

On the uniqueness of weak solutions for the 3D Navier–Stokes equations

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Abstract

In this paper, we improve some known uniqueness results of weak solutions for the 3D Navier–Stokes equations. The proof uses the Fourier localization technique and the losing derivative estimates.

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1. Introduction

We consider the three-dimensional Navier–Stokes equations in \mathbb{R}^3

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0) = u_0(x), \end{cases} \quad (1.1)$$

where $u = (u^1(t, x), u^2(t, x), u^3(t, x))$ and $p = p(t, x)$ denote the unknown velocity vector and the unknown scalar pressure of the fluid respectively, while $u_0(x)$ is a given initial velocity vector satisfying $\operatorname{div} u_0 = 0$.

In a seminal paper [21], J. Leray proved the global existence of weak solution with finite energy, that is,

$$u(t, x) \in \mathcal{L}_T \stackrel{\text{def}}{=} L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \quad \text{for any } T > 0.$$

It is well known that weak solution is unique and regular in two spatial dimensions. In three dimensions, however, the question of regularity and uniqueness of weak solution is an outstanding open problem in mathematical fluid mechanics. In this paper, we are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that they become regular and unique. Let us firstly recall the definition of weak solution.

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Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. A measurable function u is called a weak solution of (1.1) on $(0, T) \times \mathbb{R}^3$ if it satisfies the following conditions:

- (1) $u \in \mathcal{L}_T \cap C_w([0, T]; L^2)$, where $C_w([0, T]; L^2)$ consists of all weak continuous functions with respect to time in $L^2(\mathbb{R}^3)$;
- (2) $\operatorname{div} u = 0$ in the sense of distribution;
- (3) For any function $\psi \in C_0^\infty([s, t] \times \mathbb{R}^3)$ with $\operatorname{div} \psi = 0$, there holds

$$\int_s^t \int_{\mathbb{R}^3} \{u \cdot \psi_t - \nabla u \cdot \nabla \psi + \nabla \psi : (u \otimes u)\}(t', x) dx dt' = \int_{\mathbb{R}^3} u(t, x) \cdot \psi(t, x) dx - \int_{\mathbb{R}^3} u(s, x) \cdot \psi(s, x) dx.$$

In addition, if u satisfies the energy inequality

$$\|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(t')\|_2^2 dt' \leq \|u_0\|_2^2,$$

it is also called a Leray–Hopf weak solution.

The Leray–Hopf weak solutions are unique and regular in the class

$$\begin{aligned} \mathcal{P} &= L^q(0, T; L^r) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 1, \quad 3 \leq r \leq \infty \quad [11,14,15,25,27], \\ \text{or } \mathcal{P} &= L^q(0, T; W^{1,r}) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 2, \quad \frac{3}{2} < r \leq \infty \quad [1], \\ \text{or } \mathcal{P} &= L^q(0, T; W^{s,r}) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 1 + s, \quad \frac{3}{1+s} < r \leq \infty, \quad s \geq 0 \quad [26]. \end{aligned}$$

Recently, there are many researches devoted to refine the above results. First of all, we have the following refined regularity criterion in the framework of Besov spaces: the weak solutions are regular in the class

$$\mathcal{P} = C([0, T]; B_{\infty,\infty}^{-1}) \quad \text{or} \quad \mathcal{P} = L^q(0, T; B_{p,\infty}^r),$$

with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leq \infty$, and $-1 < r \leq 1$, see [4,8,17,18]. Concerning the refined uniqueness criterion of weak solutions, Kozono and Taniuchi [16] proved the uniqueness of the Leray–Hopf weak solutions in the class

$$\mathcal{P} = L^2(0, T; BMO).$$

Gallagher and Planchon [12] proved the uniqueness in the class

$$\mathcal{P} = L^q(0, T; \dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{2}{q}}) \quad \text{with } \frac{3}{p} + \frac{2}{q} > 1.$$

Lemarié-Rieusset [19] proved the uniqueness in the class

$$\mathcal{P} = C([0, T]; X_1^{(0)}) \quad \text{or} \quad \mathcal{P} = L^{\frac{2}{1-r}}(0, T; X_r) \quad \text{with } r \in [0, 1).$$

Finally, Germain [13] proved the uniqueness in the class

$$\mathcal{P} = C([0, T]; X_1^{(0)}) \quad \text{or} \quad \mathcal{P} = L^{\frac{2}{1-r}}(0, T; X_r) \quad \text{with } r \in [-1, 1).$$

Here $B_{p,q}^s$ denotes the Besov space and

$$X_s := \begin{cases} \mathbf{M}(\dot{H}^s, L^2), & \text{if } s \in (0, 1], \\ \Lambda^s BMO, & \text{if } s \in (-1, 0], \\ \mathbf{Lip}, & \text{if } s = -1, \end{cases}$$

where $\mathbf{M}(\dot{H}^s, L^2)$ is the space of distributions such that their pointwise product with a function in \dot{H}^s belongs to L^2 , $\Lambda^s = (1 - \Delta)^{\frac{s}{2}}$. $X_s^{(0)}$ denotes the closure of the Schwartz class in X_s . We want to point out that

$$X_s \hookrightarrow \Lambda^s BMO, \quad \text{if } s \in (0, 1]. \tag{1.2}$$

We refer to [13] for more properties about X_s . The key step of their proofs is to find a path space \mathcal{P} so that the trilinear form

$$F(u, v, w) := \int_0^T \int_{\mathbb{R}^3} u \cdot \nabla v \cdot w \, dx \, dt$$

is continuous from $(\mathcal{L}_T)^2 \times \mathcal{P}$ to \mathbb{R} . Germain also pointed out that the path space \mathcal{P} he found is optimal in some sense (see [13, p. 400] for precise meaning).

The purpose of this paper is to improve the above uniqueness results.

Theorem 1.2. *Let $u_0, v_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} v_0 = 0$. Let u and v be two Leray–Hopf weak solutions of (1.1) on $(0, T)$ with the initial data u_0 and v_0 respectively. Assume that*

$$u \in L^q(0, T; B_{p,\infty}^r),$$

with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leq \infty$, $r \in (0, 1]$, and $(p, r) \neq (\infty, 1)$. Then there holds

$$\|u(t) - v(t)\|_2^2 + \int_0^t \|\nabla(u - v)(t')\|_2^2 \, dt' \leq \|u_0 - v_0\|_2^2 \exp \left\{ C \int_0^t (e + \|u(t')\|_{B_{p,\infty}^r})^q \, dt' \right\}.$$

In particular, if $u_0 = v_0$, then $u = v$ a.e. on $(0, T) \times \mathbb{R}^3$.

Remark 1.3. Due to the embedding relation

$$B_{p,q}^s \subsetneq B_{p,\infty}^s, \quad q < +\infty \quad \text{and} \quad \Lambda^{-r} BMO \subsetneq B_{\infty,\infty}^r,$$

Theorem 1.2 is an improvement of the corresponding results given by Gallagher and Planchon [12] and Germain [13]. The proof only uses an important observation that if $u \in L^q(0, T; B_{p,\infty}^r)$ with (p, q, r) as in Theorem 1.2, then u can be decomposed as

$$u = u^l + u^h \quad \text{with } u^l \in L^1(0, T; \mathbf{Lip}) \text{ and } u^h \in L^{\tilde{q}}(0, T; L^{\tilde{p}})$$

for some \tilde{p}, \tilde{q} satisfying $\frac{2}{\tilde{q}} + \frac{3}{\tilde{p}} = 1$, $\tilde{p} > 3$, see Lemma 3.1.

In the case where either $r \leq 0$ or $(p, r) = (\infty, 1)$, using Bony’s decomposition and the losing derivative estimates, we prove

Theorem 1.4. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Let u and v be two weak solutions of (1.1) on $(0, T)$ with the same initial data u_0 . Assume that u and v satisfy one of the following two conditions:*

(a) $u \in L^{q_1}(0, T; B_{p_1,\infty}^{r_1})$ and $v \in L^{q_2}(0, T; B_{p_2,\infty}^{r_2})$, where

$$\frac{2}{q_1} + \frac{3}{p_1} = 1 + r_1, \quad \frac{2}{q_2} + \frac{3}{p_2} = 1 + r_2,$$

with $r_1, r_2 \in (-1, 0]$, $r_1 + r_2 > -1$, $\frac{3}{1+r_1} < p_1 \leq \infty$, $\frac{3}{1+r_2} < p_2 \leq \infty$.

(b) $u, v \in L^1(0, T; B_{\infty,\infty}^1)$.

Then $u = v$ a.e. on $(0, T) \times \mathbb{R}^3$.

Remark 1.5. Due to the embedding relation

$$X_s \subsetneq \Lambda^s BMO \subsetneq B_{\infty,\infty}^{-s}, \quad s \in (0, 1],$$

the condition imposed on weak solution in Theorem 1.4 is weaker than that of Germain [13] and Lemarié-Rieusset [19]. However, the price to pay is to impose the conditions on both weak solutions.

Remark 1.6. The main novelty of Theorem 1.4 is that weak solutions are unique in the class $L^1(0, T; B_{\infty,\infty}^1)$. In particular, from the inequality

$$\|u\|_{B_{\infty,\infty}^1} \leq C(\|u\|_2 + \|\operatorname{curl} u\|_{B_{\infty,\infty}^0}) \quad (\text{see Section 2 for its proof}), \tag{1.3}$$

we can obtain the *Beale–Kato–Majda* type uniqueness criterion: if weak solutions u and v with the same initial data satisfy

$$\operatorname{curl} u, \operatorname{curl} v \in L^1(0, T; B_{\infty,\infty}^0),$$

then $u = v$ on $(0, T) \times \mathbb{R}^3$. Secondly, Theorem 1.4 allows us to impose different conditions on both weak solutions. Thirdly, we do not impose the energy inequality on weak solutions.

Remark 1.7. Chemin and Lemarié-Rieusset [6,20] proved the uniqueness of weak solutions in the class $C([0, T]; B_{\infty,\infty}^{-1})$. While, Theorem 1.4 gives the uniqueness in the class $L^1(0, T; B_{\infty,\infty}^1)$. It is natural to expect that the uniqueness also holds in the class $L^{\frac{2}{1+r}}(0, T; B_{\infty,\infty}^r)$ for $r \in (-1, 1)$ from the viewpoint of interpolation. This problem remains unknown for the case of $r \in (-1, -\frac{1}{2}]$.

Remark 1.8. The result (b) in Theorem 1.4 is also valid for the Euler equation. In detail, let $u, v \in C_\omega([0, T]; L^2(\mathbb{R}^3)) \cap L^1([0, T]; B_{\infty,\infty}^1(\mathbb{R}^3))$ be two weak solutions of the Euler equation with the same initial data, then $u = v$ a.e. on $[0, T)$.

Notation. Throughout the paper, C stands for a generic constant. We will use the notation $A \lesssim B$ to denote the relation $A \leq CB$, and $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p .

2. Preliminaries

Let us firstly recall some basic facts on the Littlewood–Paley decomposition, one may check [5] for more details. Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that for any $\xi \in \mathbb{R}^3$,

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1. \tag{2.1}$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, the frequency localization operator Δ_j and S_j are defined by

$$\begin{aligned} \Delta_j f &= \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy, \quad \text{for } j \geq 0, \\ S_j f &= \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x - y) dy, \end{aligned}$$

and

$$\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2.$$

With our choice of φ , one can easily verify that

$$\begin{aligned} \Delta_j \Delta_k f &= 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \\ \Delta_j (S_{k-1} f \Delta_k f) &= 0 \quad \text{if } |j - k| \geq 5. \end{aligned} \tag{2.2}$$

For any $f \in \mathcal{S}'(\mathbb{R}^3)$, we have by (2.1) that

$$f = S_0(f) + \sum_{j \geq 0} \Delta_j f, \tag{2.3}$$

which is called the Littlewood–Paley decomposition. In the sequel, we will constantly use the Bony’s decomposition from [2] that

$$uv = T_u v + T_v u + R(u, v), \tag{2.4}$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u, v) = \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v,$$

and we also denote

$$T'_u v = T_u v + R(u, v).$$

With the introduction of Δ_j , let us recall the definition of the inhomogenous Besov space from [29]:

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the inhomogenous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} := \begin{cases} (\sum_{j=-1}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_p, & \text{for } q = \infty. \end{cases}$$

Let us point out that $B^s_{\infty,\infty}$ is the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$ and the following inclusion relations hold

$$\mathbf{Lip} \subsetneq B^1_{\infty,\infty}, \quad \Lambda^{-s} BMO \subsetneq B^s_{\infty,\infty} \quad \text{for } s \in \mathbb{R}.$$

We refer to [13,29] for more properties.

The following Bernstein’s inequalities will be frequently used throughout the paper.

Lemma 2.2. (See [5].) Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p$, then there hold

$$\begin{aligned} \text{supp } \hat{f} \subset \{|\xi| \leq C2^j\} &\Rightarrow \|\partial^\alpha f\|_q \leq C2^{j|\alpha|+3j(\frac{1}{p}-\frac{1}{q})} \|f\|_p, \\ \text{supp } \hat{f} \subset \left\{ \frac{1}{C}2^j \leq |\xi| \leq C2^j \right\} &\Rightarrow \|f\|_p \leq C2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_p. \end{aligned}$$

Here the constant C is independent of f and j .

We conclude this section by a proof of the inequality (1.3). Using Lemma 2.2, we have

$$\|\Delta_{-1} u\|_\infty \leq C \|u\|_2,$$

and for $j \geq 0$,

$$2^j \|\Delta_j u\|_\infty \leq C \|\Delta_j \nabla u\|_\infty.$$

Due to the Biot–Savart law [23], ∇u can be written as

$$\nabla u(x) = Cw(x) + K * w(x), \quad w = \text{curl } u,$$

where C is a constant matrix, and K is a matrix valued function with homogeneous of degree -3 . So, we get that for $j \geq 0$,

$$2^j \|\Delta_j u\|_\infty \leq C \|\Delta_j w\|_\infty,$$

where we used the fact that

$$\|\Delta_j(Tf)\|_p \leq C \|\Delta_j f\|_p, \quad \text{for } j \geq 0, 1 \leq p \leq \infty,$$

if T is a singular integral operator of convolution type with smooth kernel [28]. Then the inequality (1.3) is concluded from the definition of Besov space.

3. Proofs of theorems

This section is devoted to the proof of Theorems 1.2 and 1.4.

3.1. The proof of Theorem 1.2

The proof is based on the following decomposition lemma which may be independent of interest.

Lemma 3.1. *Assume that $u \in L^q(0, T; B_{p,\infty}^r)$ with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leq \infty$, $r \in (0, 1]$, and $(p, r) \neq (\infty, 1)$. Then u can be decomposed as*

$$u = u^l + u^h \quad \text{with } u^l \in L^1(0, T; \mathbf{Lip}) \text{ and } u^h \in L^{\tilde{q}}(0, T; L^{\tilde{p}})$$

for some \tilde{p}, \tilde{q} satisfying $\frac{2}{\tilde{q}} + \frac{3}{\tilde{p}} = 1$, $\tilde{p} > 3$.

Proof. Fix $N \in \mathbb{N}$ to be determined later on. We set

$$u^l = S_N u, \quad u^h = u - u^l.$$

By the definition of S_N and Lemma 2.2, we have

$$\|\nabla u^l\|_\infty \leq C \sum_{j \leq N-1} 2^{j(1+\frac{3}{p})} \|\Delta_j u\|_p \leq C 2^{2(1-\frac{1}{q})N} \|u\|_{B_{p,\infty}^r}. \tag{3.1}$$

Due to the conditions on (p, q, r) , we can choose \tilde{p} such that

$$\tilde{p} > \max(3, p) \quad \text{and} \quad \frac{3}{p} - \frac{3}{\tilde{p}} - r < 0.$$

Thus, by Lemma 2.2

$$\|u^h\|_{\tilde{p}} \leq \sum_{j \geq N} 2^{(\frac{3}{p}-\frac{3}{\tilde{p}})j} \|\Delta_j u\|_p \leq C 2^{(\frac{3}{p}-\frac{3}{\tilde{p}}-r)N} \|u\|_{B_{p,\infty}^r}. \tag{3.2}$$

Now we choose

$$N = \left\lceil \frac{q}{2} \log_2 (e + \|u\|_{B_{p,\infty}^r}) \right\rceil + 1.$$

Then by (3.1), we have

$$\int_0^T \|\nabla u^l(t)\|_\infty dt \leq C \int_0^T (e + \|u(t)\|_{B_{p,\infty}^r})^q dt < +\infty. \tag{3.3}$$

On the other hand, from (3.2) we get that

$$\int_0^T \|u^h(t)\|_{\tilde{B}_p^q}^q dt \leq C \int_0^T (e + \|u(t)\|_{B_{p,\infty}^r})^q dt < +\infty. \tag{3.4}$$

Hence, we complete the proof of Lemma 3.1 by (3.3) and (3.4). \square

Lemma 3.2. *Let u, v be as in Theorem 1.2. Set $w = u - v$. Then for any $t \in [0, T]$, there holds*

$$\langle u(t), v(t) \rangle + 2 \int_0^t \langle \nabla u, \nabla v \rangle dt' = \langle u_0, v_0 \rangle + \int_0^t \langle w \cdot \nabla u, w \rangle dt'.$$

Proof. Lemma 3.1 ensures that the trilinear form

$$F(u, v, w) := \int_0^T \int_{\mathbb{R}^3} u \cdot \nabla w \cdot v \, dx \, dt$$

is continuous from $(\mathcal{L}_T)^2 \times L^q(0, T; B_{p,\infty}^r)$ to \mathbb{R} . Then the lemma can be proved by following the argument of Lemma 4.4 in [13]. Here we omit the details. \square

Now we are in position to prove Theorem 1.2. Since u and v are Leray–Hopf weak solutions, there hold

$$\begin{aligned} \|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(t')\|_2^2 dt' &\leq \|u_0\|_2^2, \\ \|v(t)\|_2^2 + 2 \int_0^t \|\nabla v(t')\|_2^2 dt' &\leq \|v_0\|_2^2. \end{aligned}$$

On the other hand, Lemma 3.2 yields that

$$\langle u(t), v(t) \rangle + 2 \int_0^t \langle \nabla u, \nabla v \rangle dt' = \langle u_0, v_0 \rangle + \int_0^t \langle w \cdot \nabla u, w \rangle dt'.$$

Combining the above inequalities, we obtain

$$\begin{aligned} \|w(t)\|_2^2 + 2 \int_0^t \|\nabla w(t')\|_2^2 dt' &= \|u(t)\|_2^2 + \|v(t)\|_2^2 - 2\langle u, v \rangle(t) + 2 \int_0^t \|\nabla u(t')\|_2^2 dt' \\ &\quad + 2 \int_0^t \|\nabla v(t')\|_2^2 dt' - 4 \int_0^t \langle \nabla u, \nabla v \rangle(t') dt' \\ &\leq \|u_0 - v_0\|_2^2 - 2 \int_0^t \langle w \cdot \nabla u, w \rangle dt'. \end{aligned} \tag{3.5}$$

We decompose $u = u^l + u^h$ as in Lemma 3.1 and rewrite

$$\int_0^t \langle w \cdot \nabla u, w \rangle dt' = \int_0^t \langle w \cdot \nabla u^l, w \rangle dt' + \int_0^t \langle w \cdot \nabla u^h, w \rangle dt'.$$

We get by Hölder inequality that

$$\left| \int_0^t \langle w \cdot \nabla u^l, w \rangle dt' \right| \leq \int_0^t \|w(t')\|_2^2 \|\nabla u^l(t')\|_\infty dt'. \tag{3.6}$$

Integration by parts, we get

$$\int_0^t \langle w \cdot \nabla u^h, w \rangle dt' = - \int_0^t \langle w \cdot \nabla w, u^h \rangle dt',$$

from which and the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned} \left| \int_0^t \langle w \cdot \nabla u^h, w \rangle dt' \right| &\leq \int_0^t \|\nabla w\|_2 \|w\|_{\frac{2\tilde{p}}{\tilde{p}-2}} \|u^h\|_{\tilde{p}} dt' \\ &\leq C \int_0^t \|\nabla w\|_2 \|w\|_2^{1-\frac{3}{\tilde{p}}} \|\nabla w\|_2^{\frac{3}{\tilde{p}}} \|u^h\|_{\tilde{p}} dt' \\ &\leq C \left(\int_0^t \|w(t')\|_2^2 \|u^h(t')\|_{\tilde{p}}^{\tilde{q}} dt' \right)^{\frac{1}{\tilde{q}}} \left(\int_0^t \|\nabla w(t')\|_2^2 dt' \right)^{1-\frac{1}{\tilde{q}}} \\ &\leq C \int_0^t \|w(t')\|_2^2 \|u^h(t')\|_{\tilde{p}}^{\tilde{q}} dt' + \int_0^t \|\nabla w(t')\|_2^2 dt'. \end{aligned}$$

This together with (3.5) and (3.6) gives

$$\|w(t)\|_2^2 + \int_0^t \|\nabla w(t')\|_2^2 dt' \leq \|u_0 - v_0\|^2 + C \int_0^t \|w(t')\|_2^2 (\|\nabla u^l(t')\|_\infty + \|u^h(t')\|_{\tilde{p}}^{\tilde{q}}) dt'.$$

This jointed with the Gronwall inequality produces that

$$\begin{aligned} \|w(t)\|_2^2 + \int_0^t \|\nabla w(t')\|_2^2 dt' &\leq \|u_0 - v_0\|^2 \exp \left\{ C \int_0^t (\|\nabla u^l(t')\|_\infty + \|u^h(t')\|_{\tilde{p}}^{\tilde{q}}) dt' \right\} \\ &\leq \|u_0 - v_0\|_2^2 \exp \left\{ C \int_0^t (e + \|u(t')\|_{B_{p,\infty}^r})^q dt' \right\}. \end{aligned}$$

This finishes the proof of Theorem 1.2.

3.2. The proof of Theorem 1.4

Assume that u and v are two weak solutions of (1.1) on $(0, T)$ with the initial data u_0 . Let $w = u - v$, w satisfies the equation in the sense of distribution

$$w_t - \Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla \tilde{p} = 0, \tag{3.7}$$

for some pressure \tilde{p} . We get by taking the operation Δ_j on both sides of (3.7) that

$$\partial_t \Delta_j w - \Delta \Delta_j w + \Delta_j (w \cdot \nabla u) + \Delta_j (v \cdot \nabla w) + \nabla \Delta_j \tilde{p} = 0. \tag{3.8}$$

Multiplying (3.8) by $\Delta_j w$, we get by Lemma 2.2 for $j \geq -1$ that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j w(t)\|_2^2 + ca_j 2^{2j} \|\Delta_j w(t)\|_2^2 \leq -\langle \Delta_j (w \cdot \nabla u), \Delta_j w \rangle - \langle \Delta_j (v \cdot \nabla w) - v \cdot \nabla \Delta_j w, \Delta_j w \rangle, \tag{3.9}$$

with $a_{-1} = 0$ and $a_j = 1$ for $j \geq 0$. Here we used the fact that

$$\langle \Delta_j(v \cdot \nabla w), \Delta_j w \rangle = \langle \Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w, \Delta_j w \rangle.$$

Case 1. u and v satisfy the assumption (a).

Due to $r_1 + r_2 > -1$, one of r_1 and r_2 must be bigger than $-\frac{1}{2}$. Without loss of generality, we assume that $r_1 > -\frac{1}{2}$.

Step 1. Estimate of $\langle \Delta_j(w \cdot \nabla u), \Delta_j w \rangle$.

Using the Bony’s decomposition (2.4), we have

$$\Delta_j(w \cdot \nabla u) = \Delta_j(T_{w^i} \partial_i u) + \Delta_j(T_{\partial_i u} w^i) + \Delta_j R(w^i, \partial_i u).$$

Considering the support of the Fourier transform of the term $T_{w^i} \partial_i u$, we have

$$\Delta_j(T_{w^i} \partial_i u) = \sum_{|j'-j| \leq 4} \Delta_j(S_{j'-1} w^i \partial_i \Delta_{j'} u). \tag{3.10}$$

This gives by Lemma 2.2 that

$$\begin{aligned} \|\Delta_j(T_{w^i} \partial_i u)\|_2 &\lesssim \sum_{|j'-j| \leq 4} 2^{j'} \sum_{k \leq j'-2} \|\Delta_k w\|_{\frac{2p_1}{p_1-2}} \|\Delta_{j'} u\|_{p_1} \lesssim \sum_{|j'-j| \leq 4} 2^{j'} \sum_{k \leq j'-2} 2^{k \frac{3}{p_1}} \|\Delta_k w\|_2 \|\Delta_{j'} u\|_{p_1} \\ &\lesssim 2^{j(1-r_1)} \|u\|_{B_{p_1, \infty}^{r_1}} \sum_{j' \leq j+2} 2^{j' \frac{3}{p_1}} \|\Delta_{j'} w\|_2. \end{aligned} \tag{3.11}$$

Similarly, we have

$$\Delta_j(T_{\partial_i u} w^i) = \sum_{|j'-j| \leq 4} \Delta_j(S_{j'-1}(\partial_i u) \Delta_{j'} w^i). \tag{3.12}$$

Applying Lemma 2.2 to (3.12) yields that

$$\begin{aligned} \|\Delta_j(T_{\partial_i u} w^i)\|_2 &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^k \|\Delta_k u\|_{\infty} \|\Delta_{j'} w\|_2 \\ &\lesssim 2^{j(1-r_1 + \frac{3}{p_1})} \|u\|_{B_{p_1, \infty}^{r_1}} \sum_{|j'-j| \leq 4} \|\Delta_{j'} w\|_2. \end{aligned} \tag{3.13}$$

Since $\operatorname{div} w = 0$, we have

$$\Delta_j R(w^i, \partial_i u) = \sum_{j', j'' \geq j-3; |j'-j''| \leq 1} \partial_i \Delta_j (\Delta_{j'} w^i \Delta_{j''} u), \tag{3.14}$$

from which and Lemma 2.2, it follows that

$$\begin{aligned} \|\Delta_j R(w^i, \partial_i u)\|_{\frac{2p_1}{p_1+2}} &\lesssim \sum_{j', j'' \geq j-3; |j'-j''| \leq 1} 2^j \|\Delta_{j'} w\|_2 \|\Delta_{j''} u\|_{p_1} \\ &\lesssim 2^j \|u\|_{B_{p_1, \infty}^{r_1}} \sum_{j' \geq j-3} 2^{-j' r_1} \|\Delta_{j'} w\|_2. \end{aligned} \tag{3.15}$$

Summing up (3.11)–(3.15), we obtain

$$\begin{aligned} |\langle \Delta_j(w \cdot \nabla u), \Delta_j w \rangle| &\lesssim 2^{j(1-r_1)} \|u\|_{B_{p_1, \infty}^{r_1}} \sum_{j' \leq j} 2^{j' \frac{3}{p_1}} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 \\ &\quad + 2^{j(1+\frac{3}{p_1})} \|u\|_{B_{p_1, \infty}^{r_1}} \sum_{j' \geq j} 2^{-j' r_1} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2. \end{aligned} \tag{3.16}$$

Step 2. Estimate of $\langle \Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w, \Delta_j w \rangle$.

Using the Bony’s decomposition (2.4), we write

$$\begin{aligned} \Delta_j(v \cdot \nabla w) &= \Delta_j(T_{v^i} \partial_i w) + \Delta_j(T_{\partial_i w} v^i) + \Delta_j R(v^i, \partial_i w), \\ v \cdot \nabla \Delta_j w &= T_{v^i} \partial_i \Delta_j w + T'_{\partial_i \Delta_j w} v^i. \end{aligned}$$

Then we have

$$\Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w = [\Delta_j, T_{v^i}] \partial_i w + \Delta_j(T_{\partial_i w} v^i) + \Delta_j R(v^i, \partial_i w) - T'_{\partial_i \Delta_j w} v^i.$$

Similar arguments as in deriving (3.11) and (3.15), we have

$$\|\Delta_j(T_{\partial_i w} v^i)\|_2 \lesssim 2^{-jr_2} \|v\|_{B_{p_2, \infty}^{r_2}} \sum_{j' \leq j+2} 2^{j'(1+\frac{3}{p_2})} \|\Delta_{j'} w\|_2, \tag{3.17}$$

$$\|\Delta_j R(v^i, \partial_i w)\|_{\frac{2p_2}{p_2+2}} \lesssim 2^j \|v\|_{B_{p_2, \infty}^{r_2}} \sum_{j' \geq j-3} 2^{-j'r_2} \|\Delta_{j'} w\|_2. \tag{3.18}$$

In view of the definition of $T'_{\partial_i \Delta_j w} v^i$,

$$T'_{\partial_i \Delta_j w} v^i = \sum_{j' \geq j-2} S_{j'+2} \Delta_j \partial_i w \Delta_{j'} v^i,$$

and note that $S_{j'+2} \Delta_j w = \Delta_j w$ for $j' > j$, we get

$$\langle T'_{\partial_i \Delta_j w} v^i, \Delta_j w \rangle = \sum_{j-2 \leq j' \leq j} \langle S_{j'+2} \Delta_j \partial_i w \Delta_{j'} v^i, \Delta_j w \rangle,$$

from which and Lemma 2.2, it follows that

$$|\langle T'_{\partial_i \Delta_j w} v^i, \Delta_j w \rangle| \lesssim 2^{j(1+\frac{3}{p_2}-r_2)} \|v\|_{B_{p_2, \infty}^{r_2}} \|\Delta_j w\|_2^2. \tag{3.19}$$

Now, we turn to estimate $[T_{v^i}, \Delta_j] \partial_i w$. In view of the definition of Δ_j , we write

$$\begin{aligned} [T_{v^i}, \Delta_j] \partial_i w &= \sum_{|j'-j| \leq 4} [S_{j'-1} v^i, \Delta_j] \partial_i \Delta_{j'} w \\ &= \sum_{|j'-j| \leq 4} 2^{3j} \int_{\mathbb{R}^3} h(2^j(x-y)) (S_{j'-1} v^i(x) - S_{j'-1} v^i(y)) \partial_i \Delta_{j'} w(y) dy \\ &= \sum_{|j'-j| \leq 4} 2^{4j} \int_{\mathbb{R}^3} \int_0^1 y \cdot \nabla S_{j'-1} v^i(x-\tau y) d\tau \partial_i h(2^j y) \Delta_{j'} w(x-y) dy, \end{aligned} \tag{3.20}$$

from which and the Minkowski inequality, we deduce that

$$\begin{aligned} \|[T_{v^i}, \Delta_j] \partial_i w\|_2 &\lesssim \sum_{|j'-j| \leq 4} \|\nabla S_{j'-1} v\|_{\infty} \|\Delta_{j'} w\|_2 \\ &\lesssim 2^{j(1+\frac{3}{p_2}-r_2)} \|v\|_{B_{p_2, \infty}^{r_2}} \sum_{|j'-j| \leq 4} \|\Delta_{j'} w\|_2. \end{aligned} \tag{3.21}$$

Summing up (3.17)–(3.21), we obtain

$$\begin{aligned} |\langle \Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w, \Delta_j w \rangle| &\lesssim 2^{-jr_2} \|v\|_{B_{p_2, \infty}^{r_2}} \sum_{j' \leq j} 2^{j'(1+\frac{3}{p_2})} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 \\ &\quad + 2^{j(1+\frac{3}{p_2})} \|v\|_{B_{p_2, \infty}^{r_2}} \sum_{j' \geq j} 2^{-j'r_2} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2. \end{aligned} \tag{3.22}$$

Under the assumption (a), we can choose s such that

$$-r_1 < s < \min(1 + r_1, 1 + r_2). \tag{3.23}$$

From (3.9), (3.16) and (3.22), it follows that

$$\begin{aligned} & 2^{-2js} \|\Delta_j w(t)\|_2^2 + a_j 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt' \\ & \lesssim \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}} 2^{j(1-r_1-2s)} \sum_{j' \leq j} 2^{j' \frac{3}{p_1}} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 dt' \\ & \quad + \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}} 2^{j(1+\frac{3}{p_1}-2s)} \sum_{j' \geq j} 2^{-j' r_1} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 dt' \\ & \quad + \int_0^t \|v\|_{B_{p_2, \infty}^{r_2}} 2^{-j(r_2+2s)} \sum_{j' \leq j} 2^{j'(1+\frac{3}{p_2})} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 dt' \\ & \quad + \int_0^t \|v\|_{B_{p_2, \infty}^{r_2}} 2^{j(1+\frac{3}{p_2}-2s)} \sum_{j' \geq j} 2^{-j' r_2} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2 dt' \\ & := \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \tag{3.24}$$

We set

$$W(t) = \sup_{j \geq -1} 2^{-js} \|\Delta_j w(t)\|_2.$$

Using (3.23) and the Young’s inequality, we obtain

$$\begin{aligned} \text{I} & \leq \sum_{j' \leq j} 2^{(j'-j)(r_1+s)} \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}} W(t')^{\frac{2}{q_1}} (2^{j'(1-s)} \|\Delta_{j'} w\|_2)^{1-\frac{2}{q_1}} 2^{j(1-s)} \|\Delta_j w\|_2 dt' \\ & \leq C \left(\int_0^t \|u\|_{B_{p_1, \infty}^{r_1}}^{q_1} W(t')^2 dt' \right)^{\frac{1}{q_1}} \left(\sup_{j \geq -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt' \right)^{\frac{1}{q_1}} \\ & \leq C \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}}^{q_1} W(t')^2 dt' + \delta \sup_{j \geq -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt', \end{aligned}$$

and for II, we have

$$\begin{aligned} \text{II} & \leq \sum_{j' \geq j} 2^{(j'-j)(s-1-r_1)} \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}} W(t')^{\frac{2}{q_1}} 2^{j'(1-s)} \|\Delta_{j'} w\|_2 (2^{j(1-s)} \|\Delta_j w\|_2)^{1-\frac{2}{q_1}} dt' \\ & \leq C \int_0^t \|u\|_{B_{p_1, \infty}^{r_1}}^{q_1} W(t')^2 dt' + \delta \sup_{j \geq -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt', \end{aligned}$$

and similarly for IV,

$$\text{IV} \leq C \int_0^t \|v\|_{B_{p_2, \infty}^{r_2}}^{q_2} W(t')^2 dt' + \delta \sup_{j \geq -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt',$$

and for III,

$$\begin{aligned} \text{III} &\leq \sum_{j' \leq j} 2^{(j'-j)(1+r_2+s)} \int_0^t \|v\|_{B_{p_2, \infty}^{r_2}} W(t')^{\frac{2}{q_2}} (2^{j'(1-s)} \|\Delta_{j'} w\|_2)^{1-\frac{2}{q_2}} 2^{j(1-s)} \|\Delta_j w\|_2 dt' \\ &\leq C \int_0^t \|v\|_{B_{p_2, \infty}^{r_2}}^{q_2} W(t')^2 dt' + \delta \sup_{j \geq -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2 dt'. \end{aligned}$$

Collecting these estimates with (3.24) implies that

$$W(t)^2 \leq C \int_0^t (\|u(t')\|_{B_{p_1, \infty}^{q_1}} + \|v(t')\|_{B_{p_2, \infty}^{q_2}}) W(t')^2 dt'.$$

This together with the Gronwall inequality shows that

$$W(t) = 0, \quad \text{i.e. } u = v = 0.$$

This completes the proof of case (a).

Case 2. u and v satisfy the assumption (b).

Since u and v are non-Lipschitz vectors, we will use the idea of the losing derivative estimate which was firstly introduced by Chemin and Lerner [7]. We can refer to [9] for a systematic study. Recently, Danchin and Paicu [10] applied this idea to prove the uniqueness of weak solution for the 2-D Boussinesq equations with partial viscosity. The present proof is motivated by [10]. We also refer to [3,22,24] for the other applications about the losing derivative estimate.

Let $s \in (0, 1)$. For $\lambda > 0$, we set

$$W_j^\lambda(t) = 2^{-js} e^{-\lambda \varepsilon_j(t)} \|\Delta_j w(t)\|_2,$$

where $\varepsilon_j(t)$ is defined by

$$\varepsilon_j(t) = \int_0^t 2^{j'} \sum_{j' \leq j+4} (\|\Delta_{j'} u(t')\|_\infty + \|\Delta_{j'} v(t')\|_\infty) dt'.$$

We get by (3.9) that

$$\begin{aligned} &\frac{d}{dt} W_j^\lambda(t) + \lambda \varepsilon_j'(t) W_j^\lambda(t) + a_j 2^{2j} W_j^\lambda(t) \\ &\lesssim 2^{-js} e^{-\lambda \varepsilon_j(t)} \left(\|\Delta_j (w \cdot \nabla u)\|_2 + \left\| \Delta_j (v \cdot \nabla w) - v \cdot \nabla \Delta_j w + \sum_{j' > j} \partial_i \Delta_j w \Delta_{j'} v^i \right\|_2 \right). \end{aligned} \tag{3.25}$$

Here we used the fact that

$$\langle \partial_i \Delta_j w \Delta_{j'} v^i, \Delta_j w \rangle = -\langle \Delta_{j'} \partial_i v^i \Delta_j w, \Delta_j w \rangle = 0.$$

Since $W_j^\lambda(0) = 0$, we get by integrating (3.25) on $[0, t]$ that

$$\begin{aligned}
 &W_j^\lambda(t) + \lambda \int_0^t \varepsilon'_j(t') W_j^\lambda(t') dt' + a_j 2^{2j} \int_0^t W_j^\lambda(t') dt' \\
 &\lesssim 2^{-js} \int_0^t e^{-\lambda \varepsilon_j(t')} \|\Delta_j(w \cdot \nabla u)(t')\|_2 dt' \\
 &\quad + 2^{-js} \int_0^t e^{-\lambda \varepsilon_j(t')} \left\| \Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w + \sum_{j' > j} \partial_i \Delta_j w \Delta_{j'} v^i \right\|_2(t') dt'.
 \end{aligned} \tag{3.26}$$

Step 1. Estimate of $\|\Delta_j(w \cdot \nabla u)\|_2$.

Using the Bony’s decomposition (2.4), we write

$$\Delta_j(w \cdot \nabla u) = \Delta_j(T_{w^i} \partial_i u) + \Delta_j(T_{\partial_i u} w^i) + \Delta_j R(w^i, \partial_i u).$$

By (3.10) and Lemma 2.2, we get

$$\begin{aligned}
 \|\Delta_j(T_{w^i} \partial_i u)\|_2 &\lesssim \sum_{|j'-j| \leq 4} 2^{j'} \sum_{k \leq j'-2} \|\Delta_k w\|_2 \|\Delta_{j'} u\|_\infty \lesssim \sum_{|j'-j| \leq 4} 2^{j'} \sum_{k \leq j'-2} 2^{ks} e^{\lambda \varepsilon_k(t)} W_k^\lambda(t) \|\Delta_{j'} u\|_\infty \\
 &\lesssim \sum_{j' \leq j+2} 2^{j's} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_j(t).
 \end{aligned} \tag{3.27}$$

By (3.12), (3.14) and Lemma 2.2, we have

$$\begin{aligned}
 \|\Delta_j(T_{\partial_i u} w^i)\|_2 &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^k \|\Delta_k u\|_\infty \|\Delta_{j'} w\|_2 \\
 &\lesssim \sum_{|j'-j| \leq 4} 2^{j's} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \sum_{k \leq j'-2} 2^k \|\Delta_k u\|_\infty \\
 &\lesssim \sum_{|j'-j| \leq 4} 2^{j's} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_j(t),
 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
 \|\Delta_j R(w^i, \partial_i u)\|_2 &\lesssim \sum_{j', j'' \geq j-3; |j'-j''| \leq 1} 2^j \|\Delta_{j'} w\|_2 \|\Delta_{j''} u\|_\infty \\
 &\lesssim \sum_{j', j'' \geq j-3; |j'-j''| \leq 1} 2^{j's+j} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \|\Delta_{j''} u\|_\infty \\
 &\lesssim \sum_{j' \geq j-3} 2^{j'(s-1)+j} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_j(t).
 \end{aligned} \tag{3.29}$$

Summing up (3.27)–(3.29), we obtain

$$\begin{aligned}
 2^{-js} \int_0^t e^{-\lambda \varepsilon_j(t')} \|\Delta_j(w \cdot \nabla u)(t')\|_2 dt' &\lesssim \sum_{j' \leq j} 2^{(j'-j)s} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_j(t') dt' \\
 &\quad + \sum_{j' \geq j} 2^{-(j'-j)(1-s)} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_j(t') dt'.
 \end{aligned} \tag{3.30}$$

Step 2. Estimate of $\|\Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w + \sum_{j' > j} \partial_i \Delta_j w \Delta_{j'} v^i\|_2$.

Using the Bony’s decomposition (2.4), we write

$$\Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w = [\Delta_j, T_{v^i}] \partial_i w + \Delta_j(T_{\partial_i w} v^i) + \Delta_j R(v^i, \partial_i w) - T'_{\partial_i \Delta_j w} v^i.$$

Similar to the proof of (3.27) and (3.29), we get

$$\|\Delta_j(T_{\partial_i w} v^i)\|_2 \lesssim \sum_{j' \leq j+2} 2^{j's} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_{j'}(t), \tag{3.31}$$

$$\|\Delta_j R(v^i, \partial_i w)\|_2 \lesssim \sum_{j' \geq j-3} 2^{j'(s-1)+j} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_{j'}(t). \tag{3.32}$$

Using the formula (3.20) again, we have

$$\|[\Delta_j, T_{v^i}] \partial_i w\|_2 \lesssim \sum_{|j'-j| \leq 4} 2^{j's} e^{\lambda \varepsilon_{j'}(t)} W_{j'}^\lambda(t) \varepsilon'_{j'}(t). \tag{3.33}$$

Note that

$$T'_{\partial_i \Delta_j w} v^i - \sum_{j' > j} \partial_i \Delta_j w \Delta_{j'} v^i = \sum_{j-2 \leq j' \leq j} S_{j'+2} \Delta_j \partial_i w \Delta_{j'} v^i,$$

it gives by Lemma 2.2 that

$$\left\| T'_{\partial_i \Delta_j w} v^i - \sum_{j' > j} \partial_i \Delta_j w \Delta_{j'} v^i \right\|_2 \lesssim 2^{js} e^{\lambda \varepsilon_j(t)} W_j^\lambda(t) \varepsilon'_j(t). \tag{3.34}$$

Summing up (3.31)–(3.34), we obtain

$$\begin{aligned} & 2^{-js} \int_0^t e^{-\lambda \varepsilon_j(t')} \|\Delta_j(v \cdot \nabla w) - v \cdot \nabla \Delta_j w\|_2 dt' \\ & \lesssim \sum_{j' \leq j} 2^{(j'-j)s} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt' \\ & \quad + \sum_{j' \geq j} 2^{-(j'-j)(1-s)} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt'. \end{aligned} \tag{3.35}$$

From (3.26), (3.30) and (3.35), it follows that

$$\begin{aligned} & W_j^\lambda(t) + \lambda \int_0^t \varepsilon'_j(t') W_j^\lambda(t') dt' + a_j 2^{2j} \int_0^t W_j^\lambda(t') dt' \\ & \lesssim \sum_{j' \leq j} 2^{(j'-j)s} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt' \\ & \quad + \sum_{j' \geq j} 2^{-(j'-j)(1-s)} \int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt' \\ & := \text{I} + \text{II}. \end{aligned} \tag{3.36}$$

Write

$$\varepsilon'_j(t') = \varepsilon'_{j'}(t') + (\varepsilon'_j(t') - \varepsilon'_{j'}(t')),$$

and note that $\varepsilon'_j(t') - \varepsilon'_{j'}(t') \geq 0$ for $j \geq j'$, we obtain

$$I \lesssim \sum_{j' \leq j} 2^{(j'-j)s} \int_0^t W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt' + \frac{1}{\lambda} \sum_{j' \leq j} 2^{(j'-j)s} \sup_{t' \in [0,t]} W_{j'}^\lambda(t'), \tag{3.37}$$

here we used the inequality

$$\int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))} (\varepsilon'_{j'}(t') - \varepsilon'_{j'}(t')) dt' \lesssim \frac{1}{\lambda}, \quad \text{for } j' \leq j.$$

Since $\varepsilon_{j'}(t') - \varepsilon_j(t')$ is an increasing function in t' for $j' \geq j$, we have

$$\Pi \lesssim \sum_{j' \geq j} 2^{-(j'-j)(1-s)} e^{\lambda(\varepsilon_{j'}(t) - \varepsilon_j(t))} \int_0^t W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt'. \tag{3.38}$$

Let us for the moment assume that

$$\lambda(\|u\|_{L^1(0,t;B_{\infty,\infty}^1)} + \|v\|_{L^1(0,t;B_{\infty,\infty}^1)}) < (1-s) \log 2. \tag{3.39}$$

Notice that

$$\varepsilon_{j'}(t) - \varepsilon_j(t) \leq (j' - j)(\|u\|_{L^1(0,t;B_{\infty,\infty}^1)} + \|v\|_{L^1(0,t;B_{\infty,\infty}^1)}),$$

which together with (3.38) ensures that

$$\Pi \lesssim \int_0^t W_{j'}^\lambda(t') \varepsilon'_{j'}(t') dt'. \tag{3.40}$$

Summing up (3.36), (3.37) and (3.40), we obtain

$$\begin{aligned} & \sup_{j \geq -1, t' \in [0,t]} W_j^\lambda(t') + \lambda \sup_{j \geq -1} \int_0^t \varepsilon'_j(t') W_j^\lambda(t') dt' + \sup_{j \geq -1} 2^{2j} \int_0^t W_j^\lambda(t') dt' \\ & \leq C \sup_{j \geq -1} \int_0^t \varepsilon'_j(t') W_j^\lambda(t') dt' + \frac{C}{\lambda} \sup_{j \geq -1, t' \in [0,t]} W_j^\lambda(t'), \end{aligned}$$

from which, we get by taking λ big enough that

$$\sup_{j \geq -1, t' \in [0,t]} W_j^\lambda(t') = 0.$$

On the other hand, the assumption (b) ensures that we can choose $t > 0$ small enough such that (3.39) holds. Thus, $u = v$ on $[0, t]$, and then we can conclude that $u = v$ on $[0, T]$ by a standard continuity argument. The proof of case (b) is completed.

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