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A regularity criterion for the dissipative quasi-geostrophic equations ☆

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Abstract

We establish a regularity criterion for weak solutions of the dissipative quasi-geostrophic equations (with dissipation $(-\Delta)^{\gamma/2}$, $0 < \gamma \leqslant 1$). More precisely, we show that if $\theta \in L_t^{r_0}((0,T); B_{p,\infty}^{\alpha}(\mathbb{R}^2))$ with $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}$ is a weak solution of the 2D quasi-geostrophic equation, then θ is a classical solution in $(0,T] \times \mathbb{R}^2$. This result extends the regularity result of Constantin and Wu [P. Constantin, J. Wu, Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation, Ann. I. H. Poincaré – AN (2007), doi:10.1016/j.anihpc.2007.10.001] to scaling invariant spaces.

MSC: 35Q35

Keywords: Regularity criteria; Quasi-geostrophic equations

1. Introduction

In this paper we obtain a regularity criterion for weak solutions of the 2D dissipative quasi-geostrophic equations. We consider the following initial value problem

$$\begin{cases} \theta_t + u \cdot \nabla\theta + (-\Delta)^{\gamma/2}\theta = 0, & x \in \mathbb{R}^2,\ t \in (0,\infty), \\ \theta(0,x) = \theta_0(x), \end{cases} \tag{1.1}$$

where $\gamma \in (0,2]$ is a fixed parameter and the velocity $u = (u_1, u_2)$ is divergence free and determined by the Riesz transforms of the potential temperature θ:

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) = \big(-\partial_{x_2}(-\Delta)^{-1/2}\theta, \partial_{x_1}(-\Delta)^{-1/2}\theta\big).$$

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The 2D quasi-geostrophic equation is an important model in geophysical fluid dynamics used in meteorology and oceanography (see, for example, Pedlosky [25]). It is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency.

The main mathematical question concerning the initial value problem (1.1) is whether there exists a global in time smooth solution to (1.1) evolving from any given smooth initial data. Before we recall the known results in this direction we note that cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$ are called subcritical, critical and supercritical, respectively. Existence of a global weak solution was established by Resnick [27]. Furthermore, in the subcritical case, Constantin and Wu [10] proved that every sufficiently smooth initial data give rise to a unique global smooth solution. In the critical case, $\gamma = 1$, Constantin, Cordoba and Wu [8] established the existence of a unique global classical solution corresponding to any initial data that are small in L^∞. The assumption requiring smallness in L^∞ was removed by Caffarelli and Vasseur [1], Kiselev, Nazarov and Volberg [22] and Dong and Du [16]. In [22] the authors proved persistence of a global solution in C^∞ corresponding to any C^∞ periodic initial data. Dong and Du in [16] adapted the method of [22] and obtained global well-posedness for the critical 2D dissipative quasi-geostrophic equations with H^1 initial data in the whole space. On the other hand, Caffarelli and Vasseur established regularity of Leray–Hopf solution by proving the following three claims:

(1) Every Leray–Hopf weak solution corresponding to initial data $\theta_0 \in L^2$ is in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \times (0,\infty))$.
(2) The L^∞ solutions are Hölder regular i.e. they are in C^γ for some $\gamma > 0$.
(3) Every Hölder regular solution is a classical solution in $C^{1,\beta}$.

While the main question addressing global in time existence is settled in the critical case, it still remains open in the supercritical case, $\gamma < 1$. In this case Chae and Lee [4], Wu [29,31], Chen, Miao and Zhang [6], Hmidi and Keraani [20] established existence of a global solution in Besov spaces evolving from small initial data (see also [24,21]). Recently, Constantin and Wu in [11] implemented the approach of [1] in the supercritical case. They proved that every Leray–Hopf weak solution corresponding to initial data $\theta_0 \in L^2$ is in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \times (0,\infty))$ and hence the claim (1) is valid in the supercritical case. Concerning an analogue of the claim (2), Constantin and Wu in [11] proved that L^∞ solutions are Hölder continuous under the additional assumption that the velocity $u \in C^{1-\gamma}$. In a separate paper [12] Constantin and Wu considered the step (3) of the above approach and established a conditional regularity result of the type: if a Leray–Hopf solution is in the sub-critical space $L^\infty((t_0,t_1); C^\delta(\mathbb{R}^2))$ for some $\delta > 1-\gamma$ on the time interval $[t_0,t_1]$, then such a solution is a classical solution on $(t_0,t_1]$.

In this paper we extend the conditional regularity result of [12] to scaling invariant mixed time–space Besov spaces $L^{r_0}((0,T); B^\alpha_{p,\infty})$ with

$$\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}. \tag{1.2}$$

More precisely, we show that if

$$\theta \in L_t^{r_0}\big((0,T); B^\alpha_{p,\infty}\big(\mathbb{R}^2\big)\big)$$

is a weak solution of the 2D quasi-geostrophic equation (1.1), then θ is a classical solution of (1.1) in $(0,T] \times \mathbb{R}^2$. Significance of this space is that it is a critical space, by which we mean scaling invariant under the scaling transformation

$$\theta_\lambda = \lambda^{\gamma-1}\theta\big(\lambda x, \lambda^\gamma t\big).$$

Since the following embedding relations

$$L_t^\infty L_x^2 \cap L_t^\infty C_x^\delta \hookrightarrow L_t^\infty L_x^2 \cap L_t^\infty \dot{B}_{p,\infty}^{\delta(1-\frac{2}{p})} \hookrightarrow L_t^{r_0} B^\alpha_{p,\infty}$$

hold for sufficiently large p and r_0, our regularity result can be understood as an extension of the regularity result of Constantin and Wu [12] to critical spaces.

In order to prove the regularity result we first establish local existence and uniqueness of weak solutions to (1.1) in certain mixed time–space Besov spaces of Chemin type $\tilde{L}^r B^\alpha_{p,q}$ (for a definition of this space, see Section 2). We prove such existence and uniqueness results following the approach of Chen et al. [6]. We choose α according to (1.2) which in turn implies that the space $B^{\alpha}_{p,q}$ itself is subcritical. Therefore the time of existence depends only on the norm of the initial data and not on the profile. We combine the local existence (stated in Proposition 3.1) and uniqueness of weak solutions (stated in Proposition 3.3) to prove regularity by using a contradiction argument in the spirit of the work of Giga [19] in the context of the Navier–Stokes equations.

We recall that the first conditional regularity result for solutions to (1.1) was obtained by Constantin, Majda and Tabak [9]. Recently Chae established a conditional regularity result in Sobolev spaces in [3] and in Triebel–Lizorkin spaces in [2], while Dong and Chen in [14] extended the regularity criterion of Chae [3] to Besov spaces by proving that a solution to (1.1) is regular on the time interval $(0,T]$ if

$$\nabla\theta \in L^r\big((0,T);\dot{B}^0_{p,\infty}\big) \quad \text{with } \frac{2}{p}+\frac{\gamma}{r}=\gamma,\ \frac{4}{\gamma}\leqslant p\leqslant\infty.$$

In comparison with [14] we require less regularity for θ. We note that these conditional regularity results are in the spirit of the conditional regularity results available for the 3D Navier–Stokes equations e.g. [23,26,28,18,7].

Organization of the paper

The paper is organized as follows. In Section 2 we introduce the notation that shall be used throughout the paper and we review known estimates on the nonlinear term. In Section 3 we state the main results of the paper. Then in Section 4 we prove the existence and regularity results, while in Appendix A we fill out details of the existence result stated in Section 3.

2. Notation and preliminaries

2.1. Notation and spaces

We recall that for any $\beta\in\mathbb{R}$ the fractional Laplacian $(-\Delta)^\beta$ is defined via its Fourier transform:

$$\widehat{(-\Delta)^\beta f}(\xi)=|\xi|^{2\beta}\hat{f}(\xi).$$

We note that by a weak solution to (1.1) we mean $\theta(t,x)$ in $(0,\infty)\times\mathbb{R}^2$ such that for any smooth function $\phi(t,x)$ satisfying $\phi(t,\cdot)\in\mathcal{S}$ for each t, the identity

$$\int_{\mathbb{R}^2}\theta(T,\cdot)\phi(T,\cdot)\,dx-\int_{\mathbb{R}^2}\theta(0,\cdot)\phi(0,\cdot)\,dx-\int_0^T\!\!\int_{\mathbb{R}^2}\theta\phi_t\,dx\,dt-\int_0^T\!\!\int_{\mathbb{R}^2}u\theta\nabla\phi\,dx\,dt+\int_0^T\!\!\int_{\mathbb{R}^2}\theta\Lambda^\gamma\phi\,dx\,dt=0$$

holds for any $T>0$.

Before we recall the definition of the spaces that will be used throughout the paper, we shall review the Littlewood–Paley decomposition. For any integer j, define Δ_j to be the Littlewood–Paley projection operator with $\Delta_j v=\phi_j * v$, where

$$\hat{\phi}_j(\xi)=\hat{\phi}\big(2^{-j}\xi\big),\quad \hat{\phi}\in C_0^\infty\big(\mathbb{R}^2\setminus\{0\}\big),\ \hat{\phi}\geqslant 0,$$
$$\operatorname{supp}\hat{\phi}\subset\big\{\xi\in\mathbb{R}^2\ \big|\ 1/2\leqslant|\xi|\leqslant 2\big\},\quad \sum_{j\in\mathbb{Z}}\hat{\phi}_j(\xi)=1\quad\text{for }\xi\neq 0.$$

Formally, we have the Littlewood–Paley decomposition

$$v(t,\cdot)=\sum_{j\in\mathbb{Z}}\Delta_j v(t,\cdot).$$

Also denote

$$\Lambda=(-\Delta)^{1/2},\qquad \bar{\Delta}_{-1}=\sum_{j<0}\Delta_j.$$

As usual, for any $p\in[1,\infty)$ and $s\geqslant 0$, we denote by $\dot{W}^s_p$ and W^s_p, respectively the homogeneous and inhomogeneous Sobolev spaces with norms

$$\|v\|_{\dot W^s_p} := \left\| \left(\sum_{k\in\mathbb{Z}} \left|2^{ks}\Delta_k v\right|^2 \right)^{1/2} \right\|_{L^p} \sim \left\| \Lambda^s v \right\|_{L^p},$$
$$\|v\|_{W^s_p} := \|v\|_{\dot W^s_p} + \|v\|_{L^p}.$$

When $p = 2$, we use $\dot H^s$ and H^s instead of $\dot W^s_p$ and W^s_p. For any $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, we denote by $\dot B^s_{p,q}$ and $B^s_{p,q}$, respectively the homogeneous and inhomogeneous Besov spaces equipped with norms

$$\|v\|_{\dot B^s_{p,q}} := \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j v\|^q_{L^p})^{1/q}, & \text{for } q<\infty,\\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j v\|_{L^p}, & \text{for } q=\infty,\end{cases}$$
$$\|v\|_{B^s_{p,q}} := \begin{cases} (\sum_{j\geqslant 0} 2^{jsq}\|\Delta_j v\|^q_{L^p})^{1/q} + \|\bar\Delta_{-1} v\|_{L^p}, & \text{for } q<\infty,\\ \sup_{j\geqslant 0} 2^{js}\|\Delta_j v\|_{L^p} + \|\bar\Delta_{-1} v\|_{L^p}, & \text{for } q=\infty.\end{cases}$$

If $s > 0$, we have

$$B^s_{p,q} = \dot B^s_{p,q} \cap L^p, \qquad \|v\|_{B^s_{p,q}} \sim \|v\|_{\dot B^s_{p,q}} + \|v\|_{L^p}.$$

For $s \in \mathbb{R}$, $1 \leqslant p, q, r \leqslant \infty$, I an interval in $\mathbb{R}$, the homogeneous mixed time–space Besov space $\tilde L^r(I; \dot B^s_{p,q})$ is the space of distributions in $\mathcal{D}(I; \mathcal{S}'_0(\mathbb{R}^d))$ such that

$$\|f\|_{\tilde L^r(I;\dot B^s_{p,q})} := \left\| 2^{sj} \left(\int_I \|\Delta_j f(t)\|^r_{L^p(\mathbb{R}^d)}\, dt \right)^{1/r} \right\|_{l^q(\mathbb{Z})} < \infty$$

(usual modification applied if $r = \infty$ or $q = \infty$). Also the inhomogeneous time–space Besov norm is given by

$$\|f\|_{\tilde L^r(I;B^s_{p,q})} := \|f\|_{L^r(I;L^p(\mathbb{R}^d))} + \|f\|_{\tilde L^r(I;\dot B^s_{p,q})}.$$

These spaces were introduced by Chemin [5].

2.2. Preliminaries

The following Bernstein's inequality is well known.

Lemma 2.1.

(i) *Let $p \in [1,\infty]$ and $s \in \mathbb{R}$. Then for any $j \in \mathbb{Z}$, we have*

$$\lambda 2^{js}\|\Delta_j v\|_{L^p} \leqslant \left\|\Lambda^s \Delta_j v\right\|_{L^p} \leqslant \lambda' 2^{js}\|\Delta_j v\|_{L^p} \tag{2.1}$$

with some constants λ and λ' depending only on p and s.

(ii) *Moreover, for $1 \leqslant p \leqslant q \leqslant \infty$, there exists a positive constant C depending only on p and q such that*

$$\|\Delta_j v\|_{L^q} \leqslant C 2^{(1/p-1/q)dj}\|\Delta_j v\|_{L^p}. \tag{2.2}$$

Now we recall the generalized Bernstein's inequality and a lower bound for an integral involving fractional Laplacian which will be used in the paper. They can be found in [30,21] and [6].

Lemma 2.2.

(i) *Let $p \in [2,\infty)$ and $\gamma \in [0,2]$. Then for any $j \in \mathbb{Z}$, we have*

$$\lambda 2^{\gamma j/p}\|\Delta_j v\|_{L^p} \leqslant \left\|\Lambda^{\gamma/2}\left(|\Delta_j v|^{p/2}\right)\right\|^{2/p}_{L^2} \leqslant \lambda' 2^{\gamma j/p}\|\Delta_j v\|_{L^p} \tag{2.3}$$

with some positive constants λ and λ' depending only on p and γ.

(ii) *Moreover, we have*

$$\int_{\mathbb{R}^2} \big(\Lambda^\gamma v\big)|v|^{p-2}v \geqslant c\big\|\Lambda^{\gamma/2}|v|^{p/2}\big\|_{L^2}^2, \tag{2.4}$$

and

$$\int_{\mathbb{R}^2} \big(\Lambda^\gamma \Delta_j v\big)|\Delta_j v|^{p-2}\Delta_j v \geqslant c2^{\gamma j}\|\Delta_j v\|_{L^p}^p, \tag{2.5}$$

with some positive constant c depending only on p and γ.

Next we recall the commutator estimate that shall be used throughout the paper.

Lemma 2.3. *Let $d \geqslant 1$ be an integer, $p, q \in [1,\infty]$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1$, $\rho_1 < 1$, $\rho_2 < 1$ and u be a divergence free vector field. Assume in addition that*

$$\rho_1 + \rho_2 + d\min\left(1, \frac{2}{p}\right) > 0, \qquad \rho_1 + \frac{d}{p} > 0.$$

Then for any $j \in \mathbb{Z}$ we have

$$\big\|[u, \Delta_j]\cdot\nabla v\big\|_{L_t^r(L^p(\mathbb{R}^d))} \leqslant Cc_j 2^{-j(\frac{d}{p}+\rho_1+\rho_2-1)}\|\nabla u\|_{\tilde{L}_t^{r_1}(\dot{B}_{p,q}^{\frac{d}{p}+\rho_1-1}(\mathbb{R}^d))}\|\nabla v\|_{\tilde{L}_t^{r_2}(\dot{B}_{p,q}^{\frac{d}{p}+\rho_2-1}(\mathbb{R}^d))}, \tag{2.6}$$

where C is a positive constant independent of j and $\{c_j\} \in l^q$ satisfying $\|c_j\|_{l^q} \leqslant 1$. Here

$$[u, \Delta_j]\cdot\nabla v = u\cdot\Delta_j(\nabla v) - \Delta_j(u\cdot\nabla v).$$

Proof. See [6] and [13]. □

Also we state the following result about a product of two functions in Besov spaces. For a proof, see, for example, [6].

Lemma 2.4. *Let $s > -\frac{d}{p} - 1$, $s < s_1 < \frac{d}{p}$, $2 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1$ and u be a divergence free vector field. Then*

$$\|u\cdot\nabla v\|_{\tilde{L}_t^r(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_t^{r_1}(\dot{B}_{p,q}^{s_1})}\|\nabla v\|_{\tilde{L}_t^{r_2}(\dot{B}_{p,q}^{s+\frac{d}{p}-s_1})}.$$

If $s_1 = \frac{d}{p}$ or $s_1 = s$, then q has to be taken to be 1.

3. Formulation of results

In this section we formulate existence and uniqueness results that shall be used in the proof of our main regularity result. Also we formulate the main regularity result.

First we state the local well-posedness result for (1.1).

Proposition 3.1. *Let $\gamma \in (0,1]$, $p \in [2,\infty)$, $q \in [1,\infty]$ and $r_0 \in [2,\infty)$. Denote by $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}$. Assume $\theta_0 \in B_{p,q}^\alpha(\mathbb{R}^2)$. Then there exists $T \geqslant c\|\theta_0\|_{\dot{B}_{p,q}^\alpha}^{-r_0}$ for some constant $c > 0$ such that the initial value problem for* (1.1) *has a unique weak solution*

$$\theta(t,x) \in \tilde{L}^2\big((0,T); B_{p,q}^{\alpha+\frac{\gamma}{2}}\big) \cap \tilde{L}^\infty\big((0,T); B_{p,q}^\alpha\big).$$

For any $r \in [2,\infty]$,

$$\|\theta\|_{\tilde{L}_t^r B_{p,q}^{\alpha+\frac{\gamma}{r}}((0,T)\times\mathbb{R}^2)} \leqslant C\|\theta_0\|_{B_{p,q}^\alpha} \tag{3.1}$$

with a positive constant C independent of r, and θ is smooth in $(0,T)\times\mathbb{R}^2$. Moreover, if $q<\infty$, we also have

$$\theta(t,x)\in C\big([0,T);B^{\alpha}_{p,q}\big).$$

Remark 3.2. From the proof, it is clear that if $r_0>2$ then the unique solution θ is actually in

$$\tilde L^1\big((0,T);B^{\alpha+\gamma}_{p,q}\big)\cap\tilde L^\infty\big((0,T);B^{\alpha}_{p,q}\big).$$

Moreover, for any $r\in[1,\infty]$ estimate (3.1) holds. However, we will not use this in our main theorem.

An analogous local well-posedness result in the critical space $B^{\frac{2}{p}+1-\gamma}_{p,q}$ was established in [6] (see also [31] for a similar well-posedness result and several uniqueness results, and [24,21] for local well-posedness results in Sobolev spaces). However, we remark that with θ_0 in the critical space the time of existence T depends on the profile of θ_0 instead of the norm.

The next proposition is about the uniqueness of weak solutions in mixed time–space Besov spaces.

Proposition 3.3. *Let $\gamma\in(0,1]$, $p\in[2,\infty)$, $T\in(0,\infty)$ and $r_0\in[2,\infty)$. Denote by $\alpha=\frac{2}{p}+1-\gamma+\frac{\gamma}{r_0}$.*

(a) *Let $q\in[1,\infty)$. If $\theta,\theta'\in\tilde L^{r_0}_tB^{\alpha}_{p,q}((0,T)\times\mathbb{R}^2)$ are two weak solutions of (1.1) with the same initial data, then $\theta=\theta'$ in $[0,T)\times\mathbb{R}^2$.*
(b) *Let $q=\infty$. If $\theta,\theta'\in L^{r_0}_tB^{\alpha}_{p,q}((0,T)\times\mathbb{R}^2)$ are two weak solutions of (1.1) with the same initial data, then $\theta=\theta'$ in $[0,T)\times\mathbb{R}^2$.*

The following regularity criteria is our main result. Roughly speaking, it says weak solutions in certain critical time–space Besov spaces are regular.

Theorem 3.4. *Let $\gamma\in(0,1]$, $p\in[2,\infty)$, $T\in(0,\infty)$ and $r_0\in[2,\infty)$. Denote by $\alpha=\frac{2}{p}+1-\gamma+\frac{\gamma}{r_0}$. If*

$$\theta\in L^{r_0}_t\big((0,T);B^{\alpha}_{p,\infty}\big(\mathbb{R}^2\big)\big)$$

is a weak solution of (1.1), then θ is in $C^\infty((0,T]\times\mathbb{R}^2)$, and thus it is a classical solution of (1.1) in the region $(0,T]\times\mathbb{R}^2$.

4. Proofs of existence, uniqueness and regularity

In this section we present proofs of the above stated results. In order to prove Propositions 3.1 and 3.3 we modify accordingly the approach used by Chen et al. [6].

4.1. Proof of Proposition 3.1

4.1.1. A priori estimate

We apply the operator Δ_j to the first equation in (1.1) to obtain

$$\partial_t\Delta_j\theta+\Delta_j(u\cdot\nabla\theta)+\Lambda^\gamma\Delta_j\theta=0, \tag{4.1}$$

which is equivalent to

$$\partial_t\Delta_j\theta+u\cdot\nabla\Delta_j\theta+\Lambda^\gamma\Delta_j\theta=[u,\Delta_j]\cdot\nabla\theta. \tag{4.2}$$

Now we multiply (4.2) by $|\Delta_j\theta|^{p-2}\Delta_j\theta$ and integrate in x. Since u is divergence free, the integration by parts yields

$$\int_{\mathbb{R}^2}u\cdot\nabla\Delta_j\theta|\Delta_j\theta|^{p-2}\Delta_j\theta\,dx=0.$$

Hence we have

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p+\int_{\mathbb{R}^2}\big(\Lambda^\gamma\Delta_j\theta\big)|\Delta_j\theta|^{p-2}\Delta_j\theta\,dx=\int_{\mathbb{R}^2}[u,\Delta_j]\cdot\nabla\theta|\Delta_j\theta|^{p-2}\Delta_j\theta\,dx. \tag{4.3}$$

Now we use Lemma 2.2 to obtain a lower bound on the second term on the left-hand side of (4.3) and Hölder's inequality to get an upper bound on the right-hand side of (4.3) to derive

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^p}+\lambda 2^{\gamma j}\|\Delta_j\theta\|_{L^p}\leqslant C\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L^p}, \tag{4.4}$$

where $\lambda=\lambda(p,\gamma)>0$. Gronwall's inequality applied on (4.4) implies

$$\|\Delta_j\theta\|_{L^p}\leqslant e^{-\lambda 2^{\gamma j}t}\big\|\Delta_j\theta(0)\big\|_{L^p}+C\int_0^t e^{-\lambda 2^{\gamma j}(t-s)}\big\|\big([u,\Delta_j]\cdot\nabla\theta\big)(s)\big\|_{L^p}\,ds. \tag{4.5}$$

Fix $r\in[2,\infty]$. We take the L_t^r norm over the interval of time $(0,T)$ to obtain:

$$\|\Delta_j\theta\|_{L_t^rL_x^p((0,T)\times\mathbb{R}^2)}\leqslant I_1+I_2, \tag{4.6}$$

where

$$I_1=\big\|e^{-\lambda 2^{\gamma j}t}\big\|_{L_t^r(0,T)}\big\|\Delta_j\theta(0)\big\|_{L_x^p},$$

$$I_2=\left\|\int_0^t e^{-\lambda 2^{\gamma j}(t-s)}\big\|\big([u,\Delta_j]\cdot\nabla\theta\big)(s)\big\|_{L_x^p}\,ds\right\|_{L_t^r(0,T)}.$$

Since

$$\big\|e^{-\lambda 2^{\gamma j}t}\big\|_{L_t^r(0,T)}\lesssim\left(\frac{1-e^{-r\lambda 2^{\gamma j}T}}{r\lambda 2^{\gamma j}}\right)^{\frac{1}{r}}\lesssim\lambda^{-\frac{1}{r}}2^{-\frac{\gamma}{r}j},$$

we can bound I_1 from above as follows

$$I_1\lesssim\lambda^{-\frac{1}{r}}2^{-\frac{\gamma}{r}j}\big\|\Delta_j\theta(0)\big\|_{L_x^p}. \tag{4.7}$$

In order to estimate I_2 we use Young's inequality to obtain

$$I_2\lesssim\big\|e^{-\lambda 2^{\gamma j}t}\big\|_{L_t^1(0,T)}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L_t^rL_x^p((0,T)\times\mathbb{R}^2)}. \tag{4.8}$$

Since

$$\frac{1-e^{-\lambda 2^{\gamma j}T}}{\lambda 2^{\gamma j}}\lesssim 2^{-\gamma j},$$

as well as

$$\frac{1-e^{-\lambda 2^{\gamma j}T}}{\lambda 2^{\gamma j}}\lesssim T,$$

we have

$$\frac{1-e^{-\lambda 2^{\gamma j}T}}{\lambda 2^{\gamma j}}\leqslant 2^{-\frac{\gamma}{r_3}j}T^{1-\frac{1}{r_3}},$$

where r_3 is arbitrary real number such that $r_3>1$ and will be chosen later. Hence (4.8) implies

$$I_2\lesssim 2^{-\frac{\gamma}{r_3}j}T^{1-\frac{1}{r_3}}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L_t^rL_x^p((0,T)\times\mathbb{R}^2)}. \tag{4.9}$$

Now (4.6) combined with (4.7) and (4.9) gives

$$\|\Delta_j\theta\|_{L_t^rL_x^p((0,T)\times\mathbb{R}^2)}\lesssim\lambda^{-\frac{1}{r}}2^{-\frac{\gamma}{r}j}\big\|\Delta_j\theta(0)\big\|_{L_x^p}+2^{-\frac{\gamma}{r_3}j}T^{1-\frac{1}{r_3}}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L_t^rL_x^p((0,T)\times\mathbb{R}^2)}. \tag{4.10}$$

After we multiply (4.10) by $2^{(\alpha+\frac{\gamma}{r})j}$ and take $l^q(\mathbb{Z})$ norm we infer:

$$\|\theta\|_{\tilde{L}^r(\dot{B}^{\alpha+\frac{\gamma}{r}}_{p,q})} \lesssim \lambda^{-\frac{1}{r}}\big\|\theta(0)\big\|_{\dot{B}^{\alpha}_{p,q}} + T^{1-\frac{1}{r_3}}\big\|2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j}[u,\Delta_j]\cdot\nabla\theta\big\|_{L^r_tL^p_x((0,T)\times\mathbb{R}^2)}\big\|_{l^q}. \tag{4.11}$$

In order to estimate $\|2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j}[u,\Delta_j]\cdot\nabla\theta\|_{L^r_tL^p_x((0,T)\times\mathbb{R}^2)}\|_{l^q}$ we apply Lemma 2.3 with

$$v=\theta,\quad d=2,\quad r_1=r_2=2r,\quad \rho_1=\rho_2=1-\gamma+\frac{\gamma}{2r}+\frac{\gamma}{r_0}<1$$

and use the boundedness of the Riesz transforms to obtain

$$\begin{aligned}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L^r_tL^p_x((0,T)\times\mathbb{R}^2)} &\lesssim c_j2^{-(\alpha+\frac{\gamma}{r_0}+\frac{\gamma}{r}-\gamma)j}\|u\|_{\tilde{L}^{r_1}(\dot{B}^{\alpha+\frac{\gamma}{r_1}}_{p,q})}\|\theta\|_{\tilde{L}^{r_2}(\dot{B}^{\alpha+\frac{\gamma}{r_2}}_{p,q})}\\ &\lesssim c_j2^{-(\alpha+\frac{\gamma}{r_0}+\frac{\gamma}{r}-\gamma)j}\|\theta\|_{\tilde{L}^{r_1}(\dot{B}^{\alpha+\frac{\gamma}{r_1}}_{p,q})}\|\theta\|_{\tilde{L}^{r_2}(\dot{B}^{\alpha+\frac{\gamma}{r_2}}_{p,q})},\end{aligned}$$

where $c_j\in l^q$ is such that $\|c_j\|_{l^q}\leqslant 1$. Therefore

$$2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L^r_tL^p_x((0,T)\times\mathbb{R}^2)} \lesssim c_j2^{(-\frac{\gamma}{r_3}-\frac{\gamma}{r_0}+\gamma)j}\|\theta\|_{\tilde{L}^{r_1}(\dot{B}^{\alpha+\frac{\gamma}{r_1}}_{p,q})}\|\theta\|_{\tilde{L}^{r_2}(\dot{B}^{\alpha+\frac{\gamma}{r_2}}_{p,q})}. \tag{4.12}$$

After we choose r_3 such that

$$1=\frac{1}{r_3}+\frac{1}{r_0}, \tag{4.13}$$

we observe that (4.12) implies

$$\big\|2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j}\big\|[u,\Delta_j]\cdot\nabla\theta\big\|_{L^r_tL^p_x((0,T)\times\mathbb{R}^2)}\big\|_{l^q} \lesssim \|\theta\|_{\tilde{L}^{r_1}(\dot{B}^{\alpha+\frac{\gamma}{r_1}}_{p,q})}\|\theta\|_{\tilde{L}^{r_2}(\dot{B}^{\alpha+\frac{\gamma}{r_2}}_{p,q})}. \tag{4.14}$$

Now we combine (4.11) and (4.14) together with (4.13) to conclude

$$\|\theta\|_{\tilde{L}^r(\dot{B}^{\alpha+\frac{\gamma}{r}}_{p,q})} \lesssim \lambda^{-\frac{1}{r}}\big\|\theta(0)\big\|_{\dot{B}^{\alpha}_{p,q}} + T^{\frac{1}{r_0}}\|\theta\|_{\tilde{L}^{r_1}(\dot{B}^{\alpha+\frac{\gamma}{r_1}}_{p,q})}\|\theta\|_{\tilde{L}^{r_2}(\dot{B}^{\alpha+\frac{\gamma}{r_2}}_{p,q})}, \tag{4.15}$$

which is our main a priori estimate. In particular, if we denote by

$$\Lambda(\theta,T)=\|\theta\|_{\tilde{L}^2(\dot{B}^{\alpha+\frac{\gamma}{2}}_{p,q})}+\|\theta\|_{\tilde{L}^\infty(\dot{B}^{\alpha}_{p,q})},$$

we then have

$$\Lambda(\theta,T)\lesssim\big\|\theta(0)\big\|_{\dot{B}^{\alpha}_{p,q}}+T^{\frac{1}{r_0}}\Lambda(\theta,T)^2. \tag{4.16}$$

With a help of the a priori estimate (4.15), it is standard to construct a solution of (1.1) by using approximations. For the sake of completeness, we give a sketch of a proof in Appendix A. We refer to [15] and [17] for the proof of the smoothness of θ in $(0,T]\times\mathbb{R}^2$.

4.1.2. Uniqueness

The proof of the uniqueness part of Proposition 3.1 is not much different from that of Proposition 3.3. We refer the reader to the next section for details.

4.2. Proof of Proposition 3.3

Here we establish the uniqueness result for weak solutions to (1.1), i.e. Proposition 3.3. Suppose that θ and θ' are two solutions to (1.1) in $\tilde{L}^{r_0}_tB^{\alpha}_{p,q}((0,T)\times\mathbb{R}^2)$ which correspond to the same initial data $\theta_0(x)$. We denote $\delta\theta=\theta-\theta'$ and $\delta u=u-u'$, where $u'=(-\mathcal{R}_2\theta',\mathcal{R}_1\theta')$. Then it follows that:

$$\begin{cases}\partial_t\delta\theta+u\cdot\nabla\delta\theta+\delta u\cdot\nabla\theta'+\Lambda^\gamma\delta\theta=0, & x\in\mathbb{R}^2,\ t>0,\\ \delta u=\mathcal{R}^\perp\delta\theta,\\ \delta\theta(x,0)=0.\end{cases} \tag{4.17}$$

We follow the strategy used to derive (4.4) to obtain

$$\frac{d}{dt}\|\Delta_j\delta\theta\|_{L^p}+\lambda 2^{\gamma j}\|\Delta_j\delta\theta\|_{L^p}\leqslant C\big(\big\|[u,\Delta_j]\cdot\nabla\delta\theta\big\|_{L^p}+\big\|\Delta_j(\delta u\cdot\nabla\theta')\big\|_{L^p}\big). \tag{4.18}$$

Since $\delta\theta(x,0)=0$, Gronwall's inequality applied on (4.18) implies

$$\|\Delta_j\delta\theta\|_{L^p}\leqslant C\int_0^t e^{-\lambda 2^{\gamma j}(t-s)}\big(\big\|\big([u,\Delta_j]\cdot\nabla\delta\theta\big)(s)\big\|_{L^p}+\big\|\big(\Delta_j\big(\delta u\cdot\nabla\theta'\big)\big)(s)\big\|_{L^p}\big)\,ds.$$

We take the $L_t^{r_0}$ norm over the interval of time $(0,T)$ and use Young's inequality to obtain:

$$\begin{aligned}&\|\Delta_j\delta\theta\|_{L_t^{r_0}L_x^p((0,T)\times\mathbb{R}^2)}\\&\quad\leqslant C\big\|e^{-\lambda 2^{\gamma j}t}\big\|_{L_t^{r'}(0,T)}\big(\big\|[u,\Delta_j]\cdot\nabla\delta\theta\big\|_{L_t^{\frac{r_0}{2}}L_x^p((0,T)\times\mathbb{R}^2)}+\big\|\Delta_j(\delta u\cdot\nabla\theta')\big\|_{L_t^{\frac{r_0}{2}}L_x^p((0,T)\times\mathbb{R}^2)}\big),\end{aligned} \tag{4.19}$$

where $\frac{1}{r'}=1-\frac{1}{r_0}$.

Now let us pick $\eta\in(0,\frac{2}{p})$ such that

$$1-\frac{\gamma}{r'}-\eta+\frac{4}{p}>0. \tag{4.20}$$

We bound $\|e^{-\lambda 2^{\gamma j}t}\|_{L_t^{r'}(0,T)}$ from above by $2^{-\frac{\gamma}{r'}j}$, then multiply (4.19) by $2^{(\frac{2}{p}-\eta)j}$ and take l^q norm with respect to j to infer:

$$\|\delta\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\eta})}\lesssim C(I_3+I_4), \tag{4.21}$$

where

$$\begin{aligned}I_3&=\big\|2^{(\frac{2}{p}-\eta-\frac{\gamma}{r'})j}\big\|[u,\Delta_j]\cdot\nabla\delta\theta\big\|_{L_t^{\frac{r_0}{2}}L_x^p((0,T)\times\mathbb{R}^2)}\big\|_{l^q(\mathbb{Z})},\\I_4&=\|\delta u\cdot\nabla\theta'\|_{\tilde L_t^{\frac{r_0}{2}}\dot B_{p,q}^{\frac{2}{p}-\eta-\frac{\gamma}{r'}}((0,T)\times\mathbb{R}^2)}.\end{aligned}$$

In order to estimate I_3 we apply Lemma 2.3 with

$$v=\delta\theta,\quad d=2,\quad (r_1,r_2)=(r_0,r_0),\quad (\rho_1,\rho_2)=\left(1-\frac{\gamma}{r'},-\eta\right)$$

and the boundedness of the Riesz transforms as follows

$$\begin{aligned}\big\|[u,\Delta_j]\cdot\nabla\delta\theta\big\|_{L_t^{\frac{r_0}{2}}L_x^p((0,T)\times\mathbb{R}^2)}&\lesssim c_j2^{-(\frac{2}{p}-\frac{\gamma}{r'}-\eta)j}\|u\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\frac{\gamma}{r'}+1})}\|\delta\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\eta})}\\&\lesssim c_j2^{-(\frac{2}{p}-\frac{\gamma}{r'}-\eta)j}\|\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\frac{\gamma}{r'}+1})}\|\delta\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\eta})},\end{aligned}$$

where $c_j\in l^q$ is such that $\|c_j\|_{l^q}\leqslant 1$. Since

$$\frac{2}{p}-\frac{\gamma}{r'}+1=\alpha,$$

we obtain

$$I_3\lesssim\|\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\alpha})}\|\delta\theta\|_{\tilde L^{r_0}(\dot B_{p,q}^{\frac{2}{p}-\eta})}. \tag{4.22}$$

On the other hand to estimate I_4 we use Lemma 2.4 with

$$s=\frac{2}{p}-\frac{\gamma}{r'}-\eta,\qquad s_1=\frac{2}{p}-\eta,$$

and the boundedness of the Riesz transforms to obtain

$$I_4 \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \|\theta'\|_{\tilde{L}^{r_0}(\dot{B}^{\alpha}_{p,q})}. \tag{4.23}$$

Now we combine (4.21), (4.22) and (4.23) to conclude

$$\|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \big(\|\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\alpha}_{p,q})} + \|\theta'\|_{\tilde{L}^{r_0}(\dot{B}^{\alpha}_{p,q})}\big). \tag{4.24}$$

We first look at part (a) of the proposition, i.e. the case $q < \infty$. As $T \to 0$, the terms in the parenthesis on the right-hand side of (4.24) go to 0. For part (b), i.e. $q = \infty$, from (4.24) and the Minkowski's inequality we get

$$\|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \big(\|\theta\|_{L^{r_0}(\dot{B}^{\alpha}_{p,q})} + \|\theta'\|_{L^{r_0}(\dot{B}^{\alpha}_{p,q})}\big). \tag{4.25}$$

As $T \to 0$, the terms in the parenthesis on the right-hand side of (4.25) go to 0. Thus in both cases if T is chosen small enough, then

$$\|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}^{\frac{2}{p}-\eta}_{p,q})((0,T)\times\mathbb{R}^2)} = 0,$$

which in turn implies $\delta\theta = 0$. Now the standard continuity argument can be employed to show that $\delta\theta(x,t) = 0$ for all $x \in \mathbb{R}^2$ and $t \geqslant 0$.

4.3. Proof of Theorem 3.4

We prove the theorem by a contradiction. Assume θ is not a regular solution in $(0,T] \times \mathbb{R}^2$. Without loss of generality, one may assume T is the first blowup time. Since $\theta \in L_t^{r_0} B^{\alpha}_{p,\infty}$, for almost all $s \in (0,T)$ we have $\theta(s,\cdot) \in B^{\alpha}_{p,\infty}$. For any such s, consider the initial value problem (1.1) with initial data $\theta_0 = \theta(s,\cdot)$. By applying the local well-posedness result (Proposition 3.1), (1.1) has a unique weak solution

$$\bar{\theta} \in \tilde{L}^2\big((0,T_s); B^{\alpha+\frac{\gamma}{2}}_{p,\infty}\big) \cap \tilde{L}^{\infty}\big((0,T_s); B^{\alpha}_{p,\infty}\big) \cap \tilde{L}^{r_0}\big((0,T_s); B^{\alpha+\frac{\gamma}{r_0}}_{p,\infty}\big)$$

for some

$$T_s \geqslant c\big\|\theta(s,\cdot)\big\|^{-r_0}_{\dot{B}^{\alpha}_{p,\infty}} \tag{4.26}$$

with a constant $c > 0$ independent of s. Moreover, by simple embedding relations we have

$$\bar{\theta} \in \tilde{L}^{r_0}\big((0,T_s); B^{\alpha+\frac{\gamma}{r_0}}_{p,\infty}\big) \hookrightarrow \tilde{L}^{r_0}\big((0,T_s); B^{\alpha}_{p,r_0}\big) \hookrightarrow L^{r_0}\big((0,T_s); B^{\alpha}_{p,\infty}\big).$$

Now we apply the uniqueness result Proposition 3.3 and get $\bar{\theta}(\cdot,\cdot) = \theta(s+\cdot,\cdot)$. The last equality and (4.26) imply that

$$T - s \geqslant c\big\|\theta(s,\cdot)\big\|^{-r_0}_{\dot{B}^{\alpha}_{p,\infty}}.$$

Therefore, for almost all $s \in (0,T)$, we have

$$\big\|\theta(s,\cdot)\big\|_{\dot{B}^{\alpha}_{p,\infty}} \geqslant c^{\frac{1}{r_0}}(T-s)^{-\frac{1}{r_0}},$$

which contradicts the condition $\theta \in L_t^{r_0}((0,T); B^{\alpha}_{p,\infty}(\mathbb{R}^2))$. The theorem is proved.

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Appendix A

The appendix is devoted to the proof of the existence part in Theorem 3.1. Consider the following successive approximations: $\theta^0 \equiv u^0 \equiv 0$, and for $k = 0, 1, 2, \dots,$

$$\begin{cases} \partial_t \theta^{k+1} + u^k \cdot \nabla \theta^{k+1} + (-\Delta)^{\gamma/2} \theta^{k+1} = 0, \quad x \in \mathbb{R}^2,\ t \in (0, \infty), \\ u^{k+1} = \left(-\mathcal{R}_2 \theta^{k+1}, \mathcal{R}_1 \theta^{k+1}\right), \\ \theta^{k+1}(0, x) = \theta_0(x). \end{cases} \tag{A.1}$$

Similar to (4.16), we have

$$\Lambda\left(\theta^{k+1}, T\right) \lesssim \left\|\theta(0)\right\|_{\dot{B}^{\alpha}_{p,q}} + T^{\frac{1}{r_0}} \Lambda\left(\theta^k, T\right) \Lambda\left(\theta^{k+1}, T\right). \tag{A.2}$$

If we choose $T = c\|\theta_0\|^{-r_0}_{\dot{B}^{\alpha}_{p,q}}$ for small $c > 0$ depending on λ and the implicit constant in (A.2), it then holds that for any $k = 0, 1, 2, \dots,$

$$\Lambda\left(\theta^k, T\right) \lesssim \|\theta_0\|_{\dot{B}^{\alpha}_{p,q}}. \tag{A.3}$$

Due to the L^p maximum principle for (1.1), we also have

$$\left\|\theta^k\right\|_{\tilde{L}^2(B^{\alpha+\frac{\gamma}{2}}_{p,q})} + \left\|\theta^k\right\|_{\tilde{L}^\infty(B^{\alpha}_{p,q})} \lesssim \|\theta_0\|_{B^{\alpha}_{p,q}}. \tag{A.4}$$

In order to get a contraction we closely follow the argument in Section 4.2. Denote $\delta\theta^k = \theta^{k+1} - \theta^k$ and $\delta u^k = u^{k+1} - u^k$. Then it follows that:

$$\begin{cases} \partial_t \delta\theta^k + u^k \cdot \nabla \delta\theta^k + \delta u^{k-1} \cdot \nabla \theta^k + \Lambda^\gamma \delta\theta^k = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ \delta u^k = \mathcal{R}^\perp \delta\theta^k, \\ \delta\theta^k(x, 0) = 0. \end{cases} \tag{A.5}$$

We follow the strategy used to derive (4.4) to obtain

$$\frac{d}{dt}\left\|\Delta_j \delta\theta^k\right\|_{L^p} + \lambda 2^{\gamma j}\left\|\Delta_j \delta\theta^k\right\|_{L^p} \leqslant C\left(\left\|\left[u^k, \Delta_j\right] \cdot \nabla \delta\theta^k\right\|_{L^p} + \left\|\Delta_j\left(\delta u^{k-1} \cdot \nabla \theta^k\right)\right\|_{L^p}\right). \tag{A.6}$$

Since $\delta\theta^k(x, 0) = 0$, Gronwall's inequality applied on (A.6) implies

$$\left\|\Delta_j \delta\theta^k\right\|_{L^p} \leqslant C \int_0^t e^{-\lambda 2^{\gamma j}(t-s)}\left(\left\|\left(\left[u^k, \Delta_j\right] \cdot \nabla \delta\theta^k\right)(s)\right\|_{L^p} + \left\|\left(\Delta_j\left(\delta u^{k-1} \cdot \nabla \theta^k\right)\right)(s)\right\|_{L^p}\right) ds.$$

We take the L^∞_t norm over the interval of time $(0, T)$ and use Young's inequality to obtain:

$$\begin{aligned} \left\|\Delta_j \delta\theta^k\right\|_{L^\infty_t L^p_x((0,T)\times\mathbb{R}^2)} \leqslant C\left\|e^{-\lambda 2^{\gamma j} t}\right\|_{L^1_t(0,T)}\Big(&\left\|\left[u^k, \Delta_j\right] \cdot \nabla \delta\theta^k\right\|_{L^\infty_t L^p_x((0,T)\times\mathbb{R}^2)} \\ &+ \left\|\Delta_j\left(\delta u^{k-1} \cdot \nabla \theta^k\right)\right\|_{L^\infty_t L^p_x((0,T)\times\mathbb{R}^2)}\Big). \end{aligned} \tag{A.7}$$

Now let us pick $\eta \in (0, \frac{2}{p})$ such that

$$1 - \gamma - \eta + \frac{4}{p} > 0.$$

We bound $\|e^{-\lambda 2^{\gamma j} t}\|_{L^1_t(0,T)}$ from above by $2^{(\frac{\gamma}{r_0} - \gamma) j} T^{\frac{1}{r_0}}$ as in Section 4.1.1, then multiply (A.7) by $2^{(\frac{2}{p} - \eta) j}$ and take l^q norm with respect to j to infer:

$$\left\|\delta\theta^k\right\|_{\tilde{L}^\infty(\dot{B}^{\frac{2}{p}-\eta}_{p,q})} \lesssim C T^{\frac{1}{r_0}}(I_5 + I_6), \tag{A.8}$$

where

$$I_5 = \left\| 2^{(\frac{2}{p}-\eta+\frac{\gamma}{r_0}-\gamma)j} \left\| [u^k, \Delta_j] \cdot \nabla \delta\theta^k \right\|_{L_t^\infty L_x^p((0,T)\times\mathbb{R}^2)} \right\|_{l^q(\mathbb{Z})},$$
$$I_6 = \left\| \delta u^{k-1} \cdot \nabla \theta^k \right\|_{\tilde{L}_t^\infty \dot{B}_{p,q}^{\frac{2}{p}-\eta+\frac{\gamma}{r_0}-\gamma}((0,T)\times\mathbb{R}^2)}.$$

In order to estimate I_5 we apply Lemma 2.3 with

$$u = u^k, \quad v = \delta\theta^k, \quad d = 2, \quad (r_1, r_2) = (\infty, \infty), \quad (\rho_1, \rho_2) = \left(1 + \frac{\gamma}{r_0} - \gamma, -\eta\right)$$

and the boundedness of the Riesz transforms as follows

$$\begin{aligned} \left\| [u^k, \Delta_j] \cdot \nabla \delta\theta^k \right\|_{L_t^\infty L^p((0,T)\times\mathbb{R}^2)} &\lesssim c_j 2^{-(\frac{2}{p}+\frac{\gamma}{r_0}-\gamma-\eta)j} \left\| u^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)} \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \\ &\lesssim c_j 2^{-(\frac{2}{p}+\frac{\gamma}{r_0}-\gamma-\eta)j} \left\| \theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)} \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})}, \end{aligned}$$

where $c_j \in l^q$ is such that $\|c_j\|_{l^q} \leqslant 1$. Thus, we obtain

$$I_5 \lesssim \left\| \theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)} \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})}. \tag{A.9}$$

On the other hand to estimate I_6 we use Lemma 2.4 with

$$s = \frac{2}{p} + \frac{\gamma}{r_0} - \gamma - \eta, \qquad s_1 = \frac{2}{p} - \eta$$

and the boundedness of the Riesz transforms to obtain

$$I_6 \lesssim \left\| \delta\theta^{k-1} \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \left\| \theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)}. \tag{A.10}$$

Now we combine (A.8)–(A.10) and (A.3) to conclude

$$\begin{aligned} \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} &\leqslant C T^{\frac{1}{r_0}} \left\| \theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)} \left(\left\| \delta\theta^{k-1} \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} + \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \right) \\ &\leqslant C T^{\frac{1}{r_0}} \|\theta_0\|_{\dot{B}_{p,q}^\alpha} \left(\left\| \delta\theta^{k-1} \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} + \left\| \delta\theta^k \right\|_{\tilde{L}^\infty(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \right). \end{aligned} \tag{A.11}$$

We choose $T = c\|\theta_0\|_{\dot{B}_{p,q}^\alpha}^{-r_0}$ for even smaller $c > 0$ such that

$$C T^{\frac{1}{r_0}} \|\theta_0\|_{\dot{B}_{p,q}^\alpha} \leqslant \frac{1}{3}.$$

Therefore, $\{\theta^k\}$ is a Cauchy sequence in $\tilde{L}^\infty((0,T); \dot{B}_{p,q}^{\frac{2}{p}-\eta})$, and it converges to a function θ in the same space.

Moreover, θ satisfies (1.1) in the sense of distributions and

$$\|\theta\|_{\tilde{L}^2(B_{p,q}^{\alpha+\frac{\gamma}{2}})} + \|\theta\|_{\tilde{L}^\infty(B_{p,q}^\alpha)} \lesssim \|\theta_0\|_{B_{p,q}^\alpha}. \tag{A.12}$$

As in [17], θ is smooth in $(0,T) \times \mathbb{R}^2$ and satisfies the first equation of (1.1) in the same region in the classical sense.

We claim $\theta \in C([0,T); B_{p,q}^\alpha)$ if $q < \infty$. Observe that from (4.1), Lemmas 2.4 and 2.1(i) we know for $j = 1, 2, 3, \ldots,$

$$\partial_t \Delta_j \theta \in L^\infty\left((0,T); B_{p,q}^\alpha\right).$$

It follows immediately that

$$\Delta_j \theta \in C\left([0,T); B_{p,q}^\alpha\right). \tag{A.13}$$

On the other hand, (A.12) implies that as $k \to \infty$

$$\sum_{|j| \leqslant k} \Delta_j \theta \to \theta \quad \text{in } L^\infty\left((0,T); B_{p,q}^\alpha\right).$$

This together with (A.13) proves the claim.

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