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Nonlinear Schrödinger equation on real hyperbolic spaces

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Abstract

We consider the Schrödinger equation with no radial assumption on real hyperbolic spaces $\mathbb{H}^n$. We obtain in all dimensions $n \geqslant 2$ sharp dispersive and Strichartz estimates for a large family of admissible pairs. As a first consequence, we obtain strong well-posedness results for NLS. Specifically, for small initial data, we prove L^2 and H^1 global well-posedness for any subcritical power (in contrast with the Euclidean case) and with no gauge invariance assumption on the nonlinearity F. On the other hand, if F is gauge invariant, L^2 charge is conserved and hence, as in the Euclidean case, it is possible to extend local L^2 solutions to global ones. The corresponding argument in H^1 requires conservation of energy, which holds under the stronger condition that F is defocusing. Recall that global well-posedness in the gauge invariant case was already proved by Banica, Carles and Staffilani, for small radial L^2 data or for large radial H^1 data. The second application of our global Strichartz estimates is *scattering* for NLS both in L^2 and in H^1, with no radial or gauge invariance assumption. Notice that, on Euclidean spaces $\mathbb{R}^n$, this is only possible for the critical power $\gamma = 1 + \frac{4}{n}$ and can be false for subcritical powers while, on hyperbolic spaces $\mathbb{H}^n$, global existence and scattering of small L^2 solutions hold for all powers $1 < \gamma \leqslant 1 + \frac{4}{n}$. If we restrict to defocusing nonlinearities F, we can extend the H^1 scattering results of Banica, Carles and Staffilani to the nonradial case. Also there is no distinction anymore between short range and long range nonlinearities: the geometry of hyperbolic spaces makes every power-like nonlinearity short range.

Résumé

Nous étudions l'équation de Schrödinger sur les espaces hyperboliques réels $\mathbb{H}^n$, sans aucune hypothèse de radialité. Nous commençons par établir une inégalité dispersive optimale en toute dimension $n \geqslant 2$, ainsi qu'une inégalité de Strichartz pour une grande famille de paires admissibles. Nous en déduisons que l'équation semi-linéaire est fortement bien posée dans L^2 ou dans H^1, pour des données initiales petites et pour des non-linéarités relativement générales, en particulier pour toutes les puissance sous-critiques (contrairement au cas euclidien) et sans hypothèse d'invariance par changement de jauge. Dans ce dernier cas, on a conservation de la charge et les solutions L^2 locales se prolongent en solutions L^2 globales ; le phénomène analogue dans H^1 repose sur la conservation de l'énergie, qui est vérifiée pour des non-linéarités défocalisantes. Rappelons que Banica, Carles et Staffilani avaient précédemment montré que l'équation semi-linéaire était globalement bien posée pour des non-linéarités invariantes par changement de jauge et pour des données radiales petites dans L^2 ou arbitraires dans H^1. Comme seconde application, nous montrons qu'il y a diffusion (*scattering*) dans L^2 et dans H^1, à nouveau sans hypothèse de radialité ou d'invariance par changement de jauge. Rappelons que dans $\mathbb{R}^n$ ceci n'est possible que pour l'exposant critique $\gamma = 1 + \frac{4}{n}$ et peut être faux pour des exposants sous-critiques, tandis que sur l'espace hyperboliques $\mathbb{H}^n$, on a existence globale et diffusion pour tout exposant $1 < \gamma \leqslant 1 + \frac{4}{n}$

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(et pour des conditions initiales petites dans L^2). Dans le cas défocalisant, nous pouvons étendre au cas non radial les résultats de diffusion H^1 de Banica, Carles, Staffilani. Observons également que, sur l'espace hyperbolique, toutes les non-linéarités de type puissance n'ont qu'un effet à courte portée, contrairement au cas euclidien.

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1. Introduction

The nonlinear Schrödinger equation (NLS) in Euclidean space $\mathbb{R}^n$

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = F\big(u(t,x)\big), \\ u(0,x) = f(x) \end{cases} \tag{1}$$

has motivated a number of mathematical results in the last 30 years. Indeed, this equation (especially in the *cubic* case $F(u) = \pm u|u|^2$) seems ubiquitous in physics and appears in many different contexts, including nonlinear optics, the theory of Bose–Einstein condensates and of water waves. In particular a detailed scattering theory for NLS has been developed.

An essential tool in the study of (1) is the *dispersive estimate*

$$\big\|e^{it\Delta} f\big\|_{L^\infty(\mathbb{R}^n)} \leqslant C|t|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}$$

for the linear homogeneous Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0) = f. \end{cases} \tag{2}$$

This estimate is classical and follows directly from the representation formula for the fundamental solution. A well-known procedure (introduced by Kato [15], Ginibre and Velo [9], and perfected by Keel and Tao [16]) then leads to the *Strichartz estimates*

$$\|u\|_{L^p(I;L^q(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'}(\mathbb{R}^n))} \tag{3}$$

for the linear inhomogeneous Cauchy problem

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = F(t,x), \\ u(0,x) = f(x). \end{cases}$$

The estimates (3) hold for any bounded or unbounded time interval $I \subseteq \mathbb{R}$ and for all pairs $(p,q), (\tilde{p},\tilde{q}) \in [2,\infty] \times [2,\infty)$ satisfying the *admissibility condition*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{4}$$

Notice that both endpoints $(p,q) = (\infty,2)$ and $(p,q) = (2, \frac{2n}{n-2})$ are included in dimension $n \geqslant 3$ while only the first one is included in dimension $n = 2$.

The issue of well-posedness for the nonlinear Cauchy problem (1) has been widely studied, at least for a power nonlinearity $F(u) = \pm|u|^\gamma$ or $F(u) = \pm|u|^{\gamma-1}u$ and for suitable ranges of the exponent $\gamma > 1$. Here is a brief account of the classical theory. In the model case $F(u) = |u|^\gamma$, we have

- local well-posedness in L^2 in the subcritical case $\gamma < 1 + \frac{4}{n}$;
- global well-posedness in L^2 in the critical case $\gamma = 1 + \frac{4}{n}$ for small data;
- local well-posedness in H^1 in the subcritical case $\gamma < 1 + \frac{4}{n-2}$;
- global well-posedness in H^1 in the critical case $\gamma = 1 + \frac{4}{n-2}$ for small data.

Notice that the value of the critical exponent depends on the dimension n. On the other hand, in the model case $F(u) = |u|^{\gamma-1}u$, Eq. (1) is *gauge invariant* and *defocusing*, which implies L^2 and H^1 conservation laws. Thus, in addition to the previous results, we have

- global well-posedness in L^2 in the subcritical case $\gamma < 1 + \frac{4}{n}$;
- global well-posedness in H^1 in the subcritical case $\gamma < 1 + \frac{4}{n-2}$.

Global existence for arbitrary data in the critical case remains an open problem, although several results are available (Bourgain [5], Tao, Visan and Zhang [19], ...).

The results above are proved essentially by a fixed point argument in a suitable mixed space $L^p(\mathbb{R}; L^q(\mathbb{R}^n))$, using Strichartz estimates in combination with conservation laws when available.

As a byproduct, this method shows that solutions $u(t,x)$ to (1) are small in a suitable L^q_x sense as $t \to \pm\infty$. Hence, asymptotically, the contribution of the nonlinearity is dominated by the linear part and the *nonlinear* equation (1) becomes close to the *linear* equation (2). This basic observation is at the origin of scattering theory for NLS. By L^2 *scattering* we mean that, for every global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^n))$, there exist *scattering data* $u_\pm \in L^2(\mathbb{R}^n)$ such that

$$\big\|u(t) - e^{it\Delta}u_\pm\big\|_{L^2_x} \to 0 \quad \text{as } t \to \pm\infty.$$

The definition of H^1 *scattering* is analogous.

The classical scattering theory for NLS, in the defocusing case $F(u) = |u|^{\gamma-1}u$, can be summarized as follows:

- scattering in L^2 holds in the critical case $\gamma = 1 + \frac{4}{n}$ for small data;
- scattering in H^1 holds for $1 + \frac{4}{n} < \gamma < 1 + \frac{4}{n-2}$;
- scattering in H^1 fails for $1 < \gamma \leqslant 1 + \frac{2}{n}$.

This paper is a contribution to the study of Strichartz estimates and NLS on a manifold M. Several results have been obtained for this problem and quite general classes of manifolds. The geometry of M plays obviously an essential role: on a compact or positively curved manifold, one expects weaker decay properties and hence weaker results for NLS; on the other hand, on a noncompact negatively curved manifold, one expects better dispersion properties than in the Euclidean case and hence stronger well-posedness and scattering results for NLS.

The compact case has been studied extensively by Burq, Gérard and Tzvetkov [6] after earlier results by Bourgain [5] on the torus. In general one obtains Strichartz estimates

$$\big\|e^{it\Delta}f\big\|_{L^p(I;L^q(M))} \leqslant C(I)\|f\|_{H^{1/p}(M)}$$

which are local in time and with a loss of smoothness in space. As a consequence, the results for NLS are weaker than on $\mathbb{R}^n$. Let us mention in particular the local well-posedness theory in $H^s(\mathbb{T}^n)$ developed by Bourgain in the early nineties, extended ten years later to general compact manifolds by Burq, Gérard and Tzvetkov, and improved in some special cases such as spheres $\mathbb{S}^n$ [6] or 4-dimensional compact manifolds [8].

In this paper we shall restrict our attention to real hyperbolic spaces $M = \mathbb{H}^n$ of dimension $n \geqslant 2$. Actually our results extend straightforwardly to all hyperbolic spaces i.e. Riemannian symmetric spaces of noncompact type and rank one (they extend furthermore to Damek–Ricci spaces and this will be the subject of a forthcoming work). Consider the following linear Cauchy problem on $\mathbb{H}^n$:

$$\begin{cases} i\partial_t u(t,x) + \Delta_{\mathbb{H}^n} u(t,x) = F(t,x), & t \in \mathbb{R},\ x \in \mathbb{H}^n, \\ u(0,x) = f(x). \end{cases} \tag{5}$$

On one hand, Banica [3] (see also [17]) obtained the following weighted dispersive estimate, for radial solutions to the homogeneous equation (5) in dimension $n \geqslant 3$:

$$w(x)\big|u(t,x)\big| \leqslant C\big(|t|^{-\frac{n}{2}} + |t|^{-\frac{3}{2}}\big)\int_{\mathbb{H}^n} \big|f(y)\big|w(y)^{-1}\,dy.$$

Here $w(x) = \frac{\sinh r}{r}$, where r denotes the geodesic distance from x to the origin. On the other hand, Pierfelice [18] obtained the following sharp weighted Strichartz estimate, for radial solutions to the inhomogeneous equation (5) in dimension $n \geqslant 3$:

$$\left\| w^{\frac{1}{2}-\frac{1}{q}} u \right\|_{L_t^p L_x^q} \leqslant C \|f\|_{L_x^2} + C \left\| w^{\frac{1}{\tilde{q}}-\frac{1}{2}} F \right\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

Here $w(r) = (\frac{\sinh r}{r})^{n-1}$ is the jacobian of the exponential map and $(\frac{1}{p}, \frac{1}{q})$, $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ belong to the interval $I_n = \{(\frac{1}{p}, \frac{1}{q}) \in [0, \frac{1}{2}] \times (0, \frac{1}{2}] \mid \frac{2}{p} + \frac{n}{q} = \frac{n}{2}\}$. Actually this result was established in the more general setting of Damek–Ricci spaces and it implies unweighted estimates for a wider range of indices, as pointed out by Banica, Carles and Staffilani [4].

Our first main result is the following dispersive estimate (Theorem 3.4), which holds for general functions (no radial assumption) in dimension $n \geqslant 2$.

Dispersive estimate. Let $q, \tilde{q} \in (2, \infty]$. Then, for $0 < |t| < 1$, we have

$$\left\| u(t) \right\|_{L^q(\mathbb{H}^n)} \leqslant C |t|^{-\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\} n} \|f\|_{L^{\tilde{q}'}(\mathbb{H}^n)}$$

while, for $|t| \geqslant 1$, we have

$$\left\| u(t) \right\|_{L^q(\mathbb{H}^n)} \leqslant C |t|^{-\frac{3}{2}} \|f\|_{L^{\tilde{q}'}(\mathbb{H}^n)}.$$

If $q = \tilde{q} = 2$, we have of course L^2 conservation $\|u(t)\|_{L^2(\mathbb{H}^n)} = \|f\|_{L^2(\mathbb{H}^n)}$ for all $t \in \mathbb{R}$. Our second main result is the following Strichartz estimate (Theorem 3.6), which is deduced from the previous estimate and holds under the same general assumptions.

Strichartz estimate. Assume that $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ belong to the triangle $T_n = \{(\frac{1}{p}, \frac{1}{q}) \in (0, \frac{1}{2}] \times (0, \frac{1}{2}) \mid \frac{2}{p} + \frac{n}{q} \geqslant \frac{n}{2}\} \cup \{(0, \frac{1}{2})\}$. Then

$$\|u\|_{L_t^p L_x^q} \leqslant C \|f\|_{L_x^2} + C \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

Notice that the set T_n of admissible pairs for $\mathbb{H}^n$ is much wider than the corresponding set I_n for $\mathbb{R}^n$ (which is just the lower edge of the triangle). This striking phenomenon was already observed in [4] for radial solutions. It can be regarded as an effect of hyperbolic geometry on dispersion.

Next we apply these estimates to study well-posedness and scattering for the nonlinear Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = F(u), \\ u(0) = f. \end{cases} \tag{6}$$

Throughout our paper, we shall use the following (standard) terminology about the nonlinearity $F = F(u)$:

- F is *power-like* if there exist constants $\gamma > 1$ and $C \geqslant 0$ such that

$$\begin{cases} \left|F(u)\right| \leqslant C|u|^\gamma, \\ \left|F(u) - F(v)\right| \leqslant C\big(|u|^{\gamma-1} + |v|^{\gamma-1}\big)|u - v|, \end{cases} \tag{7}$$

- F is *gauge invariant* if

$$\operatorname{Im}\big\{F(u)\bar{u}\big\} = 0, \tag{8}$$

- F is *defocusing* if there exists a C^1 function $G = G(v) \geqslant 0$ such that

$$F(u) = G'\big(|u|^2\big)u. \tag{9}$$

Notice that gauge invariance implies L^2 conservation of *charge* or *mass*:

$$\int_{\mathbb{H}^n} \left|u(t,x)\right|^2 dx = \int_{\mathbb{H}^n} \left|f(x)\right|^2 dx \quad \forall t,$$

while the defocusing assumption implies H^1 conservation of *energy*:

$$\int_{\mathbb{H}^n} \left|\nabla u(t,x)\right|^2 dx + \int_{\mathbb{H}^n} G\big(\left|u(t,x)\right|^2\big)\, dx = \text{constant}.$$

If we specialize to the model cases $F = \pm|u|^\gamma$ and $F = \pm|u|^{\gamma-1}u$, then the gauge invariant nonlinearities are

$$F = \pm|u|^{\gamma-1}u$$

and the defocusing ones

$$F = +|u|^{\gamma-1}u.$$

Let us first summarize our well-posedness results (Theorems 4.2 and 4.4).

Well-posedness for NLS. Consider the Cauchy problem (6) with a power-like nonlinearity F of order γ.

- Assume $\gamma \leqslant 1 + \frac{4}{n}$. Then the problem is globally well-posed for small L^2 data. For arbitrary L^2 data, it is locally well-posed if $\gamma < 1 + \frac{4}{n}$.
- Assume $\gamma \leqslant 1 + \frac{4}{n-2}$. Then the problem is globally well-posed for small H^1 data. For arbitrary H^1 data, it is locally well-posed if $\gamma < 1 + \frac{4}{n-2}$.
- If F is gauge invariant and $\gamma < 1 + \frac{4}{n}$, the problem is globally well-posed for arbitrary L^2 data. If F is defocusing and $\gamma < 1 + \frac{4}{n-2}$, the problem is globally well-posed for arbitrary H^1 data.

Similar results were obtained in [4] for radial functions and nonlinearities $F = |u|^{\gamma-1}u$. As expected, they are better for hyperbolic spaces than for Euclidean spaces. For instance, on $\mathbb{H}^n$ we have global well-posedness for small L^2 data, for any power $1 < \gamma \leqslant 1 + \frac{4}{n}$, while on $\mathbb{R}^n$ we must assume in addition gauge invariance. Of course, under this condition, we can also handle arbitrarily large data, using conservation laws, as in the Euclidean case.

Let us next summarize our scattering results.

Scattering for NLS. Consider the Cauchy problem (6) with a power-like nonlinearity F of order γ.

- Assume $\gamma \leqslant 1 + \frac{4}{n}$. Then, for all small data $f \in L^2$, the unique global solution $u(t,x)$ has the scattering property: there exist $u_\pm \in L^2$ such that

$$\left\|u(t) - e^{it\Delta_{\mathbb{H}^n}} u_\pm\right\|_{L^2(\mathbb{H}^n)} \to 0 \quad \text{as } t \to \pm\infty.$$

- Assume $\gamma \leqslant 1 + \frac{4}{n-2}$. Then, for all small data $f \in H^1$, the unique global solution $u(t,x)$ has the scattering property: there exist $u_\pm \in H^1$ such that

$$\left\|u(t) - e^{it\Delta_{\mathbb{H}^n}} u_\pm\right\|_{H^1(\mathbb{H}^n)} \to 0 \quad \text{as } t \to \pm\infty.$$

- Assume $\gamma < 1 + \frac{4}{n-2}$ and the defocusing condition. Then, for all data $f \in H^1$ at $t = \pm\infty$, the NLS has a unique global solution $u(t,x)$ with the following scattering property:

$$\left\|u(t) - e^{it\Delta_{\mathbb{H}^n}} f\right\|_{H^1(\mathbb{H}^n)} \to 0 \quad \text{as } t \to \pm\infty.$$

Notice that on $\mathbb{H}^n$ we have small data scattering for all powers $1 < \gamma \leqslant 1 + \frac{4}{n}$ (respectively $1 < \gamma \leqslant 1 + \frac{4}{n-2}$). This is in sharp contrast with $\mathbb{R}^n$, where scattering is known to fail for the range $1 < \gamma \leqslant 1 + \frac{2}{n}$.

The results in this paper were presented by the second author at the *Convegno Nazionale di Analisi Armonica* (Caramanico, 22–25 May 2007) and by the first author at the Conference (in honor of Sigurdur Helgason on the occasion of his 80th birthday) *Integral Geometry, Harmonic Analysis and Representation Theory* (Reykjavik, 15–18 August 2007) and at the DFG–JSPS Joint Seminar *Infinite Dimensional Harmonic Analysis IV* (Tokyo, 10–14 September 2007).

A related preprint [14] was produced independently and simultaneously by Ionescu and Staffilani. While we are mostly interested in sharp dispersive and Strichartz estimates, with applications to general nonlinearities and scattering, their main aim is scattering in H^1 and the Morawetz inequality in the defocusing case. Thus our works, although overlapping, are complementary rather than concurrent.

2. Real hyperbolic spaces

In this paper, we consider the simplest class of Riemannian symmetric spaces of noncompact type, namely real hyperbolic spaces $\mathbb{H}^n$ of dimension $n \geqslant 2$. We refer to Helgason's books [10–12] for their structure, geometric properties, and for harmonic analysis on these spaces. Recall that $\mathbb{H}^n$ can be realized as the upper sheet

$$\begin{cases} x_0^2 - x_1^2 - \cdots - x_n^2 = 1, \\ x_0 \geqslant 1, \end{cases}$$

of hyperboloid in $\mathbb{R}^{1+n}$, equipped with the Riemannian metric

$$d\ell^2 = -dx_0^2 + dx_1^2 + \cdots + dx_n^2,$$

or as the homogeneous space G/K, where $G = \mathrm{SO}(1,n)^0$ and $K = \mathrm{SO}(n)$. In geodesic polar coordinates, the Riemannian metric is given by

$$d\ell^2 = dr^2 + (\sinh r)^2 \, d\ell^2_{\mathbb{S}^{n-1}},$$

the Riemannian volume by

$$dv = (\sinh r)^{n-1} \, dr \, dv_{\mathbb{S}^{n-1}},$$

and the Laplace–Beltrami operator by

$$\widetilde{\Delta} = \Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\coth r \, \partial_r + (\sinh r)^{-2} \Delta_{\mathbb{S}^{n-1}}.$$

Inhomogeneous Sobolev spaces on $\mathbb{H}^n$ (and on more general manifolds) are defined by

$$H^{s,q}(\mathbb{H}^n) = (I - \widetilde{\Delta})^{-\frac{s}{2}} L^q(\mathbb{H}^n) \quad (1 < q < \infty,\ s \in \mathbb{R}).$$

Using L^q spectral analysis (see for instance [1]), they can be also defined as well by

$$H^{s,q}(\mathbb{H}^n) = (-\widetilde{\Delta})^{-\frac{s}{2}} L^q(\mathbb{H}^n).$$

Moreover, for $s = N \in \mathbb{N}$, $H^{s,q}(\mathbb{H}^n)$ coincides with

$$W^{N,q}(\mathbb{H}^n) = \left\{ f \in L^q(\mathbb{H}^n) \;\middle|\; \left|\nabla^j f\right| \in L^q(\mathbb{H}^n)\ \forall 0 \leqslant j \leqslant N \right\},$$

where ∇ denotes the covariant derivative. Recall eventually the Sobolev embedding theorem:

$$H^{s,q}(\mathbb{H}^n) \subset H^{\tilde{s},\tilde{q}}(\mathbb{H}^n) \quad \text{if } s - \tilde{s} \geqslant n\left(\frac{1}{q} - \frac{1}{\tilde{q}}\right) > 0.$$

3. Dispersive and Strichartz estimates on $\mathbb{H}^n$

Consider first the homogeneous linear Schrödinger equation on the hyperbolic space $\mathbb{H}^n$ of dimension $n \geqslant 2$:

$$\begin{cases} i\partial_t u(t,x) + \widetilde{\Delta} u(t,x) = 0, \\ u(0,x) = f(x), \end{cases}$$

whose solution is given by

$$u(t,x) = e^{it\widetilde{\Delta}} f(x) = f * s_t(x) = \int_{\mathbb{H}^n} s_t\big(d(x,y)\big) f(y) \, dy.$$

The convolution kernel s_t is a bi-K-invariant function on G i.e. a radial function on $G/K = \mathbb{H}^n$, which can be expressed as an inverse spherical Fourier transform:

$$s_t(r) = \text{const.}\, e^{-i(\frac{n-1}{2})^2 t} \int_{-\infty}^{+\infty} e^{-it\lambda^2} \varphi_\lambda(r) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}.$$

For hyperbolic spaces $\mathbb{H}^n$ (and more generally for Damek–Ricci spaces), this expression can be made more explicit, using the inverse Abel transform:

$$s_t(r) = \text{const.}(it)^{-\frac{1}{2}} e^{-i(\frac{n-1}{2})^2 t} \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} e^{\frac{i}{4}\frac{r^2}{t}}. \tag{10}$$

Here $(it)^{-\frac{1}{2}} = e^{-i\frac{\pi}{4}\operatorname{sign}(t)}|t|^{-\frac{1}{2}}$ and, in the even dimensional case, the fractional derivative reads

$$\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} e^{\frac{i}{4}\frac{r^2}{t}} = \frac{1}{\sqrt{\pi}} \int_{|r|}^{+\infty} \left(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}} e^{\frac{i}{4}\frac{s^2}{t}} \frac{\sinh s\, ds}{\sqrt{\cosh s - \cosh r}}. \tag{11}$$

Proposition 3.1. *There exists a constant $C > 0$ such that the following pointwise kernel estimate holds, for every $t \in \mathbb{R}^*$ and $r \geqslant 0$:*

$$|s_t(r)| \leqslant C \begin{cases} |t|^{-3/2}(1+r)e^{-\frac{n-1}{2}r} & \text{if } |t| \geqslant 1+r, \\ |t|^{-n/2}(1+r)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}r} & \text{if } |t| \leqslant 1+r. \end{cases} \tag{12}$$

Remark 3.2. In dimension $n = 3$, this estimate boils down to

$$|s_t(r)| \leqslant C|t|^{-3/2}(1+r)e^{-r}$$

and was well known (see for instance [3]). In other dimensions, it is sharper than the kernel estimates obtained previously [3,4]. Our estimate can be rewritten as follows:

$$|s_t(r)| \leqslant C \begin{cases} |t|^{-3/2}\varphi_0(r) & \text{if } |t| \geqslant 1+r, \\ |t|^{-n/2} j(r)^{-1/2} & \text{if } |t| \leqslant 1+r, \end{cases}$$

using the ground spherical function $\varphi_0(r) \asymp (1+r)e^{-\frac{n-1}{2}r}$ and the jacobian of the exponential map $j(r) = (\frac{\sinh r}{r})^{n-1} \asymp (\frac{e^r}{1+r})^{n-1}$.

Proof of Proposition 3.1. We shall assume $t > 0$ for simplicity and we shall resume in part the analysis carried out in [2] for the heat kernel. Consider first the odd dimensional case. Set $m = \frac{n-1}{2}$ and let us expand

$$\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m e^{\frac{i}{4}\frac{r^2}{t}} = e^{\frac{i}{4}\frac{r^2}{t}} \sum_{j=1}^m t^{-j} f_j(r). \tag{13}$$

The functions $f_j(r)$ involved are linear combinations of products $\varphi_{\ell_1}(r)\cdots\varphi_{\ell_j}(r)$, where

$$\varphi_\ell(r) = \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^\ell r^2$$

and $\ell_1, \ldots, \ell_j \in \mathbb{N}^*$ are such that $\ell_1 + \cdots + \ell_j = m$. Using the elementary global estimate

$$\varphi_\ell(r) = \mathrm{O}\big((1+r)e^{-\ell r}\big),$$

we are lead to the conclusion:

$$|s_t(r)| \lesssim t^{-\frac{1}{2}} \sum_{j=1}^m \left(\frac{1+r}{t}\right)^j e^{-mr} \asymp t^{-\frac{1}{2}} \left\{\frac{1+r}{t} + \left(\frac{1+r}{t}\right)^m\right\} e^{-mr}.$$

Because of the fractional derivative (11), the even dimensional case $n = 2m$ is more delicate to handle. According to the above estimate of (13), we have

$$\left|s_t(r)\right| \lesssim t^{-\frac{1}{2}} \int_r^{+\infty} \left\{\frac{1+s}{t} + \left(\frac{1+s}{t}\right)^m\right\} e^{-ms} \frac{\sinh s\, ds}{\sqrt{\cosh s - \cosh r}}. \tag{14}$$

Here and throughout the proof, we make repeated use of the following elementary estimates:

$$\sinh s \asymp \frac{s}{1+s} e^s, \tag{15}$$

and

$$\begin{aligned}\cosh s - \cosh r &= 2 \sinh \frac{s-r}{2} \sinh \frac{s+r}{2}\\ &\asymp \frac{s-r}{1+s-r} e^{\frac{s-r}{2}} \frac{s+r}{1+s+r} e^{\frac{s+r}{2}}\\ &\asymp \frac{s-r}{1+s-r} \frac{s}{1+s} e^s \quad \text{or} \quad \begin{cases} \frac{s^2-r^2}{1+r} e^r & \text{if } r \leqslant s \leqslant r+1,\\ e^s & \text{if } s \geqslant r+1.\end{cases}\end{aligned} \tag{16}$$

Thus (14) becomes

$$\left|s_t(r)\right| \lesssim t^{-\frac{1}{2}} \int_r^{+\infty} \left\{\frac{1+s}{t} + \left(\frac{1+s}{t}\right)^m\right\} e^{-(m-\frac{1}{2})s} \frac{\sqrt{1+s-r}}{\sqrt{s-r}} \frac{\sqrt{s}}{\sqrt{1+s}}\, ds. \tag{17}$$

After performing the change of variables $s = r + u$ and using the trivial inequalities

$$\frac{\sqrt{r+u}}{\sqrt{1+r+u}} \leqslant 1, \qquad 1 + r + u \leqslant (1+r)(1+u),$$

we obtain eventually

$$\left|s_t(r)\right| \lesssim t^{-\frac{1}{2}} \left\{\frac{1+r}{t} + \left(\frac{1+r}{t}\right)^m\right\} e^{-(m-\frac{1}{2})r}. \tag{18}$$

This allows us to conclude that

$$\left|s_t(r)\right| \leqslant C t^{-\frac{1}{2}} \frac{1+r}{t} e^{-(m-\frac{1}{2})r} \tag{19}$$

when $t \geqslant 1 + r$. If $t \leqslant 1 + r$, the polynomial power m in (18) must be brought down to $m - \frac{1}{2}$. For this purpose, let us rewrite more carefully the expansion

$$\left(-\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^m e^{\frac{i}{4}\frac{s^2}{t}} = \sum_{0<j<m} t^{-j} f_j(s) e^{\frac{i}{4}\frac{s^2}{t}} + t^{-(m-1)} \left(-\frac{i}{2} \frac{s}{\sinh s}\right)^{m-1} \left(-\frac{1}{\sinh s} \frac{\partial}{\partial s}\right) e^{\frac{i}{4}\frac{s^2}{t}}.$$

The contribution of the sum (which does not occur in dimension $n = 2$) can be handled as above and is

$$\mathrm{O}\left(t^{-\frac{1}{2}} \left\{\frac{1+r}{t} + \left(\frac{1+r}{t}\right)^{m-1}\right\} e^{-(m-\frac{1}{2})r}\right).$$

Thus it remains for us to show that the integral

$$I(t,r) = \int_r^{+\infty} \left(\frac{s}{\sinh s}\right)^{m-1} \left(\frac{\partial}{\partial s} e^{\frac{i}{4}\frac{s^2}{t}}\right) \frac{ds}{\sqrt{\cosh s - \cosh r}}$$

is $\mathrm{O}(t^{-\frac{1}{2}}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r})$ when $t \leqslant 1 + r$. Let us split

$$I(t,r) = I_1(t,r) + I_2(t,r) + I_3(t,r)$$

according to

$$\int_r^{+\infty} = \int_r^{\sqrt{r^2+t}} + \int_{\sqrt{r^2+t}}^{r+1} + \int_{r+1}^{+\infty}.$$

The first integral is easy to handle. We simply differentiate $\frac{\partial}{\partial s} e^{\frac{i}{4}\frac{s^2}{t}} = \frac{i}{2}\frac{s}{t} e^{\frac{i}{4}\frac{s^2}{t}}$ and use the elementary estimates (15), (16) together with the fact that $s \in [r, r+1]$. As a result,

$$\left|I_1(t,r)\right| \lesssim t^{-1}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r} \int_r^{\sqrt{r^2+t}} \frac{s\,ds}{\sqrt{s^2-r^2}} = t^{-\frac{1}{2}}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r}.$$

Let us turn to the second and third integrals, that we integrate by parts:

$$\begin{aligned} I_2(t,r) + I_3(t,r) = {} & e^{\frac{i}{4}\frac{s^2}{t}} \left(\frac{s}{\sinh s}\right)^{m-1} \frac{1}{\sqrt{\cosh s - \cosh r}} \left\{ \Big|_{s=\sqrt{r^2+t}}^{s=r+1} + \Big|_{s=r+1}^{s=+\infty} \right\} \\ & + \left\{ \int_{\sqrt{r^2+t}}^{r+1} + \int_{r+1}^{+\infty} \right\} e^{\frac{i}{4}\frac{s^2}{t}} \left\{ (m-1)\left(\frac{s}{\sinh s}\right)^{m-2} \frac{s\coth s - 1}{\sinh s} (\cosh s - \cosh r)^{-\frac{1}{2}} \right. \\ & \left. + \frac{1}{2}\left(\frac{s}{\sinh s}\right)^{m-1} (\cosh s - \cosh r)^{-\frac{3}{2}} \sinh s \right\} ds. \end{aligned}$$

The boundary terms are estimated as $I_1(t,r)$:

$$\left(\frac{s}{\sinh s}\right)^{m-1} \frac{1}{\sqrt{\cosh s - \cosh r}} \bigg|_{s=\sqrt{r^2+t}} \asymp t^{-\frac{1}{2}}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r}.$$

The integral terms are bounded by

$$(1+r)^{m-2} e^{-(m-\frac{1}{2})r} \int_{\sqrt{r^2+t}}^{r+1} \left\{ \left(\frac{1+r}{s^2-r^2}\right)^{\frac{1}{2}} + \left(\frac{1+r}{s^2-r^2}\right)^{\frac{3}{2}} \right\} s\,ds + \int_{r+1}^{+\infty} (1+s)^{m-1} e^{-(m-\frac{1}{2})s}\,ds. \tag{20}$$

Here we have used (15), (16) and the elementary estimate $\frac{s\coth s - 1}{\sinh s} \asymp s e^{-s}$. In the expression between braces, the first factor is dominated by the second one. Thus the first integral in (20) is bounded by

$$(1+r)^{\frac{3}{2}} \int_{\sqrt{r^2+t}}^{r+1} \left(s^2-r^2\right)^{-\frac{3}{2}} s\,ds = (1+r)^{\frac{3}{2}} \left\{ -\left(s^2-r^2\right)^{-\frac{1}{2}} \Big|_{s=\sqrt{r^2+t}}^{s=r+1} \right\} \lesssim t^{-\frac{1}{2}}(1+r)^{\frac{3}{2}}.$$

The second integral in (20) is estimated as (17):

$$\int_{r+1}^{+\infty} (1+s)^{m-1} e^{-(m-\frac{1}{2})s}\,ds = \int_1^{+\infty} (1+r+u)^{m-1} e^{-(m-\frac{1}{2})(r+u)}\,du \lesssim (1+r)^{m-1} e^{-(m-\frac{1}{2})r}.$$

As a conclusion, we obtain

$$I(t,r) \lesssim t^{-\frac{1}{2}}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r}.$$

Thus we have shown that

$$\left|s_t(r)\right| \lesssim t^{-m}(1+r)^{m-\frac{1}{2}} e^{-(m-\frac{1}{2})r}$$

when $0 < t \leqslant 1 + r$. □

Corollary 3.3. *Let $2 < q < \infty$ and $1 \leqslant \alpha \leqslant \infty$. Then there exists a constant $C > 0$ such that the following kernel estimate holds, with respect to Lorentz norms*:

$$\|s_t\|_{L^{q,\alpha}} \leqslant C \begin{cases} |t|^{-n/2} & \text{if } 0 < |t| \leqslant 1, \\ |t|^{-3/2} & \text{if } |t| \geqslant 1. \end{cases} \tag{21}$$

Proof. Recall that Lorentz spaces $L^{q,\alpha}(\mathbb{H}^n)$ are variants of the classical Lebesgue spaces, whose norms are defined by

$$\|f\|_{L^{q,\alpha}} = \begin{cases} [\int_0^{+\infty} \{s^{1/q} f^*(s)\}^\alpha \frac{ds}{s}]^{1/\alpha} & \text{if } 1 \leqslant \alpha < \infty, \\ \sup_{s>0} s^{1/q} f^*(s) & \text{if } \alpha = \infty, \end{cases}$$

where f^* denotes the decreasing rearrangement of f. In particular, if f is a positive radial decreasing function on $\mathbb{H}^n$, then $f^* = f \circ V^{-1}$, where

$$V(r) = C \int_0^r (\sinh s)^{n-1}\, ds \asymp \begin{cases} r^n & \text{as } r \to 0, \\ e^{(n-1)r} & \text{as } r \to +\infty \end{cases}$$

is the volume of a ball of radius $r > 0$ in $\mathbb{H}^n$. Hence

$$\begin{aligned} \|f\|_{L^{q,\alpha}} &= \left[\int_0^{+\infty} \bigl\{V(r)^{1/q} f(r)\bigr\}^\alpha \frac{V'(r)}{V(r)}\, dr \right]^{1/\alpha} \\ &\asymp \left[\int_0^1 f(r)^\alpha r^{\frac{\alpha n}{q} - 1}\, dr \right]^{1/\alpha} + \left[\int_1^{+\infty} f(r)^\alpha e^{\frac{\alpha(n-1)}{q} r}\, dr \right]^{1/\alpha} \end{aligned}$$

if $1 \leqslant \alpha < \infty$ and

$$\|f\|_{L^{q,\infty}} = \sup_{r>0} V(r)^{1/q} f(r) \asymp \sup_{0<r<1} r^{\frac{n}{q}} f(r) + \sup_{r \geqslant 1} e^{\frac{n-1}{q} r} f(r).$$

The Lorentz norm estimate (21) follows from these considerations and from the pointwise estimate (12). □

Let us turn to L^q mapping properties of the Schrödinger propagator $e^{it\widetilde{\Delta}}$ on $\mathbb{H}^n$. Recall that $e^{it\widetilde{\Delta}}$ is a one parameter group of unitary operators on $L^2(\mathbb{H}^n)$.

Theorem 3.4. *Let $2 < q, \tilde{q} \leqslant \infty$. Then there exists a constant $C > 0$ such that the following dispersive estimates hold*:

$$\bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^{\tilde{q}'} \to L^q} \leqslant C \begin{cases} |t|^{-\max\{\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\tilde{q}}\} n} & \text{if } 0 < |t| < 1, \\ |t|^{-\frac{3}{2}} & \text{if } |t| \geqslant 1. \end{cases}$$

Remark 3.5. In the Euclidean setting, small time estimates are similar for $q = \tilde{q}$, other small time estimates do not hold, and large time estimates are drastically different.

Proof of Theorem 3.4. These estimates are obtained by interpolation and by using Corollary 3.3. Specifically, small time estimates follow from

$$\begin{cases} \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^1 \to L^q} = \|s_t\|_{L^q} \leqslant C_q |t|^{-\frac{n}{2}} & \forall q > 2, \\ \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^{q'} \to L^\infty} = \|s_t\|_{L^q} \leqslant C_q |t|^{-\frac{n}{2}} & \forall q > 2, \\ \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^2 \to L^2} = 1, \end{cases}$$

and large time estimates from

$$\begin{cases} \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^1 \to L^q} = \|s_t\|_{L^q} \leqslant C_q |t|^{-\frac{3}{2}} & \forall q > 2, \\ \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^{q'} \to L^\infty} = \|s_t\|_{L^q} \leqslant C_q |t|^{-\frac{3}{2}} & \forall q > 2, \\ \bigl\|e^{it\widetilde{\Delta}}\bigr\|_{L^{q'} \to L^q} \leqslant C_q \|s_t\|_{L^{q,1}} \leqslant C_q |t|^{-\frac{3}{2}} & \forall q > 2. \end{cases}$$

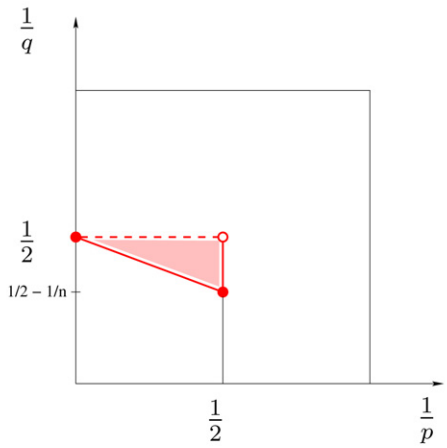

Fig. 1. Admissible set for $\mathbb{H}^n$ in dimension $n \geqslant 3$.

The key ingredient here is a sharp version of the Kunze–Stein phenomenon, due to Cowling, Meda and Setti (see [7]) and improved by Ionescu [13], which yields in particular

$$L^{q'}(K\setminus G) * L^{q'}(G/K) \subset L^{q',\infty}(K\setminus G/K) \quad \forall q>2.$$

By such an inclusion, we mean that there exists a constant $C_q>0$ such that

$$\|f*g\|_{L^{q',\infty}} \leqslant C_q\|f\|_{L^{q'}}\|g\|_{L^{q'}} \quad \forall f\in L^{p'}(K\setminus G),\ \forall g\in L^{p'}(G/K).$$

Hence by duality

$$L^{q'}(G/K) * L^{q,1}(K\setminus G/K) \subset L^{q}(G/K) \quad \forall q>2. \qquad \square$$

Consider next the inhomogeneous linear Schrödinger equation (5) on $\mathbb{H}^n$:

$$\begin{cases} i\partial_t u(t,x) + \widetilde{\Delta} u(t,x) = F(t,x), \\ u(0,x) = f(x), \end{cases}$$

whose solution is given by Duhamel's formula:

$$u(t,x) = e^{it\widetilde{\Delta}} f(x) - i\int_0^t e^{i(t-s)\widetilde{\Delta}} F(s,x)\,ds. \tag{22}$$

Strichartz estimates on $\mathbb{H}^n$ involve *admissible pairs* of indices (p,q) corresponding to the triangle

$$T_n = \left\{\left(\frac{1}{p},\frac{1}{q}\right) \in \left(0,\frac{1}{2}\right] \times \left(0,\frac{1}{2}\right) \,\Big|\, \frac{2}{p}+\frac{n}{q} \geqslant \frac{n}{2}\right\} \cup \left\{\left(0,\frac{1}{2}\right)\right\} \tag{23}$$

(see Fig. 1).

Theorem 3.6. *Assume that* (p,q) *and* $(\tilde{p},\tilde{q})$ *are admissible pairs as above. Then there exists a constant* $C>0$ *such that the following Strichartz estimate holds for solutions to the Cauchy problem* (5):

$$\|u\|_{L^p_t L^q_x} \leqslant C\left\{\|f\|_{L^2_x} + \|F\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x}\right\}. \tag{24}$$

Remark 3.7. This result was obtained previously for radial functions in [4], using sharp weighted Strichartz estimates [18] in dimension $n \geqslant 4$, the elementary kernel expression (10) in dimension $n=3$, and specific kernel estimates in dimension $n=2$. Notice that the admissible set for $\mathbb{H}^n$ is much larger than the admissible set for $\mathbb{R}^n$ (which corresponds to the lower edge of the triangle T_n). This is due to large scale dispersive effects in negative curvature. Actually it could be even larger if the region $\frac{2}{p}+\frac{n}{q}<\frac{n}{2}$ was not excluded for purely local reasons. This happens for dispersive equations on homogeneous trees and will be discussed in another paper.

Proof of Theorem 3.6. We resume the standard strategy developed by Kato [15], Ginibre and Velo [9], and Keel and Tao [16]. Consider the operator

$$Tf(t,x) = e^{it\widetilde{\Delta}} f(x)$$

and its formal adjoint

$$T^*F(x) = \int_{-\infty}^{+\infty} e^{-is\widetilde{\Delta}} F(s,x)\,ds.$$

The method consists in proving the $L_t^{p'} L_x^{q'} \to L_t^p L_x^q$ boundedness of the operator

$$TT^*F(t,x) = \int_{-\infty}^{+\infty} e^{i(t-s)\widetilde{\Delta}} F(s,x)\,ds \tag{25}$$

and of its truncated version

$$\widetilde{TT}^*F(t,x) = \int_0^t e^{i(t-s)\widetilde{\Delta}} F(s,x)\,ds, \tag{26}$$

for every admissible pair (p,q). The endpoint $(\frac{1}{p},\frac{1}{q}) = (0,\frac{1}{2})$ is settled by L^2 conservation and the endpoint $(\frac{1}{p},\frac{1}{q}) = (\frac{1}{2},\frac{1}{2}-\frac{1}{n})$ in dimension $n \geqslant 3$ will be handled at the end. Thus we are left with the pairs (p,q) such that $\frac{1}{2}-\frac{1}{n} < \frac{1}{q} < \frac{1}{2}$ and $(\frac{1}{2}-\frac{1}{q})\frac{n}{2} \leqslant \frac{1}{p} \leqslant \frac{1}{2}$. According to the dispersive estimates in Theorem 3.4, the $L_t^p L_x^q$ norms of (25) and (26) are bounded above by

$$\left\| \int_{|t-s|\geqslant 1} |t-s|^{-\frac{3}{2}} \|F(s)\|_{L_x^{q'}} \right\|_{L_t^p} + \left\| \int_{|t-s|\leqslant 1} |t-s|^{-(\frac{1}{2}-\frac{1}{q})n} \|F(s)\|_{L_x^{q'}} \right\|_{L_t^p}. \tag{27}$$

On one hand, the convolution kernel $|t-s|^{-\frac{3}{2}}\mathbb{1}_{\{|t-s|\geqslant 1\}}$ on $\mathbb{R}$ defines a bounded operator from $L_s^{p_1}$ to $L_t^{p_2}$, for all $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, in particular from $L_s^{p'}$ to L_t^p, for all $2 \leqslant p \leqslant \infty$. On the other hand, the convolution kernel $|t-s|^{-(\frac{1}{2}-\frac{1}{q})n}\mathbb{1}_{\{|t-s|\leqslant 1\}}$ defines a bounded operator from $L_s^{p_1}$ to $L_t^{p_2}$, for[1] all $1 < p_1, p_2 < \infty$ such that $0 \leqslant \frac{1}{p_1} - \frac{1}{p_2} \leqslant 1 - (\frac{1}{2}-\frac{1}{q})n$, in particular from $L_s^{p'}$ to L_t^p, for all $2 \leqslant p < \infty$ such that $\frac{1}{p} \geqslant (\frac{1}{2}-\frac{1}{q})\frac{n}{2}$. Consider eventually the endpoint $(\frac{1}{p},\frac{1}{q}) = (\frac{1}{2},\frac{1}{2}-\frac{1}{n})$ in dimension $n \geqslant 3$. The integrals over $|t-s| \geqslant 1$ are estimated as before. The integrals over $|t-s| \leqslant 1$ are handled as in [16], except that only small dyadic intervals are involved. Indices are finally decoupled, using the TT^* argument. □

4. Well-posedness results for NLS on $\mathbb{H}^n$

Strichartz estimates for inhomogeneous linear equations are used to prove local and global well-posedness results for nonlinear problems. We present here a few results in this direction for the Schrödinger equation (6)

$$\begin{cases} i\partial_t u + \widetilde{\Delta} u = F(u), \\ u(0) = f, \end{cases}$$

on $M = \mathbb{H}^n$, with a power-like nonlinearity as in (7):

$$\left|F(u)\right| \leqslant C|u|^\gamma, \qquad \left|F(u)-F(v)\right| \leqslant C\left(|u|^{\gamma-1}+|v|^{\gamma-1}\right)|u-v|.$$

Let us recall the definition of well-posedness.

[1] Actually for all $1 \leqslant p_1, p_2 \leqslant \infty$ such that $0 \leqslant \frac{1}{p_1} - \frac{1}{p_2} \leqslant 1 - (\frac{1}{2}-\frac{1}{q})n$, except for the dual endpoints $(\frac{1}{p_1},\frac{1}{p_2}) = (1,(\frac{1}{2}-\frac{1}{q})\frac{n}{2})$ and $(\frac{1}{p_1},\frac{1}{p_2}) = (1-(\frac{1}{2}-\frac{1}{q})\frac{n}{2},0)$.

Definition 4.1. Let $s \in \mathbb{R}$. The NLS equation (6) is locally well-posed in $H^s(M)$ if, for any bounded subset B of $H^s(M)$, there exist $T > 0$ and a Banach space X_T, continuously embedded into $C([-T, +T]; H^s(M))$, such that

- for any Cauchy data $f(x) \in B$, (6) has a unique solution $u(t,x) \in X_T$;
- the map $f(x) \mapsto u(t,x)$ is continuous from B to X_T.

The equation is globally well-posed if these properties hold with $T = \infty$.

In the Euclidean setting, $\gamma = 1 + \frac{4}{n}$ is known to be the critical exponent for well-posedness in $L^2(\mathbb{R}^n)$. Specifically, the NLS (1) has a unique local solution for arbitrary data $f \in L^2$ provided $\gamma < 1 + \frac{4}{n}$; in general this solution cannot be extended to a global one; this is possible under the assumption (8) of gauge invariance. In the critical case $\gamma = 1 + \frac{4}{n}$, the NLS (1) is globally well-posed for L^2 data satisfying a smallness condition. On the other hand, $\gamma = 1 + \frac{4}{n-2}$ is known to be the critical exponent for well-posedness in $H^1(\mathbb{R}^n)$. Specifically, the NLS (1) is locally well-posed in H^1 when $1 < \gamma < 1 + \frac{4}{n-2}$. Local solutions can be extended to global ones under the defocusing assumption (9). All these results are proved in a standard way using Strichartz estimates and conservations laws, when available.

In the hyperbolic setting, we have seen above that Strichartz estimates hold for a much wider range. As a consequence, well-posedness results for the NLS (6) are considerably stronger. In particular, in contrast with the Euclidean setting, global well-posedness for small data in L^2 holds for any subcritical exponent γ without the assumption of gauge invariance. Here are our well-posedness results in $L^2(\mathbb{H}^n)$.

Theorem 4.2. *If $1 < \gamma \leqslant 1 + \frac{4}{n}$, the NLS (6) is globally well-posed for small L^2 data. Moreover, in the subcritical case $1 < \gamma < 1 + \frac{4}{n}$, the NLS (6) is locally well-posed for arbitrary L^2 data.*

Proof. We resume the standard fixed point method based on Strichartz estimates. Define $u = \Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} i\partial_t u(t,x) + \widetilde{\Delta} u(t,x) = F\big(v(t,x)\big), \\ u(0,x) = f(x), \end{cases} \tag{28}$$

which is given by Duhamel's formula (22):

$$u(t,x) = e^{it\widetilde{\Delta}} f(x) + \int_0^t e^{i(t-s)\widetilde{\Delta}} F\big(v(s,x)\big)\, ds.$$

According to Theorem 3.6, we have the following Strichartz estimate

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^p L_x^q} \leqslant C\|f\|_{L_x^2} + C\big\|F(v)\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \tag{29}$$

for all $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ in the triangle T_n, which amounts to the conditions

$$\begin{cases} 2 \leqslant p, q \leqslant \infty \text{ such that } \dfrac{\beta}{p} + \dfrac{n}{q} = \dfrac{n}{2} \text{ for some } 0 < \beta \leqslant 2, \\[2ex] 2 \leqslant \tilde{p}, \tilde{q} \leqslant \infty \text{ such that } \dfrac{\tilde{\beta}}{\tilde{p}} + \dfrac{n}{\tilde{q}} = \dfrac{n}{2} \text{ for some } 0 < \tilde{\beta} \leqslant 2. \end{cases} \tag{30}$$

Moreover

$$\big\|F(v)\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\big\||v|^\gamma\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\|v\|^\gamma_{L_t^{\tilde{p}'\gamma} L_x^{\tilde{q}'\gamma}}$$

by our nonlinear assumption (7). Thus

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^p L_x^q} \leqslant C\|f\|_{L_x^2} + C\|v\|^\gamma_{L_t^{\tilde{p}'\gamma} L_x^{\tilde{q}'\gamma}}. \tag{31}$$

In order to remain within the same function space, we require in addition

$$p = \tilde{p}'\gamma, \qquad q = \tilde{q}'\gamma. \tag{32}$$

It is easily checked that all these conditions are fulfilled if we take for instance

$$0<\beta=\tilde{\beta}\leqslant 2 \quad \text{such that} \quad \gamma=1+\frac{2\beta}{n} \quad \text{and} \quad p=q=\tilde{p}=\tilde{q}=1+\gamma=2+\frac{2\beta}{n}.$$

For such a choice, Φ maps $L^\infty(\mathbb{R};L^2(\mathbb{H}^n))\cap L^p(\mathbb{R};L^q(\mathbb{H}^n))$ into itself, and actually $X=C(\mathbb{R};L^2(\mathbb{H}^n))\cap L^p(\mathbb{R};L^q(\mathbb{H}^n))$ into itself. Since X is a Banach space for the norm

$$\|u\|_X=\|u\|_{L_t^\infty L_x^2}+\|u\|_{L_t^p L_x^q},$$

it remains for us to show that Φ is a contraction in the ball

$$X_\varepsilon=\big\{u\in X \bigm| \|u\|_X\leqslant\varepsilon\big\},$$

provided $\varepsilon>0$ and $\|f\|_{L^2}$ are sufficiently small. Let $v,\tilde{v}\in X$ and $u=\Phi(v)$, $\tilde{u}=\Phi(\tilde{v})$. Arguing as above and using in addition Hölder's inequality, we estimate

$$\begin{aligned}\|u-\tilde{u}\|_X&\leqslant C\big\|F(v)-F(\tilde{v})\big\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}\\&\leqslant C\big\|\big\{|v|^{\gamma-1}+|\tilde{v}|^{\gamma-1}\big\}|v-\tilde{v}|\big\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}\\&\leqslant C\big\{\|v\|_{L_t^pL_x^q}^{\gamma-1}+\|\tilde{v}\|_{L_t^pL_x^q}^{\gamma-1}\big\}\|v-\tilde{v}\|_{L_t^pL_x^q},\end{aligned}$$

hence

$$\|u-\tilde{u}\|_X\leqslant C\big(\|v\|_X^{\gamma-1}+\|\tilde{v}\|_X^{\gamma-1}\big)\|v-\tilde{v}\|_X. \tag{33}$$

If we assume $\|v\|_X\leqslant\varepsilon$, $\|\tilde{v}\|_X\leqslant\varepsilon$ and $\|f\|_{L^2}\leqslant\delta$, then (31) and (33) yield

$$\|u\|_X\leqslant C\delta+C\varepsilon^\gamma, \qquad \|\tilde{u}\|_X\leqslant C\delta+C\varepsilon^\gamma \quad \text{and} \quad \|u-\tilde{u}\|_X\leqslant 2C\varepsilon^{\gamma-1}\|v-\tilde{v}\|_X.$$

Thus

$$\|u\|_X\leqslant\varepsilon, \qquad \|\tilde{u}\|_X\leqslant\varepsilon \quad \text{and} \quad \|u-\tilde{u}\|_X\leqslant\frac{1}{2}\|v-\tilde{v}\|_X$$

if $C\varepsilon^{\gamma-1}\leqslant\frac{1}{4}$ and $C\delta\leqslant\frac{3}{4}\varepsilon$. We conclude by applying the fixed point theorem in the complete metric space X_ε.

In the subcritical case $\gamma<1+\frac{4}{n}$, one can prove in a similar way local well-posedness in L^2 for arbitrary data f. Specifically, we restrict to a small time interval $I=[-T,+T]$ and proceed as above, except that we increase $\tilde{\beta}\in(\beta,2]$ and $\tilde{p}=\frac{\tilde{\beta}}{\beta}p$ accordingly, and that we apply in addition Hölder's inequality in time. This way, we get the Strichartz estimate

$$\|u\|_X\leqslant C\|f\|_{L^2}+CT^\lambda\|v\|_X^\gamma, \tag{34}$$

where $X=C(I;L^2(\mathbb{H}^n))\cap L^p(I;L^q(\mathbb{H}^n))$ and $\lambda=\frac{1}{p}-\frac{1}{\tilde{p}}>0$, and the related estimate

$$\|u-\tilde{u}\|_X\leqslant CT^\lambda\big(\|v\|_X^{\gamma-1}+\|v\|_X^{\gamma-1}\big)\|v-\tilde{v}\|_X. \tag{35}$$

As a consequence, we deduce that Φ is a contraction in the ball

$$X_M=\big\{u\in X \bigm| \|u\|_X\leqslant M\big\},$$

provided $M>0$ is large enough and $T>0$ small enough, more precisely $\frac{3}{4}M\geqslant C\|f\|_{L^2}$ and $CT^\lambda M^{\gamma-1}\leqslant\frac{1}{4}$. We conclude as before. □

Remark 4.3. Notice that T depends only on the L^2 norm of the initial data:

$$T=3^{\frac{\gamma-1}{\lambda}}4^{-\frac{\gamma}{\lambda}}C^{-\frac{\gamma}{\lambda}}\|f\|_{L^2}^{-\frac{\gamma-1}{\lambda}}.$$

Thus, if the nonlinearity F is gauge invariant as in (8), then L^2 conservation allows us to iterate and deduce global existence from local existence, for arbitrary data $f\in L^2$ in the subcritical case $\gamma<1+\frac{4}{n}$.

Let us turn now to our well-posedness results in $H^1(\mathbb{H}^n)$.

Theorem 4.4. *If $1 < \gamma \leqslant 1 + \frac{4}{n-2}$, the NLS* (6) *is globally well-posed for small H^1 data. Moreover, in the subcritical case $1 < \gamma < 1 + \frac{4}{n-2}$, the NLS* (6) *is locally well-posed for arbitrary H^1 data.*

Proof. Let us point out the modifications needed in order to adapt the proof of Theorem 4.2 and switch from Lebesgue spaces $L^q(\mathbb{H}^n)$ to Sobolev spaces $H^{1,q}(\mathbb{H}^n)$.

By applying $(-\widetilde{\Delta})^{\frac{1}{2}}$, (28) becomes

$$\begin{cases} i\partial_t(-\widetilde{\Delta})^{\frac{1}{2}}u(t,x) + \widetilde{\Delta}(-\widetilde{\Delta})^{\frac{1}{2}}u(t,x) = (-\widetilde{\Delta})^{\frac{1}{2}}F\big(v(t,x)\big), \\ (-\widetilde{\Delta})^{\frac{1}{2}}u(0,x) = (-\widetilde{\Delta})^{\frac{1}{2}}f(x) \end{cases}$$

and (29)

$$\|u\|_{L_t^\infty H_x^1} + \|u\|_{L_t^p H_x^{1,q}} \leqslant C\|f\|_{H_x^1} + C\big\|F(v)\big\|_{L_t^{\tilde{p}'} H_x^{1,\tilde{q}'}}.$$

It follows from our nonlinearity assumptions (7) that

$$\big\|F(v)\big\|_{L_t^{\tilde{p}'} H_x^{1,\tilde{q}'}} \leqslant C\|v\|^{\gamma}_{L_t^p H_x^{1,q}}$$

provided $\frac{1}{\tilde{p}'} = \frac{\gamma}{p}$ and $\frac{1}{\tilde{q}'} \geqslant \frac{\gamma}{q} - \frac{\gamma-1}{n}$. Using the Hölder and Sobolev inequalities, we can indeed estimate

$$\big\|\nabla_x F(v)\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\big\||v|^{\gamma-1}|\nabla_x v|\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\|v\|^{\gamma}_{L_t^p H_x^{1,q}}$$

and also

$$\big\|F(v)\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\big\||v|^{\gamma}\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \leqslant C\|v\|^{\gamma}_{L_t^p H_x^{1,q}}$$

under the weaker assumptions $\frac{1}{\tilde{p}'} = \frac{\gamma}{p}$ and $\frac{1}{\tilde{q}'} \geqslant \gamma(\frac{1}{q} - \frac{1}{n})$. Thus

$$\|u\|_{L_t^\infty H_x^1} + \|u\|_{L_t^p H_x^{1,q}} \leqslant C\|f\|_{H_x^1} + C\|v\|^{\gamma}_{L_t^p H_x^{1,q}} \tag{36}$$

for a proper choice of parameters, for instance

$$0 < \beta = \tilde{\beta} \leqslant 2 \quad \text{such that} \quad \gamma = 1 + \frac{2\beta}{n-2}, \quad p = \tilde{p} = 1 + \gamma, \quad \frac{1}{q} = \frac{1}{\tilde{q}} = \frac{1}{2} - \frac{\beta}{n}\frac{1}{1+\gamma}.$$

As a first conclusion, we obtain that $\Phi : v \longmapsto u$ maps the Banach space $X = C(\mathbb{R}; H^1(\mathbb{H}^n)) \cap L^p(\mathbb{R}; H^{1,q}(\mathbb{H}^n))$ into itself, and moreover the ball X_ε into itself, provided ε and $\|f\|_{H^1}$ are small enough.

Let us next prove existence and uniqueness of a fixed point for Φ in X_ε. Arguing as above, we can estimate

$$\begin{aligned} \|u - \tilde{u}\|_{L_t^p L_x^q} &\leqslant C\big\|F(v) - F(\tilde{v})\big\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \\ &\leqslant C\big\{\|v\|^{\gamma-1}_{L_t^p H_x^{1,q}} + \|\tilde{v}\|^{\gamma-1}_{L_t^p H_x^{1,q}}\big\}\|v - \tilde{v}\|_{L_t^p L_x^q} \\ &\leqslant 2C\varepsilon^{\gamma-1}\|v - \tilde{v}\|_{L_t^p L_x^q} \end{aligned}$$

for $v, \tilde{v} \in X_\varepsilon$ and corresponding $u = \Phi(v)$, $\tilde{u} = \Phi(\tilde{v})$. Thus Φ is a contraction in X_ε for the norm inherited from the Banach space $Y = L^p(\mathbb{R}; L^q(\mathbb{H}^n))$, provided ε is small enough. This yields uniqueness of a possible fixed point for Φ in X_ε. For existence, we use the standard iteration argument, starting from any $u_0 \in X_\varepsilon$, considering the sequence $u_j = \Phi^j(u_0)$ and getting in the limit a fixed point u in the closure of X_ε in Y. Eventually, since X is reflexive and separable, u_j has a weakly convergent subsequence $u_{j_k} \to \tilde{u}$ in X_ε and hence $u = \tilde{u}$ must belong to X_ε.

As far as local well-posedness for arbitrary data is concerned, we adapt similarly the last part of the proof of Theorem 4.2. Specifically the estimates (34) and (35) are now replaced by

$$\|u\|_X \leqslant C\|f\|_{H^1} + CT^\lambda\|v\|^{\gamma}_X$$

and

$$\|u-\tilde{u}\|_Y \leqslant CT^\lambda\big(\|v\|_X^{\gamma-1}+\|\tilde{v}\|_X^{\gamma-1}\big)\|v-\tilde{v}\|_Y,$$

where $X=C(I;H^1(\mathbb{H}^n))\cap L^p(I;H^{1,q}(\mathbb{H}^n))$ and $Y=L^p(I;L^q(\mathbb{H}^n))$. $\square$

Remark 4.5. If the nonlinearity F is defocusing as in (9), H^1 conservation allows us to iterate and deduce global existence from local existence, for arbitrary data $f\in H^1$ in the subcritical case $\gamma<1+\frac{4}{n-2}$.

5. Scattering for NLS on $\mathbb{H}^n$

A second important application of our global Strichartz estimates is scattering for the NLS (6) in L^2 and in H^1. Under the additional assumptions of radial symmetry and gauge invariance or defocusing type this was already achieved in [4], using the weighted radial Strichartz estimates obtained in [3] for $n=3$ and in [18] for $n\geqslant 3$.

Actually, using our general estimates (24), we can prove scattering for small L^2 data with no additional assumption. Notice that, in the Euclidean case, this is only possible for the critical power $\gamma=1+\frac{4}{n}$ and can be false for subcritical powers, while on the hyperbolic space global existence and scattering of small L^2 data hold for all powers $1<\gamma\leqslant 1+\frac{4}{n}$. This is an analytic effect of hyperbolic geometry, which produces a larger admissible set for the Strichartz estimates.

Theorem 5.1. *Consider the Cauchy problem* (6) *with a power-like nonlinearity of order* $1<\gamma\leqslant 1+\frac{4}{n}$. *Then global solutions $u(t,x)$ corresponding to small L^2 data have the following scattering property*: *there exist $u_\pm\in L^2$ such that*

$$\big\|u(t)-e^{it\widetilde{\Delta}}u_\pm\big\|_{L^2(\mathbb{H}^n)}\to 0 \quad \textit{as } t\to\pm\infty.$$

Proof. According to the proof of Theorem 4.2, for $1<\gamma\leqslant 1+\frac{4}{n}$ and small L^2 data, the Cauchy problem (6) has a unique solution $u(t,x)$ in $C(\mathbb{R};L^2(\mathbb{H}^n))\cap L^p(\mathbb{R};L^q(\mathbb{H}^n))$, for some suitable pair (p,q). Scattering will follow from the Cauchy criterion:

If $\|z(t_1)-z(t_2)\|_{L^2_x}\to 0$ as $t_1,t_2\to+\infty$, then there exists $z_+\in L^2$ such that $\|z(t)-z_+\|_{L^2_x}\to 0$ as $t\to+\infty$.

In our case $z(t,x)=e^{-it\widetilde{\Delta}}u(t,x)$. So if we prove that

$$\big\|e^{-it_2\widetilde{\Delta}}u(t_2)-e^{-it_1\widetilde{\Delta}}u(t_1)\big\|_{L^2_x}\to 0 \quad \text{as } t_1\leqslant t_2\to\pm\infty,$$

we can conclude that the global solution $u(t,x))$ has the scattering property stated above. Using our Strichartz estimates (24), we get

$$\begin{aligned}\big\|e^{-it_2\widetilde{\Delta}}u(t_2)-e^{-it_1\widetilde{\Delta}}u(t_1)\big\|_{L^2(\mathbb{H}^n)} &= \bigg\|\int_{t_1}^{t_2} e^{-is\widetilde{\Delta}}F\big(u(s)\big)\,ds\bigg\|_{L^2(\mathbb{H}^n)}\\ &\leqslant \|u\|^\gamma_{L^p([t_1,t_2];L^q(\mathbb{H}^n))}.\end{aligned}$$

Since $u(t,x)\in L^p(\mathbb{R};L^q(\mathbb{H}^n))$, the last expression vanishes as $t_1\leqslant t_2$ tend both to $+\infty$ or $-\infty$. $\square$

Scattering in H^1 is proved in a similar way, using Theorem 4.4 instead of Theorem 4.2.

Theorem 5.2. *Consider the Cauchy problem* (6) *with a power-like nonlinearity of order* $1<\gamma\leqslant 1+\frac{4}{n-2}$. *Then global solutions $u(t,x)$ corresponding to small H^1 data have the following scattering property*: *there exist $u_\pm\in H^1$ such that*

$$\big\|u(t)-e^{it\widetilde{\Delta}}u_\pm\big\|_{H^1(\mathbb{H}^n)}\to 0 \quad \textit{as } t\to\pm\infty.$$

Another scattering result proved in [4] is existence of the so-called wave operator. This result extends straightforwardly to the nonradial case, since it relies on the Strichartz estimates of [18] combined with the techniques of [19].

Theorem 5.3. *Assume that F is defocusing and that $\gamma < 1 + \frac{4}{n-2}$. Then, for any data $f \in H^1$ at $t = \pm\infty$, our NLS has a unique global solution $u(t,x)$ with the following scattering property*:

$$\left\| u(t) - e^{it\widetilde{\Delta}} f \right\|_{H^1(\mathbb{H}^n)} \to 0 \quad \text{as } t \to \pm\infty.$$

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