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Ann. I. H. Poincaré – AN 26 (2009) 1299–1308
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Uniqueness of values of Aronsson operators and running costs in "tug-of-war" games

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Received 12 January 2008; received in revised form 14 November 2008; accepted 14 November 2008

Available online 3 December 2008

Abstract

Let A_H be the Aronsson operator associated with a Hamiltonian $H(x,z,p)$. Aronsson operators arise from L^∞ variational problems, two person game theory, control problems, etc. In this paper, we prove, under suitable conditions, that if $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is simultaneously a viscosity solution of both of the equations

$$A_H(u) = f(x) \quad \text{and} \quad A_H(u) = g(x) \quad \text{in } \Omega, \tag{0.1}$$

where $f, g \in C(\Omega)$, then $f = g$. The assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ can be relaxed to $u \in C(\Omega)$ in many interesting situations. Also, we prove that if $f, g, u \in C(\Omega)$ and u is simultaneously a viscosity solution of the equations

$$\frac{\Delta_\infty u}{|Du|^2} = -f(x) \quad \text{and} \quad \frac{\Delta_\infty u}{|Du|^2} = -g(x) \quad \text{in } \Omega, \tag{0.2}$$

then $f = g$. This answers a question posed in Peres, Schramm, Scheffield and Wilson [Y. Peres, O. Schramm, S. Sheffield, D.B. Wilson, Tug-of-war and the infinity Laplacian, J. Amer. Math. Soc. Math. 22 (2009) 167–210] concerning whether or not the value function uniquely determines the running cost in the "tug-of-war" game.

Keywords: Aronsson operators; Infinity Laplacian operator; "Tug-of-war" games

1. Introduction

Let Ω be an open set in $\mathbb{R}^n$ and $H(x,z,p) \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. The Aronsson operator associated with H has the form

$$A_H(u) = H_p(x,u,Du) \cdot D_x\big(H(x,u,Du)\big),$$

where D_x represents the partial derivative with respect to x if we consider $H(x,u(x),Du(x))$ as a function of x. This type of operator was first introduced by G. Aronsson in 60's when he studied the L^∞ variational problems

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[1] The author was partially supported by NSF grant D0848378.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2008.11.001

[1–4]. We say that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is an *absolute minimizer for H in Ω* if for any bounded open set $V \subset \bar{V} \subset \Omega$ and $v \in W^{1,\infty}(V)$,

$$u|_{\partial V} = v|_{\partial V}$$

implies that

$$\operatorname{esssup}_V H(x, u, Du) \leqslant \operatorname{esssup}_V H(x, v, Dv).$$

Under suitable assumptions on H, it was proved that if u is an absolute minimizer for H in Ω, then it is a viscosity solution of the Aronsson equation

$$A_H(u) = 0 \quad \text{in } \Omega.$$

See for instance Barron, Jensen and Wang [6], Crandall [7] and Crandall, Wang and Yu [10]. When $H = \frac{1}{2}|p|^2$, the Aronsson operator is the famous infinity Laplacian operator $\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}$. We refer to the *User's Guide*, Crandall, Ishii and Lions [9], for definitions of viscosity solutions.

In a recent interesting paper, Peres, Schramm, Sheffield and Wilson [16], the authors derived the infinity Laplacian operator from a two-player zero-sum game, called "tug-of-war". Roughly speaking, for fixed $\epsilon > 0$, starting from $x_0 \in \Omega$, at the kth turn, the players toss a coin and the winner chooses an $x_k \in \Omega$ with $|x_k - x_{k-1}| \leqslant \epsilon$. The game ends when $x_k \in \partial\Omega$. Player I tries to maximize its payoff $F(x_k) + \frac{\epsilon^2}{2}\sum_{i=0}^{k-1} f(x_i)$ and player II tries to minimize it. Here $F \in C(\partial\Omega)$ is the *terminal payoff function* and $f \in C(\Omega)$ is the *running payoff function*. The setting of [16] is in a length space. In this paper, we only consider $\mathbb{R}^n$. Let u_ϵ be the value of the above game. Under proper assumptions on Ω, f and F, for example if Ω is bounded and f is positive, it was proved in [16] that

$$\lim_{\epsilon \to 0} u_\epsilon = u,$$

where u is the unique viscosity solution of the following equation

$$\frac{\Delta_\infty u}{|Du|^2} = -f \quad \text{in } \Omega \tag{1.1}$$

and

$$u|_{\partial\Omega} = F.$$

Since $|Du|$ might be zero in Eq. (1.1), the definition of a viscosity solution of Eq. (1.1) is little bit subtle. At the touching point where the gradient of a test function ϕ vanishes, we need to consider $\max_{v \in S^{n-1}} v \cdot D^2\phi \cdot v$ or $\min_{v \in S^{n-1}} v \cdot D^2\phi \cdot v$ depending on whether ϕ touches from above or from below. See Definition 2.1 in next section. It is natural to multiply both sides of Eq. (1.1) by $|Du|^2$ to get another equation which looks nicer

$$\Delta_\infty u + f(x)|Du|^2 = 0 \quad \text{in } \Omega. \tag{1.2}$$

Here we want to remark that these two equations are not equivalent. It is easy to show that any viscosity solution of (1.1) is also a viscosity solution of Eq. (1.2). However, except when $f(x) \equiv 0$, a viscosity solution of Eq. (1.2) might not be a viscosity solution of Eq. (1.1). For example, $u \equiv 0$ is a smooth solution of $(u')^2 u'' = (u')^2$, but not a solution of $(u')^2 u''/(u')^2 = 1$.

In Barron, Evans and Jensen [5], the authors considered generalized "tug-of-war" games where the movement of two players satisfies other dynamics. They derived that the resulting value functions satisfy PDEs which involve Aronsson-type operators. They also provided several other interesting contexts where Aronsson operators arise. A basic question about the Aronsson operator is whether it is single valued. Precisely speaking, assume that $u \in C(\Omega)$ is a viscosity solution of two equations

$$A_H(u) = f(x), \quad A_H(u) = g(x) \quad \text{in } \Omega.$$

Do we have that $f = g$? In this paper, we show that the answer is "Yes" if $H \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. The assumption that $|Du|$ is locally bounded is not necessary for a large class of H. This conclusion that Aronsson operator has unique value is not obvious at all since u lacks sufficient regularity. For example, $u = x^{\frac{4}{3}} - y^{\frac{4}{3}}$ is a viscosity solution of the infinity Laplacian equation, but it is only $C^{1,\frac{1}{3}}$. We will in fact prove a more general result.

Theorem 1.1. *Assume that $B \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $c \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $f, g \in C(\Omega)$. Suppose that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a viscosity subsolution of the following equation*

$$B(x, u, Du) \cdot D^2u \cdot B(x, u, Du) + c(x, u, Du) = f(x), \tag{1.3}$$

and is a viscosity supersolution of the following equation

$$B(x, u, Du) \cdot D^2u \cdot B(x, u, Du) + c(x, u, Du) = g(x). \tag{1.4}$$

Then

$$f \leqslant g \quad \text{in } \Omega.$$

It is clear that if $H \in C^2$, the Aronsson operator A_H satisfies structure assumptions in Theorem 1.1. Hence the Aronsson operator is single valued for locally Lipschitz continuous viscosity solutions. For suitable H, including the most interesting case $H = \frac{1}{2}|p|^2$, the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ can be relaxed to $u \in C(\Omega)$. See Corollary 3.2 and Remark 3.3.

In [16], the authors proposed an open problem which asks whether two different running payoff functions will lead to the same value function. See Problem 4 at the end of [16]. We will show that the answer is No. The following is the precise statement.

Theorem 1.2. *Assume that $f, g \in C(\Omega)$. Suppose that $u \in C(\Omega)$ is simultaneously a viscosity solution of two equations*

$$\frac{\Delta_\infty u}{|Du|^2} = -g(x), \qquad \frac{\Delta_\infty u}{|Du|^2} = -f(x) \quad \text{in } \Omega. \tag{1.5}$$

Then

$$f = g.$$

We want to stress that the above theorem cannot be deduced directly from Theorem 1.1. In fact, its proof is much more tricky. We need to employ the endpoint estimate (2.2) developed in [8] to avoid the situation where $|Du|$ is close to zero. If we want to apply Theorem 1.1 to prove Theorem 1.2, we need an open subset of Ω where $|Du|$ is bounded away from 0. The existence of such an open subset requires the continuity of $|Du|$, which can be given a meaning independent of the existence of Du itself. The author has proved this continuity if $n = 2$, but has no clue how to prove in higher dimensions.

We note that the question of whether or not a function u can simultaneously solve two distinct Hamilton–Jacobi equations $H(Du) = f$ and $H(Du) = g$ in the viscosity sense is mentioned in [9]. If $n = 1$ or H is uniformly continuous, it was proved in Evans [11] that the answer is No. Frankowska [13] also provided some sufficient conditions on H which lead to $f = g$. However for general situations, this question remains open.

Our paper is organized as follows. In Section 2, we will review some known results in [8]. In Section 3, we will prove Theorems 1.1 and 1.2.

2. Preliminaries

Viscosity solutions of Eq. (1.1) are defined as follows.

Definition 2.1. $u \in C(\Omega)$ is a viscosity supersolution of Eq. (1.1) in Ω if for any $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ satisfying

$$0 = \phi(x_0) - u(x_0) \geqslant \phi(x) - u(x) \quad \text{for all } x \in \Omega,$$

one of the following holds:

(1) $D\phi(x_0) \neq 0$ and

$$\Delta_\infty \phi(x_0) \leqslant -f(x_0)\big|D\phi(x_0)\big|^2;$$

or,

(2) $D\phi(x_0) = 0$ and

$$\min_{\{p\in\mathbb{R}^n \mid |p|=1\}} p \cdot D^2\phi(x_0) \cdot p \leqslant -f(x_0).$$

$u \in C(\Omega)$ is a viscosity subsolution of Eq. (1.1) in Ω if for any $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ satisfying

$$0 = \phi(x_0) - u(x_0) \leqslant \phi(x) - u(x) \quad \text{for all } x \in \Omega,$$

one of the following holds:

(1) $D\phi(x_0) \neq 0$ and

$$\Delta_\infty \phi(x_0) \geqslant -f(x_0)\left|D\phi(x_0)\right|^2;$$

or,

(2) $D\phi(x_0) = 0$ and

$$\max_{\{p\in\mathbb{R}^n \mid |p|=1\}} p \cdot D^2\phi(x_0) \cdot p \geqslant -f(x_0).$$

u is a viscosity solution of Eq. (1.1) in Ω if it is both a viscosity supersolution and subsolution.

A very useful tool to study the infinity Laplacian operator is “comparison with cones" which was introduced in [8] (see Definition 2.3). In this terminology, it had been proved in Jensen [12] that viscosity supersolutions (subsolutions) of the infinity Laplacian equation enjoy comparison with cones from blow (respectively, above), and that if $u \in C(\Omega)$ enjoys comparison with cones from above or from blow, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Crandall, Evans and Gariepy went on to observe that if u is upper semicontinuous and enjoys comparison with cones from above, then it is a subsolution of the infinity Laplacian equation and the quantity

$$\frac{\max_{\partial B_r(x)} u - u(x)}{r}$$

is nondecreasing with respect to r. Hence one can define

$$S_{u,+}(x) = \lim_{r\to 0^+} \frac{\max_{\partial B_r(x)} u - u(x)}{r}.$$

It turns out the function $S_{u,+}(x)$ has the following properties:

(1) $S_{u,+}(x)$ is upper-semicontinuous and

$$S_{u,+}(x) = \lim_{r\to 0} \operatorname{esssup}_{B_r(x)} |Du|. \tag{2.1}$$

(2) If u is differentiable at x, then $S_{u,+}(x) = |Du(x)|$.

(3) (Endpoint estimate.) Assume that $x_r \in \partial B_r(x)$ and $u(x_r) = \max_{\partial B_r(x)} u$, then

$$S_{u,+}(x_r) \geqslant \frac{\max_{\partial B_r(x)} u - u(x)}{r} \geqslant S_{u,+}(x). \tag{2.2}$$

Other notations we have used or will use includes:

- Ω is a bounded open subset of $\mathbb{R}^n$.
- For any set $V \in \mathbb{R}^n$, ∂V is its boundary and $\bar{V}$ is its closure.
- $B_r(x)$ is the open ball $\{y \in \mathbb{R}^n \mid |y - x| < r\}$, where $|\cdot|$ is the Euclidean norm.
- S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.
- For $p, q \in \mathbb{R}^n$, $p \cdot q$ is the usual inner product of p and q. If A is an $n \times n$ matrix, the $p \cdot A \cdot q$ means $p \cdot (Aq)$.
- For $p \in \mathbb{R}^n$, $p \otimes p$ is the $n \times n$ matrix whose (i, j) entry is $p_i p_j$.
- If $f : \Omega \to \mathbb{R}$, then Df is its gradient and $D^2 f$ is its Hessian matrix.

3. Proofs

To prove Theorem 1.1, we first prove the following lemma. Our proof heavily depends on the highly degenerate structure of Eqs. (1.3) and (1.4) and an elegant inequality in [9].

Lemma 3.1. *Let $\tau_1, \tau_2 \in \mathcal{R}$, $\tau_2 < \tau_1$, and the assumptions on B, c of Theorem* 1.1 *be satisfied. Then there does not exist a Lipschitz continuous function u in $\bar{\Omega}$ such that the following three conditions hold*:

(i) *u is a viscosity subsolution of*

$$B(x,u,Du)\cdot D^2u\cdot B(x,u,Du)+c(x,u,Du)=\tau_1 \quad in\ \Omega,$$

(ii) *u is a viscosity supersolution of*

$$B(x,u,Du)\cdot D^2u\cdot B(x,u,Du)+c(x,u,Du)=\tau_2 \quad in\ \Omega,$$

(iii) $u=0$ *on $\partial\Omega$.*

Proof. We argue by contradiction. Suppose that there exists a Lipschitz continuous function u in $\bar{\Omega}$ which satisfies (i)–(iii). Let us denote

$$\sup_{x\neq y\in\bar{\Omega}}\frac{|u(x)-u(y)|}{|x-y|}=C<+\infty; \tag{3.1}$$

Without loss of generality, we assume that there exists some $x_0\in\Omega$ such that $u(x_0)=1$. For $\epsilon>0$, let

$$u_\epsilon=\left(1+\epsilon^{\frac{3}{4}}\right)u$$

and

$$w_\epsilon(x,y)=u_\epsilon(x)-u(y)-\frac{1}{2\epsilon}|x-y|^2.$$

Let $(\bar{x},\bar{y})\in\bar{\Omega}\times\bar{\Omega}$ such that

$$w_\epsilon(\bar{x},\bar{y})=\max_{\bar{\Omega}\times\bar{\Omega}} w_\epsilon.$$

Owing to (3.1), we have that $|\bar{x}-\bar{y}|=\mathrm{O}(\epsilon)$. By (iii) and (3.1), if $(\bar{x},\bar{y})\in\partial(\Omega\times\Omega)$, then $w_\epsilon(\bar{x},\bar{y})=\mathrm{O}(\epsilon)$. Note that $w_\epsilon(x_0,x_0)=\epsilon^{\frac{3}{4}}$. Hence when ϵ is small enough, $(\bar{x},\bar{y})\in\Omega\times\Omega$. According to Crandall, Ishii and Lions [9], there exist two $n\times n$ symmetric matrices X and Y such that

$$\left(\frac{1}{\epsilon}(\hat{x}-\hat{y}),X\right)\in\bar{J}_V^{2,+}u_\epsilon(\hat{x}),\qquad\left(\frac{1}{\epsilon}(\hat{x}-\hat{y}),Y\right)\in\bar{J}_V^{2,-}u(\hat{y})$$

and

$$-\frac{3}{\epsilon}\begin{pmatrix}I_n&0\\0&I_n\end{pmatrix}\leqslant\begin{pmatrix}X&0\\0&-Y\end{pmatrix}\leqslant\frac{3}{\epsilon}\begin{pmatrix}I_n&-I_n\\-I_n&I_n\end{pmatrix}. \tag{3.2}$$

See [9] for definitions of $\bar{J}_V^{2,+}$ and $\bar{J}_V^{2,-}$. It is easy to see that u_ϵ is a viscosity subsolution of

$$B\left(x,\frac{u_\epsilon}{1+\epsilon^{\frac{3}{4}}},\frac{Du_\epsilon}{1+\epsilon^{\frac{3}{4}}}\right)\cdot D^2u_\epsilon\cdot B\left(x,\frac{u_\epsilon}{1+\epsilon^{\frac{3}{4}}},\frac{Du_\epsilon}{1+\epsilon^{\frac{3}{4}}}\right)+\left(1+\epsilon^{\frac{3}{4}}\right)c\left(x,\frac{u_\epsilon}{1+\epsilon^{\frac{3}{4}}},\frac{Du_\epsilon}{1+\epsilon^{\frac{3}{4}}}\right)=\tau_1\left(1+\epsilon^{\frac{3}{4}}\right).$$

Hence

$$B\left(\bar{x},u(\bar{x}),\frac{\bar{x}-\bar{y}}{\epsilon(1+\epsilon^{\frac{3}{4}})}\right)\cdot X\cdot B\left(\bar{x},u(\bar{x}),\frac{\bar{x}-\bar{y}}{\epsilon(1+\epsilon^{\frac{3}{4}})}\right)\left(1+\epsilon^{\frac{3}{4}}\right)c\left(\bar{x},u(\bar{x}),\frac{\bar{x}-\bar{y}}{\epsilon(1+\epsilon^{\frac{3}{4}})}\right)\geqslant\tau_1\left(1+\epsilon^{\frac{3}{4}}\right). \tag{3.3}$$

By (ii),

$$B\left(\bar{y},u(\bar{y}),\frac{\bar{x}-\bar{y}}{\epsilon}\right)\cdot Y\cdot B\left(\bar{y},u(\bar{y}),\frac{\bar{x}-\bar{y}}{\epsilon}\right)+c\left(\bar{y},u(\bar{y}),\frac{\bar{x}-\bar{y}}{\epsilon}\right)\leqslant\tau_2. \tag{3.4}$$

Owing to the right-hand side inequality in (3.2), we have that for $v_1, v_2 \in \mathbb{R}^n$,

$$v_1 \cdot X \cdot v_1 - v_2 \cdot Y \cdot v_2 \leqslant \frac{3}{\epsilon}|v_1 - v_2|^2.$$

Choosing

$$v_1 = B\left(\bar{x}, u(\bar{x}), \frac{\bar{x}-\bar{y}}{\epsilon(1+\epsilon^{3/4})}\right), \quad v_2 = B\left(\bar{y}, u(\bar{y}), \frac{\bar{x}-\bar{y}}{\epsilon}\right)$$

and using $|\bar{x} - \bar{y}| = \mathrm{O}(\epsilon)$, (3.3), (3.4), we discover that

$$\mathrm{o}(1) \geqslant \tau_1\left(1+\epsilon^{\frac{3}{4}}\right) - \tau_2,$$

where $\lim_{\epsilon\to 0} \mathrm{o}(1) = 0$. This is impossible when ϵ is small enough. So Lemma 3.1 holds. □

Proof of Theorem 1.1. For any $x_0 \in \Omega$, if $g(x_0) < f(x_0)$, then there exists $r > 0$ and $\tau_1 > \tau_2$ such that $\bar{B}_r(x_0) \subset \Omega$ and

$$g(x) < \tau_2 < \tau_1 < f(x) \quad \text{in } B_r(x_0).$$

Choose K large enough such that

$$u(x) < u(x_0) + Kr^2 \quad \text{on } \partial B_r(x_0).$$

Denote $\delta = \frac{1}{2}\min_{\partial B_r(x_0)}(u(x_0) + Kr^2 - u(x))$ and $v(x) = u(x) - u(x_0) - K|x - x_0|^2 + \delta$. Let

$$V = \left\{x \in B_r(x_0) \;\middle|\; v(x) > 0\right\}.$$

Obviously, $\bar{V} \subset B_r(x_0)$. If we define

$$\tilde{B}(x, z, p) = B\left(x, z + u(x_0) + K|x - x_0|^2 - \delta, p + 2K(x - x_0)\right)$$

and

$$\tilde{c}(x, z, p) = c\left(x, z + u(x_0) + K|x - x_0|^2 - \delta, p + 2K(x - x_0)\right) + 2K\left|\tilde{B}(x, z, p)\right|^2,$$

v is a viscosity subsolution of

$$\tilde{B}(x, v, Dv) \cdot D^2v \cdot \tilde{B}(x, v, Dv) + \tilde{c}(x, v, Dv) = \tau_1 \quad \text{in } V$$

and a viscosity supersolution of

$$\tilde{B}(x, v, Dv) \cdot D^2v \cdot \tilde{B}(x, v, Dv) + \tilde{c}(x, v, Dv) = \tau_2 \quad \text{in } V.$$

Note that in the open set V, v satisfies (i)–(iii) in Lemma 3.1. Since $\bar{V} \subset \Omega$, u is Lipschitz continuous in $\bar{V}$. Hence v is also Lipschitz continuous in $\bar{V}$. This is a contradiction. Hence $g(x_0) \geqslant f(x_0)$. So $f \leqslant g$. □

Corollary 3.2. *Suppose that $u, f, g \in C(\Omega)$ and u is simultaneously a viscosity solution of two equations*

$$\Delta_\infty u = f(x), \quad \Delta_\infty u = g(x) \quad \text{in } \Omega. \tag{3.5}$$

Then

$$f = g.$$

Proof. We argue by contradiction. If not, then there exists $x_0 \in \Omega$ such that $f(x_0) \neq g(x_0)$. Without loss of generality, we may assume that $f(x_0) > g(x_0)$. Then one of the following must occur: (i) $f(x_0) > 0$, (ii) $g(x_0) < 0$. Let us first look at case (i). Since Corollary 3.2 is a local problem, we may assume that

$$f(x) > \max\{0, g(x)\} \quad \text{for } x \in \Omega.$$

Hence u is a viscosity subsolution of the infinity Laplacian equation

$$\Delta_\infty u = 0 \quad \text{in } \Omega.$$

According to [8], $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Hence by Theorem 1.1, $f = g$ in Ω. This is a contradiction. Hence case (i) will not occur. Similarly, we can show that case (ii) will not occur either. This is a contradiction. Hence the above corollary holds. □

Remark 3.3. Gariepy, Wang and Yu [14], Yu [17] and Juutinen [15] provided a class of Hamiltonians H such that if $u \in C(\Omega)$ is a viscosity subsolution or a viscosity supersolution of the Aronsson equation

$$A_H(u) = 0 \quad \text{in } \Omega,$$

then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Hence by the proof of Corollary 3.2, for those H, the Aronsson operator A_H is also single valued under the weaker assumption $u \in C(\Omega)$.

To prove Theorem 1.2, we first prove the following lemma.

Lemma 3.4. *Suppose that $\tau_1 \neq \tau_2$. Then $u \in W^{1,\infty}(B_1(0))$ cannot be simultaneously a viscosity solution of two equations*

$$\frac{\Delta_\infty u}{|Du|^2} = \tau_1, \qquad \frac{\Delta_\infty u}{|Du|^2} = \tau_2. \tag{3.6}$$

Proof. We argue by contradiction. Without loss of generality, let us assume that $\tau_1 > \max\{0, \tau_2\}$. According to the definition of solutions of

$$\frac{\Delta_\infty u}{|Du|^2} = \tau_1 > 0,$$

u cannot be constant in any open subset of $B_1(0)$. So we may assume that $|Du|(0) = \delta > 0$, where $|Du|(x) = S_{u,+}(x)$. Let us denote

$$\operatorname{esssup}_{B_1(0)} |Du| = C. \tag{3.7}$$

Consider

$$w_\epsilon(h) = \max_{x,y \in \bar{B}_{\frac{1}{2}}(0)} \left(u(x+h) - u(y) - |y|^4 - \frac{1}{2\epsilon}|x-y|^2 \right).$$

Choose $h_\epsilon \in \partial B_{\epsilon^{3/4}}(0)$ such that

$$w_\epsilon(h_\epsilon) = \max_{h \in \partial B_{\epsilon^{3/4}}(0)} w_\epsilon.$$

Suppose that

$$w_\epsilon(h_\epsilon) = u(x_\epsilon + h_\epsilon) - u(y_\epsilon) - |y_\epsilon|^4 - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2$$

for $x_\epsilon, y_\epsilon \in \bar{B}_{\frac{1}{2}}$. Owing to (3.7), it is not hard to show that when ϵ is small

$$\frac{1}{\epsilon}|x_\epsilon - y_\epsilon| \leqslant C, \qquad |y_\epsilon|^4 \leqslant K\epsilon^{\frac{3}{4}}, \tag{3.8}$$

where K is a constant independent of ϵ. Moreover, it is clear that

$$u(x_\epsilon + h_\epsilon) = \max_{h \in \partial B_{\epsilon^{3/4}}(0)} u(x_\epsilon + h) = \max_{y \in \partial B_{\epsilon^{3/4}}(x_\epsilon)} u(y). \tag{3.9}$$

Owing to the definition of $w_\epsilon(h)$,

$$u(x_\epsilon + h_\epsilon) - u(y_\epsilon) - |y_\epsilon|^4 - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \geqslant \max_{h \in B_{\epsilon^{3/4}}(0)} \big(u(h) - u(0)\big). \tag{3.10}$$

Also,

$$\begin{aligned}
\frac{u(x_\epsilon + h_\epsilon) - u(x_\epsilon)}{\epsilon^{\frac{3}{4}}} &= \frac{u(x_\epsilon + h_\epsilon) - u(y_\epsilon)}{\epsilon^{\frac{3}{4}}} + \frac{u(y_\epsilon) - u(x_\epsilon)}{\epsilon^{\frac{3}{4}}} \\
&\geqslant \frac{u(x_\epsilon + h_\epsilon) - u(y_\epsilon)}{\epsilon^{\frac{3}{4}}} - \frac{C|x_\epsilon - y_\epsilon|}{\epsilon^{\frac{3}{4}}} \\
&\geqslant \frac{u(x_\epsilon + h_\epsilon) - u(y_\epsilon)}{\epsilon^{\frac{3}{4}}} - C^2\epsilon^{\frac{1}{4}}.
\end{aligned}$$

According to (3.10),

$$\frac{u(x_\epsilon+h_\epsilon)-u(y_\epsilon)}{\epsilon^{\frac{3}{4}}} \geqslant \max_{h\in\partial B_{\epsilon^{3/4}}(0)} \frac{u(h)-u(0)}{\epsilon^{\frac{3}{4}}} \geqslant |Du|(0)=\delta.$$

Since u is a viscosity subsolution of the infinity Laplacian equation, due to the endpoint estimate (2.2),

$$|Du|(x_\epsilon+h_\epsilon) \geqslant \frac{u(x_\epsilon+h_\epsilon)-u(x_\epsilon)}{\epsilon^{\frac{3}{4}}} \geqslant \delta - C^2\epsilon^{\frac{1}{4}}. \tag{3.11}$$

Therefore, when ϵ is small,

$$|Du|(x_\epsilon+h_\epsilon) \geqslant \frac{1}{2}\delta. \tag{3.12}$$

Obviously, when ϵ is small, both x_ϵ and y_ϵ are in the interior of $B_{\frac{1}{2}}(0)$. According to the *User's Guide* Crandall, Ishii and Lions [9], there exist two $n\times n$ symmetric matrices X and Y such that

$$\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon), X\right) \in \bar{J}_V^{2,+}u(x_\epsilon+h_\epsilon), \quad \left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon), Y\right) \in \bar{J}_V^{2,-}\big(u(y_\epsilon)+|y_\epsilon|^4\big)$$

and

$$-\frac{3}{\epsilon}\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{3}{\epsilon}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}. \tag{3.13}$$

See [9] for definitions of $\bar{J}_V^{2,+}$ and $\bar{J}_V^{2,-}$. Owing to the definition of $|Du|(x)=S_{u,+}(x)$, it is clear that

$$\frac{1}{\epsilon}|x_\epsilon-y_\epsilon| \geqslant |Du|(x_\epsilon+h_\epsilon) \geqslant \frac{\delta}{2}. \tag{3.14}$$

Since $u(\cdot+h_\epsilon)$ is a viscosity solution of Eq. (3.6), we have that

$$\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)\cdot X\cdot\frac{1}{\epsilon}(x_\epsilon-y_\epsilon) \geqslant \tau_1\frac{1}{\epsilon^2}|x_\epsilon-y_\epsilon|^2. \tag{3.15}$$

Also,

$$\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon, Y-4|y_\epsilon|^2I_n-8y_\epsilon\otimes y_\epsilon\right) \in \bar{J}_V^{2,-}u(y_\epsilon).$$

Due to Eq. (3.6),

$$\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon\right)\cdot\big(Y-4|y_\epsilon|^2I_n-8y_\epsilon\otimes y_\epsilon\big)\cdot\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon\right)$$
$$\leqslant \tau_2\left|\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon\right|^2.$$

Hence owing to (3.8),

$$\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon\right)\cdot Y\cdot\left(\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon\right) \leqslant \tau_2\frac{1}{\epsilon^2}|x_\epsilon-y_\epsilon|^2+\mathrm{o}(1), \tag{3.16}$$

where $\lim_{\epsilon\to 0}\mathrm{o}(1)=0$. Owing to the right-hand side inequality in (8), we have that for $v_1, v_2\in\mathbb{R}^n$,

$$v_1\cdot X\cdot v_1 - v_2\cdot Y\cdot v_2 \leqslant \frac{3}{\epsilon}|v_1-v_2|^2.$$

Choosing

$$v_1=\frac{1}{\epsilon}(x_\epsilon-y_\epsilon), \qquad v_2=\frac{1}{\epsilon}(x_\epsilon-y_\epsilon)-4|y_\epsilon|^2y_\epsilon$$

and using (3.14)–(3.16), one finds that

$$\frac{48|y_\epsilon|^6}{\epsilon} \geqslant (\tau_1-\tau_2)\frac{1}{\epsilon^2}|x_\epsilon-y_\epsilon|^2-\mathrm{o}(1) \geqslant (\tau_1-\tau_2)\left(\frac{\delta}{2}\right)^2-\mathrm{o}(1). \tag{3.17}$$

Owing to (3.8),

$$\frac{4|y_\epsilon|^6}{\epsilon} \leqslant \epsilon^{\frac{1}{8}}.$$

This contradicts to (3.17) when ϵ is small. □

Proof of Theorem 1.2. We argue by contradiction. If not, then there exists $x_0 \in \Omega$ such that $f(x_0) \neq g(x_0)$. Without loss of generality, we may assume that $f(x_0) > g(x_0)$. Then one of the following must occur: (i) $f(x_0) > 0$, (ii) $g(x_0) < 0$. Let us first look at case (i). Since Theorem 1.2 is a local result, we may assume that

$$f(x) > \tau_1 > \tau_2 > \max\{0, g(x)\} \quad \text{for } x \in \Omega,$$

where τ_1 and τ_2 are two positive constants. Hence u is also a viscosity supersolution of the infinity Laplacian equation

$$\Delta_\infty u = 0 \quad \text{in } \Omega.$$

According to [8], $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Choose $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega$. Then $u \in W^{1,\infty}(B_r(x_0))$. Consider

$$u_r(x) = u(rx + x_0).$$

Then $u_r \in W^{1,\infty}(B_1(0))$ and it is simultaneously a viscosity solution of two equations

$$\frac{\Delta_\infty u_r}{|Du_r|^2} = -r^2\tau_1, \quad \frac{\Delta_\infty u_r}{|Du_r|^2} = -r^2\tau_2 \quad \text{in } B_1(0). \tag{3.18}$$

This contradicts to Lemma 3.4. Similarly, we can show that case (ii) will not happen either. Hence Theorem 1.2 holds. □

Remark 3.5. By obvious modifications, our method can be used to prove that any operator like $\Delta_\infty u/f(x,u,Du)$ is single valued if $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is nonnegative or nonpositive. Here is a potential application. A very interesting problem about the infinity Laplacian operator is to find its geometric interpretation. More specifically, does there exist a function f such that $\Delta_\infty u/f(x,u,Du)$ represents some kind of curvature of the graph of u (might be in viscosity sense)? The answer of this question will justify the study of the parabolic infinity Laplacian equation. Our results implies that if there is indeed such a curvature, then it is well defined, i.e., a surface has at most one curvature.

Acknowledgement

The author would like to thank Prof. Michael Crandall for many valuable comments and suggestions which significantly improved our presentation of these results. His encouragement, as always, is deeply appreciated.

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