov–Navier–Stokes equations. Part II: Fine particles regime, Indiana Univ. Math. J. 53 (2004) 1517–1536.
[29] Y. Guo, The Vlasov–Maxwell–Boltzmann system near Maxwellians, Invent. Math. 153 (3) (2003) 593–630.
[30] V. Grandgirard, Y. Sarrazin, X. Garbet, G. Dif-Pradalier, P. Ghendrih, N. Crouseilles, G. Latu, E. Sonnendrucker, N. Besse, P. Bertrand, GYSELA, a full-f global gyrokinetic semi-Lagrangian code for ITG turbulence simulations, in: Proceedings of Theory of Fusion Plasmas, Varenna, 2006.
[31] S. Klainerman, G. Staffilani, A new approach to study the Vlasov–Maxwell system, Comm. Pure Appl. Anal. 1 (1) (2002) 103–125.
[32] S. Kullback, A lower bound for discrimination information in terms of variation, IEEE Trans. Inform. Theory 4 (1967) 126–127.
[33] R. Lai, On the one and one-half dimensional relativistic Vlasov–Maxwell–Fokker–Planck system with non-vanishing viscosity, Math. Meth. Appl. Sci. 21 (1998) 1287–1296.
[34] P. Markowich, C. Ringhofer, Quantum hydrodynamics for semiconductors in the high field case, Appl. Math. Lett. 7 (5) (1994) 37–41.
[35] J. Nieto, F. Poupaud, J. Soler, High-field limit of the Vlasov–Poisson–Fokker–Planck system, Arch. Ration. Mech. Anal. 158 (2001) 29–59.
[36] B. O'Dwyer, H.D. Victory Jr., On classical solutions of the Vlasov–Poisson–Fokker–Planck system, Indiana Univ. Math. J. 39 (1) (1990) 105–156.
[37] F. Poupaud, Runaway phenomena and fluid approximation under high fields in semiconductors kinetic theory, Z. Angew. Math. Mech. 72 (1992) 359–372.
[38] F. Poupaud, J. Soler, Parabolic limit and stability of the Vlasov–Poisson–Fokker–Planck system, Math. Models Methods Appl. Sci. 10 (7) (2000) 1027–1045.
[39] M. Puel, L. Saint-Raymond, Quasineutral limit for the relativistic Vlasov–Maxwell system, Asymptotic Anal. 40 (2004) 303–352.
[40] L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, Arch. Ration. Mech. Anal. 166 (2003) 47–80.
[41] A. Vasseur, Recent results on hydrodynamic limits, in: C.M. Dafermos, M. Pokorny (Eds.), Handbook of Differential Equations: Evolutionary Equations, vol. 4, Elsevier, 2008.
[42] H.D. Victory Jr., On the existence of global weak solutions for the Vlasov–Poisson–Fokker–Planck system, J. Math. Anal. Appl. 160 (2) (1991) 525–555.
[43] S. Wollman, An existence and uniqueness theorem for the Vlasov–Maxwell system, Comm. Pure Appl. Math. 37 (1984) 457–462.
[44] H.T. Yau, Relative entropy and hydrodynamics of Ginzburg–Landau models, Lett. Math. Phys. 22 (1) (1991) 63–80.
[45] H. Yu, Global classical solution of the Vlasov–Maxwell–Landau system near Maxwellians, J. Math. Phys. 45 (11) (2004) 4360–4376.