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On the controllability of the fifth-order Korteweg–de Vries equation ☆

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Abstract

In this paper, we consider the fifth-order Korteweg–de Vries equation in a bounded interval. We prove that this equation is locally well-posed when endowed with suitable boundary conditions, and establish a result of local controllability to the trajectories.

Résumé

Dans ce papier, nous considérons l'équation de Korteweg–de Vries du cinquième ordre en domaine borné. Nous montrons que l'équation est localement bien posée lorsque l'on impose certaines données au bord, et établissons un résultat de contrôlabilité locale aux trajectoires.

Keywords: Controllability; Fifth-order Korteweg–de Vries equation; Initial-boundary value problem

1. Introduction

In this paper, we study the controllability of the fifth-order Korteweg–de Vries equation:

$$u_t + \alpha u_{5x} + \mu u_{xxx} + \beta u u_{xxx} + \delta u_x u_{xx} + P'(u) u_x = 0, \tag{1}$$

where α, μ, β and δ are real constants and P is a cubic polynomial:

$$P(u) = pu + qu^2 + ru^3. \tag{2}$$

This class of equations was introduced by Kichenassamy and Olver [17]. It contains in particular the Kawahara equation [15] introduced to model magneto-acoustic waves, the various models derived by Olver [19] for the unidirectional propagation of waves in shallow water when the third-order term appearing in the Korteweg–de Vries equation is small, and many other models. See [17] for a discussion of them.

In this paper, we are interested in studying this equation in a bounded domain. We will both consider the Cauchy problem with boundary conditions and the boundary controllability problem. Note that there is an important literature

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concerning the Cauchy problem in the real line, see for instance [8,9,17,16,18,21] and references therein. For what concerns the boundary value problem, the Kawahara equation with homogeneous boundary conditions was investigated by Doronin and Larkin [10]. Note that the initial boundary value problem for the (third-order) Korteweg–de Vries equation has drained much attention (see in particular [1,4,5,11,14]). The controllability problem was also, up to our knowledge, completely open. The equivalent for the Korteweg–de Vries equation has also known many developments lately [2,3,6,13,22–26].

To be more precise, we will consider in the sequel that $\alpha > 0$: this is not a restriction since it suffices to make the change of variable $x' = 1 - x$ and to invert the role of the left and right boundaries. The spatial domain will be $[0, 1]$, which is not a restriction either in the present paper, since it will suffice to rescale in space to obtain a result on an interval of arbitrary length. (Note that this is not necessarily the case for the Korteweg–de Vries equation with Neumann boundary control, see [22].)

The boundary conditions that we will consider are the following:

$$u_{|x=0} = v_1, \qquad u_{|x=1} = v_2, \qquad u_{x|x=0} = v_3, \qquad u_{x|x=1} = v_4, \qquad u_{xx|x=0} = v_5. \tag{3}$$

The first and main result of this paper concerns a boundary controllability result for Eq. (1). To be more precise, we will control the system from the right endpoint (by using only v_2) and v_4 while maintaining v_1, v_3, v_5 to zero), and the type of controllability that we consider is the *local controllability to trajectories*. That is to say, we consider $T > 0$ and a fixed trajectory $\bar{u}$ of (1), and prove that for any initial state u_0 sufficiently close to $\bar{u}_{|t=0}$, there exist controls (v_2, v_4) which steer the system from u_0 to $\bar{u}_{|t=T}$.

The precise result is the following.

Theorem 1. *Let $T > 0$. Let $\bar{u} \in L^\infty(0, T; W^{3,\infty}(0, 1))$ be a trajectory of* (1) *with boundary conditions $\bar{u}_{|x=0} = \bar{u}_{x|x=0} = \bar{u}_{xx|x=0} = 0$. There exists $\varepsilon > 0$ such that for any $u_0 \in L^2(0, 1)$ such that*

$$\left\| u_0 - \bar{u}(0, \cdot) \right\|_{L^2(0,1)} \leqslant \varepsilon, \tag{4}$$

there exist two controls v_2, v_4 in $L^2(0, T)$ such that the solution u of (1) *with initial condition*

$$u_{|t=0} = u_0 \tag{5}$$

and boundary condition (3) *with controls $(0, v_2, 0, v_4, 0)$ belongs to $C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1))$ and satisfies*

$$u_{|t=T} = \bar{u}(T, \cdot). \tag{6}$$

As we will see, the solution to the controllability problem that we construct is in fact more regular than $C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1))$. Indeed, in order to prove Theorem 1, we will first take the controls (v_2, v_4) as zero and we will prove, thanks to Theorem 3 below, that the state becomes H^2_0. Then we will work with more regular solutions (belonging to $L^2(\epsilon, T; H^4(0, 1)) \cap C^0([\epsilon, T]; H^2(0, 1))$).

Remark 1. Using the reversible character of this system stated on the whole real line, it is not difficult to deduce that Eq. (1) is locally *exactly controllable* near 0 when using the five controls (for sufficiently regular states).

Remark 2. As we will see in Section 4.3, a solution of (1) with boundary condition (3) with $v_1 = v_3 = v_5 = 0$ is regularized away from the right endpoint and the initial condition. This involves that Eq. (1) cannot be locally exactly controllable by means of v_2 and v_4 only.

The next result of this paper concerns the Cauchy problem. We prove that the problem is well posed locally in time, and regularizes the state of the system when the boundary conditions are homogeneous.

Theorem 2. *Given $u_0 \in L^2(0, 1)$, $\overline{T} > 0$, $v_1, v_2 \in H^{\frac{2}{5}}(0, \overline{T})$, $v_3, v_4 \in H^{\frac{1}{5}}(0, \overline{T})$, $v_5 \in L^2(0, \overline{T})$, there exists $T \in (0, \overline{T}]$ such that the nonlinear problem* (1) *with boundary conditions* (3) *admits a unique solution $u \in L^2(0, T; H^2(0, 1)) \cap C^0([0, T]; L^2(0, 1))$ satisfying*

$$\|u\|_{L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))} \leqslant C \left(\|u_0\|_{L^2(0,1)} + \sum_{k=1}^{5} \|v_k\|_{H^{\frac{2-\lfloor (k-1)/2 \rfloor}{5}}(0,T)} \right).$$

Theorem 3. *If* $v_1 = v_2 = v_3 = v_4 = v_5 = 0$*, the solution of* (1) *is regularized in the sense that for any* $\tau \in (0, T]$*,* $u \in C^\infty([\tau, T] \times [0, 1])$ *with, for all* $k \geqslant 0$*,*

$$\|u\|_{H^k(\tau,T;H^k(0,1))} \leqslant C(\tau, k)\|u_0\|_{L^2(0,1)}. \tag{7}$$

We conclude this introduction by a remark concerning the choice of the controls among $v_1, \dots, v_5$. The controllability to the trajectories described in Theorem 1 may not take place if one chooses another set of controls, for instance when acting through v_5 only and keeping $v_1 = v_2 = v_3 = v_4 = 0$. This is given in the next example.

Proposition 1. *Let* $L > 0$ *be a solution of* $\tan(L) = L$*. The system*

$$\begin{cases} -\varphi_t - \varphi_{5x} - \varphi_{xxx} = 0 & \text{in } (0, T) \times (-L, L), \\ \varphi_{|x=-L} = \varphi_{|x=L} = \varphi_{x|x=-L} = \varphi_{x|x=L} = \varphi_{xx|x=L} = 0 & \text{in } (0, T), \\ \varphi_{|t=T} = \varphi_T & \text{on } (-L, L) \end{cases} \tag{8}$$

has solutions satisfying

$$\varphi(0, \cdot) \not\equiv 0 \quad \text{and} \quad \varphi_{xx}(t, -L) = 0 \quad \text{in } (0, T) \tag{9}$$

for all $T > 0$*. As a consequence the system*

$$\begin{cases} u_t + u_{5x} + u_{xxx} = 0 & \text{in } (0, T) \times (-L, L), \\ u_{|x=-L} = u_{|x=L} = u_{x|x=-L} = u_{x|x=L} = 0 & \text{in } (0, T), \\ u_{xx|x=-L} = v_5(t) & \text{in } (0, T) \end{cases} \tag{10}$$

is not null approximately controllable by the control v_5*.*

Proof. We introduce the following (time-independent) function:

$$g(x) = \cos(x) + \mu x^2 - \big(\cos(L) + \mu L^2\big),$$

where

$$\mu := \frac{\sin(L)}{2L} = \frac{\cos(L)}{2}.$$

It is elementary to check that

$$\begin{cases} g_{5x} + g_{xxx} = 0 & \text{in } (0, T) \times (-L, L), \\ g_{|x=-L} = g_{|x=L} = g_{x|x=-L} = g_{x|x=L} = g_{xx|x=-L} = g_{xx|x=L} = 0 & \text{in } (0, T), \end{cases}$$

hence g satisfies (8) and (9). Now the equivalence between the unique continuation of (8) and the approximate controllability of (10) is an application of the standard duality in PDE control theory, see for instance [7]. □

Let us note that this phenomenon of critical values of the length of the domain was raised by Rosier [22] for the linearized KdV equation (see [2,3,6] for further developments on this subject). Hence, according to the values of the length of the domain and of the coefficients, a similar behavior can take place here. We believe that this leads to many open and challenging problems.

The structure of the paper is the following. In Section 2, we study the initial boundary value problem for a linearized equation. This requires proving a regularizing effect on the equation $\zeta_t + \zeta_{5x} = g$. In Section 3, we study the controllability of a linearized equation. In Section 4, we use a fixed point argument to establish Theorems 2 and 3 and an inverse mapping theorem to establish Theorem 1. Finally Sections 5 and 6 are devoted to the most technical parts of the paper, namely, the proof of a Carleman estimate and a proof of the regularizing effect to the left.

2. Cauchy problem for the linearized equation

In this section, we study the well-posedness of the following linearized equation:

$$y_t + \alpha y_{5x} = \sum_{k=0}^{3} a_k(t,x)\partial_x^k y + h, \tag{11}$$

where the functions a_k satisfy

$$a_k \in L^\infty\big(0,T;W^{k,\infty}(0,1)\big), \quad \text{for } k = 0,\dots,3. \tag{12}$$

Recall that we consider $\alpha > 0$. We will state the corresponding result in Section 2.3.

For this, we will first study the adjoint system of (11):

$$\begin{cases} \psi_t + \alpha\psi_{5x} = \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k\big(a_k(t,x)\psi\big) + f & \text{in } (0,T)\times(0,1), \\ \psi_{|x=0} = \psi_{x|x=0} = 0 & \text{in } (0,T), \\ \psi_{|x=1} = \psi_{x|x=1} = \psi_{xx|x=1} = 0 & \text{in } (0,T), \\ \psi_{|t=T} = 0 & \text{in } (0,1). \end{cases} \tag{13}$$

2.1. The equation $\zeta_t + \alpha\zeta_{5x} = g$

We begin with the following proposition.

Proposition 2. *Consider $\alpha > 0$. Given $\zeta_T \in (H^5 \cap H_0^2)(0,1)$ satisfying $\zeta_{T,xx}(1) = 0$ and $g \in C^1([0,T];L^2(0,1))$, there exists a unique solution $\zeta \in C^0([0,T];H^5(0,1)) \cap C^1([0,T];L^2(0,1))$ of*

$$\begin{cases} \zeta_t + \alpha\zeta_{5x} = g & \text{in } (0,T)\times(0,1), \\ \zeta_{|x=0} = \zeta_{x|x=0} = 0 & \text{in } (0,T), \\ \zeta_{|x=1} = \zeta_{x|x=1} = \zeta_{xx|x=1} = 0 & \text{in } (0,T), \\ \zeta_{|t=T} = \zeta_T & \text{in } (0,1). \end{cases} \tag{14}$$

Proof. This follows from the standard Lumer–Phillips theory. We can introduce the operator $A : D(A) \to L^2(0,1)$:

$$D(A) = \big\{\vartheta \in \big(H^5 \cap H_0^2\big)(0,1) \;/\; \vartheta_{xx}(1) = 0\big\} \quad \text{and} \quad A\vartheta = \alpha\vartheta_{5x}.$$

Then one can see that its adjoint is defined via

$$D\big(A^*\big) = \big\{h \in \big(H^5 \cap H_0^2\big)(0,1) \;/\; h_{xx}(0) = 0\big\} \quad \text{and} \quad A^*h = -\alpha h_{5x}.$$

Then it is elementary to check that $\langle A\vartheta,\vartheta\rangle_{L^2} \leqslant 0$ and $\langle A^*h,h\rangle_{L^2} \leqslant 0$ so that Proposition 2 follows from standard operator theory [20]. □

Now we prove some estimates for the solutions of (14).

We define the spaces, for $k \geqslant 0$,

$$X_k := \big\{y \in L^2\big(0,T;H^{k+2}(0,1)\big) \cap C^0\big([0,T];H^k(0,1)\big),\ y_{xx|x=0} \in H^{k/5}(0,T)\big\},$$

endowed with their natural norm.

Proposition 3. *One has the following estimates on the solutions of* (14):

$$\|\zeta\|_{X_s} \leqslant C\|g\|_{L^2(0,T;H_0^{s-2}(0,1))}, \quad \textit{for } s \in [0,10], \tag{15}$$

$$\|\zeta\|_{X_s} \leqslant C\|g\|_{L^1(0,T;H_0^{s}(0,1))}, \quad \textit{for } s \in [0,10], \tag{16}$$

$$\|\zeta_{xxx|x=0}\|_{H^{1/5}(0,T)} + \|\zeta_{xxx|x=1}\|_{H^{1/5}(0,T)} \leqslant C\|g\|_{L^2((0,T)\times(0,1))}, \tag{17}$$

and

$$\|\zeta_{4x|x=0}\|_{L^2(0,T)} + \|\zeta_{4x|x=1}\|_{L^2(0,T)} \leqslant C\|g\|_{L^2((0,T)\times(0,1))}. \tag{18}$$

Moreover in (17) *and* (18) *one can replace* $\|g\|_{L^2((0,T)\times(0,1))}$ *by* $\|g\|_{L^1(0,T;H_0^2(0,1))}$.

Remark 3. If we interpolate (15) and (16), we also deduce

$$\|\zeta\|_{X_s} \leqslant C\|g\|_{L^{4/3}(0,T;H_0^{s-1}(0,1))}, \quad \text{for } s \in [0,10]. \tag{19}$$

Proof of Proposition 3. We consider a smooth solution of (14) and establish several estimates on it.

1. Proof of (15)–(16).

• *Estimate in* X_0. We multiply (14) by $(1+x)\zeta$:

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (1+x)\zeta^2\,dx + \alpha\zeta_{xx|x=0}^2 + \frac{5}{2}\alpha\int_0^1 \zeta_{xx}^2\,dx = -\int_0^1 (1+x)g\zeta\,dx. \tag{20}$$

It follows that

$$\|\zeta\|_{X_0} \leqslant C\|g\|_{L^2(0,T;H^{-2}(0,1))}. \tag{21}$$

It is also clear that from (20) it follows

$$\|\zeta\|_{X_0} \leqslant C\|g\|_{L^1(0,T;L^2(0,1))}. \tag{22}$$

• *Estimate in* X_5. Now we consider $g \in L^2(0,T;H_0^3(0,1))$. Observe that due to (14), for such a g, the traces of ζ_{5x} and ζ_{6x} on both sides, and the trace of ζ_{7x} on the right, vanish.

We apply the operator ∂_{5x} to the equation and we apply (21):

$$\|\zeta_{5x}\|_{X_0} \leqslant C\|g_{5x}\|_{L^2(0,T;H^{-2}(0,1))}.$$

Using the equation, this gives

$$\|\zeta_{xx|x=0}\|_{H^1(0,T)} \leqslant C\|g\|_{L^2(0,T;H_0^3(0,1))}.$$

This yields also

$$\|\zeta\|_{X_5} \leqslant C\|g\|_{L^2(0,T;H_0^3(0,1))}. \tag{23}$$

In the same way, we have

$$\|\zeta\|_{X_5} \leqslant C\|g\|_{L^1(0,T;H_0^5(0,1))}. \tag{24}$$

• *Estimate in* X_{10}. Here we consider $g \in L^2(0,T;H_0^8(0,1))$. We apply the operator ∂_{5x} to the equation and we apply (23) (since $g_{5x} \in L^2(0,T;H_0^3(0,1))$):

$$\|\zeta_{5x}\|_{X_5} \leqslant C\|g_{5x}\|_{L^2(0,T;H_0^3(0,1))}.$$

This yields as previously

$$\|\zeta\|_{X_{10}} \leqslant C\|g\|_{L^2(0,T;H_0^8(0,1))}. \tag{25}$$

Also we have

$$\|\zeta\|_{X_{10}} \leqslant C\|g\|_{L^1(0,T;H_0^{10}(0,1))}. \tag{26}$$

• *Interpolation argument*. By an interpolation argument, we deduce (15) and (16) for every $s \in [0,10]$.

2. Proof of (17). Let $\rho \in C^4([0,1];\mathbb{R})$ satisfying $\rho(x)=0$ for $x \in [0,1/2]$ and $\rho(x)=1$ for $x \in [3/4,1]$.

- We use estimate (15) for $s=1$:

$$\|\zeta\|_{X_1} \leqslant C\|g\|_{L^2(0,T;H^{-1}(0,1))}. \tag{27}$$

Multiplying (14) with $\rho\zeta_{xx}$, integrating in space and integrating by parts, we get, for almost any $t \in [0,T]$,

$$\frac{\alpha}{2}|\zeta_{xxx|x=1}|^2 = \frac{3\alpha}{2}\int_0^1 \rho_x|\zeta_{xxx}|^2\,dx - \frac{\alpha}{2}\int_0^1 \rho_{xxx}|\zeta_{xx}|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \rho|\zeta_x|^2\,dx$$
$$- \langle \rho_x\zeta_x, \zeta_t\rangle_{H_0^2(0,1)\times H^{-2}(0,1)} - \langle g, \rho\zeta_{xx}\rangle_{H^{-1}(0,1)\times H_0^1(0,1)}. \tag{28}$$

Integrating in time and thanks to (14) and (27), we get

$$\|\zeta\|_{X_1} + \|\zeta_{xxx|x=1}\|_{L^2(0,T)} + \|\zeta_{xxx|x=0}\|_{L^2(0,T)} \leqslant C\|g\|_{L^2(0,T;H^{-1}(0,1))}. \tag{29}$$

An estimate for $\zeta_{xxx|x=0}$ can be done in the same way, by employing the weight $1-\rho$.

- Now, we use estimate (15) for $s=6$:

$$\|\zeta\|_{X_6} \leqslant C\|g\|_{L^2(0,T;H_0^4(0,1))}. \tag{30}$$

In order to prove that $\zeta_{txxx|x=1} \in L^2(0,T)$, we multiply (14) by $\rho\partial_t\partial_x^7\zeta$, we integrate in space and we integrate by parts (using again what we know on the traces of ζ_{5x}, ζ_{6x} and ζ_{7x}):

$$\frac{1}{2}|\zeta_{txxx|x=1}|^2 = \frac{7}{2}\int_0^1 \rho_x|\zeta_{txxx}|^2\,dx + \int_0^1 \zeta_{txxx}(6\rho_{xx}\zeta_{txx} + 4\rho_{xxx}\zeta_{tx} + \rho_{4x}\zeta_t)\,dx$$
$$- \frac{\alpha}{2}\frac{d}{dt}\int_0^1 \big(\rho|\zeta_{6x}|^2 - \rho_{xx}|\zeta_{5x}|^2\big)\,dx + \alpha\langle \rho_x\zeta_{6x}, \zeta_{t5x}\rangle_{H_0^2\times H^{-2}} + \zeta_{txxx|x=1}g_{xxx|x=1}$$
$$- \int_0^1 \zeta_{txxx}\sum_{j=0}^4\left(\binom{4}{j}\partial_x^j g\,\partial_x^{4-j}\rho\right)dx. \tag{31}$$

As previously, the same can be done for $\zeta_{txxx|x=0}$.

Integrating in time, using Cauchy–Schwarz inequality to estimate the last term in the second line and using (14) and (30), we get

$$\|\zeta\|_{X_6} + \|\zeta_{txxx|x=1}\|_{L^2(0,T)} + \|\zeta_{txxx|x=0}\|_{L^2(0,T)} \leqslant C\|g\|_{L^2(0,T;H_0^4(0,1))}. \tag{32}$$

- An interpolation argument applied to (29) and (32) provides (17).

3. Proof of inequality (18). We multiply the equation of ζ by $\rho\zeta_{4x}$ and we integrate in space. After some integration by parts, we obtain

$$\frac{\alpha}{2}|\zeta_{4x|x=1}|^2 = \frac{\alpha}{2}\int_0^1 \rho_x|\zeta_{4x}|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \rho|\zeta_{xx}|^2\,dx - \langle \rho_{xx}\zeta_{xx}, \zeta_t\rangle_{H_0^1\times H^{-1}}$$
$$- 2\langle \rho_x\zeta_{xx}, \zeta_{tx}\rangle_{H_0^2\times H^{-2}} + \int_0^1 \rho\zeta_{4x}g\,dx. \tag{33}$$

Integrating in time this identity, we have

$$\|\zeta_{4x|x=1}\|_{L^2(0,T)} \leqslant C\big(\|\zeta\|_{L^\infty(0,T;H^2(0,1))} + \|\zeta_t\|_{L^2(0,T;H^{-1}(0,1))} + \|\zeta\|_{L^2(0,T;H^4(0,1))} + \|g\|_{L^2((0,T)\times(0,1))}\big).$$

Using (15), we have estimated $\zeta_{4x|x=1}$ as in (18). The estimate $\zeta_{4x|x=0}$ is similar by multiplying by $(1-\rho)\zeta_{4x}$.

Finally, the proof of (17) and (18) with $g \in L^1(0,T;H_0^2(0,1))$ is completely analogous. □

2.2. Well-posedness for the adjoint equation

Now we can state the following existence and regularity result for (13).

Proposition 4. *Given a_k satisfying* (12) *and $f \in L^2(0,T;L^2(0,1))$, there exists a unique solution $\psi \in L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))$ of* (13).

Proof. We use a fixed point scheme. Given $\hat{\psi} \in L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))$, we consider the solution $\psi := \mathcal{T}\hat{\psi}$ of

$$\begin{cases} \psi_t + \alpha\psi_{5x} = \displaystyle\sum_{k=0}^{3}(-1)^{k+1}\partial_x^k\big(a_k(t,x)\hat{\psi}\big) + f & \text{in } (\hat{T},T)\times(0,1),\\ \psi_{|x=0} = \psi_{x|x=0} = 0 & \text{in } (\hat{T},T),\\ \psi_{|x=1} = \psi_{x|x=1} = \psi_{xx|x=1} = 0 & \text{in } (\hat{T},T),\\ \psi_{|t=T} = 0 & \text{in } (0,1), \end{cases} \tag{34}$$

where $\hat{T} \in (0,T)$ is to be fixed later.

Using Proposition 3 (precisely (15) for $s=2$), we infer that

$$\begin{aligned} \|\mathcal{T}\hat{\psi}_1 - \mathcal{T}\hat{\psi}_2\|_{X_2} &\leqslant C\big(\|a_k\|_{L^\infty(0,T;W^{k,\infty})}\big)\|\hat{\psi}_1 - \hat{\psi}_2\|_{L^2(0,T;H^3)}\\ &\leqslant C\big(\|a_k\|_{L^\infty(0,T;W^{k,\infty})}\big)\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|_{L^4(0,T;H^3)}. \end{aligned} \tag{35}$$

Note that the constant in (15) is independent of $\hat{T} \in (0,T)$.

Then by interpolation we deduce that

$$\begin{aligned} \|\mathcal{T}\hat{\psi}_1 - \mathcal{T}\hat{\psi}_2\|_{X_2} &\leqslant C\big(\|a_k\|_{L^\infty(0,T;W^{k,\infty})}\big)\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|^{1/2}_{L^\infty(0,T;H^2)}\|\hat{\psi}_1 - \hat{\psi}_2\|^{1/2}_{L^2(0,T;H^4)}\\ &\leqslant C\big(\|a_k\|_{L^\infty(0,T;W^{k,\infty})}\big)\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|_{X_2}. \end{aligned} \tag{36}$$

It follows that $\mathcal{T}$ is contracting for sufficiently small time $\hat{T}$. Then extending the solution obtained in $(\hat{T},T)$ to a solution in $(0,T)$ is standard using the linear character of the equation. □

Furthermore, the solutions described in Proposition 4 possess the following regularity property.

Proposition 5. *Under the assumptions of Proposition* 4*, the solution ψ has the following hidden regularity*:

$$\|\psi\|_{X_2} + \|\psi_{xx|x=0,1}\|_{H^{2/5}(0,T)} + \|\psi_{xxx|x=0,1}\|_{H^{1/5}(0,T)} + \|\psi_{4x|x=0,1}\|_{L^2(0,T)} \leqslant C\|f\|_{L^2((0,T)\times(0,1))}. \tag{37}$$

Proof. This is a consequence of Propositions 3 and 4. Note that due to the contracting character of $\mathcal{T}$ and using Proposition 3, we have

$$\|\psi\|_{X_2} \lesssim \big\|\mathcal{T}(0)\big\|_{X_2} \lesssim \|f\|_{L^2((0,T)\times(0,1))}.$$

Now we can use

$$g := \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k\big(a_k(t,x)\psi\big) + f,$$

as the right-hand side in (14) to deduce (37) from Proposition 3. □

2.3. Well-posedness for the initial boundary value problem

In this subsection we give the notion of solution of

$$\begin{cases} y_t + \alpha y_{5x} = \sum_{k=0}^{3} a_k(t,x)\partial_x^k y + h & \text{in } (0,T)\times(0,1), \\ y_{|x=0} = v_1, \quad y_{|x=1} = v_2, \quad y_{x|x=0} = v_3 & \text{in } (0,T), \\ y_{x|x=1} = v_4, \quad y_{xx|x=0} = v_5 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,1), \end{cases} \tag{38}$$

where y_0, h, $v_1, \dots, v_5$ are given functions. The solution of (38) for homogeneous boundary conditions and $h \in L^2(0,T;L^2(0,1))$ is granted by Proposition 4 (replace t by $T-t$ and x by $1-x$). Hence we can suppose without loss of generality that $h = 0$.

Definition 1. Let $y_0 \in H^{-2}(0,1)$, $v_1, v_2 \in L^2(0,T)$, $v_3, v_4 \in H^{-1/5}(0,T)$ and $v_5 \in H^{-2/5}(0,T)$. We call y a solution by transposition of (38) with $h=0$, a function $y \in L^2((0,T)\times(0,1))$ such that

$$\begin{aligned} \int_0^T\int_0^1 yf\,dx\,dt = {} & \langle u_0, \psi_{|t=0}\rangle_{H^{-2}(0,1)\times H_0^2(0,1)} + \alpha\int_0^T v_1\psi_{4x|x=0}\,dt - \alpha\int_0^T v_2\psi_{4x|x=1}\,dt \\ & - \alpha\langle v_3, \psi_{xxx|x=0}\rangle_{H^{-1/5}(0,T)\times H^{1/5}(0,T)} + \alpha\langle v_4, \psi_{xxx|x=1}\rangle_{H^{-1/5}(0,T)\times H^{1/5}(0,T)} \\ & + \alpha\langle v_5, \psi_{xx|x=0}\rangle_{H^{-2/5}(0,T)\times H^{2/5}(0,T)} + \int_0^T a_{3|x=0}v_1\psi_{xx|x=0}\,dt, \quad \forall f \in L^2\big((0,T)\times(0,1)\big), \end{aligned} \tag{39}$$

where ψ is the solution of (13) associated to f.

Proposition 6. *Let a_k satisfy* (12). *There exists a unique solution by transposition of system* (38) *with $h=0$. Moreover, there exists $C>0$ such that*

$$\begin{aligned} \|y\|_{L^2((0,T)\times(0,1))} \leqslant C\big(& \|y_0\|_{H^{-2}(0,1)} + \|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} + \|v_3\|_{H^{-1/5}(0,T)} \\ & + \|v_4\|_{H^{-1/5}(0,T)} + \|v_5\|_{H^{-2/5}(0,T)}\big). \end{aligned}$$

Proof. All comes to prove

$$\psi \in C^0\big([0,T]; H_0^2(0,1)\big), \qquad \psi_{4x|x=0,1} \in L^2(0,T), \qquad \psi_{xxx|x=0,1} \in H^{1/5}(0,T),$$
$$\psi_{xx|x=0} \in H^{2/5}(0,T)$$

and the following inequality:

$$\begin{aligned} & \|\psi\|_{L^\infty(0,T;H^2(0,1))} + \|\psi_{4x|x=0,1}\|_{L^2(0,T)} + \|\psi_{xxx|x=0,1}\|_{H^{1/5}(0,T)} + \|\psi_{xx|x=0}\|_{H^{2/5}(0,T)} \\ & \quad \leqslant C\|f\|_{L^2((0,T)\times(0,T))}. \end{aligned} \tag{40}$$

This was established in Proposition 5. □

Corollary 1. *Suppose that $a_k = 0$ for $k = 0,\dots,3$. For any $T>0$, there exists $C>0$* (*nondecreasing in T*) *such that, if $v_1, v_2 \in H^{2/5}(0,T)$, $v_3, v_4 \in H^{1/5}(0,T)$ and $v_5 \in L^2(0,T)$, the above solution of system* (38) *with $h=0$ and $y_0 = 0$ belongs to $L^2(0,T;H^2(0,1))$ and satisfies that*

$$\begin{aligned} & \|y\|_{L^2(0,T;H^2(0,1))\cap C^0([0,T];L^2(0,1))} \\ & \quad \leqslant C\big(\|v_1\|_{H^{2/5}(0,T)} + \|v_2\|_{H^{2/5}(0,T)} + \|v_3\|_{H^{1/5}(0,T)} + \|v_4\|_{H^{1/5}(0,T)} + \|v_5\|_{L^2(0,T)}\big). \end{aligned}$$

Proof. Suppose that $v_1, v_2 \in H_0^1(0,T)$, $v_3, v_4 \in H_0^{4/5}(0,T)$ and $v_5 \in H_0^{3/5}(0,T)$. We apply Proposition 6 to initial state 0 and controls v_{1t}, v_{2t}, v_{3t}, v_{4t} and v_{5t}, and deduce a solution z in $L^2((0,T)\times(0,1))$. Using Proposition 3 (for ψ) in the $L^1(0,T;H_0^2(0,1))$ framework, we deduce as previously that z is moreover in $C^0([0,T];H^{-2}(0,1))$. Next we apply Proposition 6 to initial state 0 and controls v_1, v_2, v_3, v_4 and v_5, and deduce a solution y. Using Definition 1 with smooth test functions f, it is not difficult to see that $z = y_t$. Using the equation, we infer that $y \in L^2(0,T;H^5(0,1)) \cap C^0([0,T];H^3(0,1))$. Now the conclusion follows by interpolation.

The fact that the constant C can be chosen nondecreasing in time is simple: given $0 < T' < T$ and $v_1, v_2 \in H^{2/5}(0,T')$, $v_3, v_4 \in H^{1/5}(0,T')$ and $v_5 \in L^2(0,T')$, one extends these boundary values to $[0,T]$ by 0. One applies the inequality for time $[0,T]$, and one sees that one can choose a constant for T' which is not larger. □

3. Controllability of the linearized equation

3.1. Carleman estimate

We consider the following dual system:

$$\begin{cases} \varphi_t + \alpha\varphi_{5x} = \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k\big(a_k(t,x)\varphi\big) + f & \text{in } (0,T)\times(0,1),\\ \varphi_{|x=0} = \varphi_{|x=1} = \varphi_{x|x=0} = \varphi_{x|x=1} = \varphi_{xx|x=1} = 0 & \text{in } (0,T),\\ \varphi_{|t=T} = \varphi_T & \text{on } (0,1), \end{cases} \tag{41}$$

where the functions a_k satisfy (12).

A central argument in this paper consists in establishing a Carleman inequality for (41). For this let us set

$$\alpha(t,x) = \frac{\beta(x)}{t^{1/4}(T-t)^{1/4}}, \tag{42}$$

for $(t,x) \in Q$. Weight functions of this kind were first introduced by A.V. Fursikov and O.Yu. Imanuvilov; see [12]. In the above equation β is a positive, strictly decreasing and concave polynomial of degree 2 in $[0,1]$. Observe that the function α satisfies

$$C \leqslant T^{1/2}\alpha, \qquad C_0\alpha \leqslant -\alpha_x \leqslant C_1\alpha, \qquad C_0\alpha \leqslant -\alpha_{xx} \leqslant C_1\alpha \quad \text{in } (0,T)\times[0,1], \tag{43}$$

$$|\alpha_t| + |\alpha_{xt}| + |\alpha_{xxt}| \leqslant CT\alpha^5, \qquad |\alpha_{tt}| \leqslant C\big(T^2\alpha^9 + \alpha^5\big) \leqslant CT^2\alpha^9 \quad \text{in } (0,T)\times[0,1], \tag{44}$$

where C, C_0 and C_1 are positive constants independent of T.

We have

Proposition 7. *Suppose that* (12) *applies. There exists a positive constant C independent of T such that, for any $\varphi_T \in L^2(0,1)$ and $f \in L^2(0,T;L^2(0,1))$, we have*

$$\iint_Q e^{-2s\alpha}\alpha\big(|\varphi_{4x}|^2 + s^2\alpha^2|\varphi_{xxx}|^2 + s^4\alpha^4|\varphi_{xx}|^2 + s^6\alpha^6|\varphi_x|^2 + s^8\alpha^8|\varphi|^2\big)\,dt\,dx$$
$$\leqslant C\left(\int_0^T \alpha_{|x=1}e^{-2s\alpha_{|x=1}}\big(|\varphi_{4x|x=1}|^2 + s^2\alpha_{|x=1}^2|\varphi_{xxx|x=1}|^2\big)\,dt + s^{-1}\iint_Q e^{-2s\alpha}|f|^2\,dt\,dx\right), \tag{45}$$

for any $s \geqslant C(T^{1/4} + T^{1/2})$, where φ is the solution of (41).

The proof of this inequality is postponed to Section 5.

Remark 4. We will also require β to satisfy

$$\max_{x\in[0,1]}\beta(x) < \sqrt{2}\min_{x\in[0,1]}\beta(x). \tag{46}$$

This is not needed for Proposition 7 (nor for Proposition 8 below), but will be useful later.

3.2. Weighted observability estimate

Now let us deduce from Proposition 7 a slightly modified inequality, with a weight function not vanishing at $t = 0$. We begin by introducing a new weight. Set ℓ on $[0, T]$ by

$$\ell(t) := \begin{cases} \frac{T^2}{4} & \text{if } t \leqslant \frac{T}{2}, \\ t(T-t) & \text{otherwise.} \end{cases} \tag{47}$$

Now introduce

$$\gamma(t,x) = \frac{\beta(x)}{\ell(t)^{1/4}}. \tag{48}$$

Proposition 8. *Suppose that* (12) *applies. There exist two positive constants s_0 and $C > 0$ depending on T such that, for any $\varphi_T \in L^2(0,1)$ and any $f \in L^2(0,T;L^2(0,1))$, we have*

$$\iint_Q e^{-2s_0\gamma}\gamma^9|\varphi|^2\,dx\,dt + \int_0^1 \big|\varphi(0,x)\big|^2\,dx$$
$$\leqslant C\bigg(\iint_Q e^{-2s_0\gamma}|f|^2\,dt\,dx + \int_0^T \gamma_{|x=1}e^{-2s_0\gamma_{|x=1}}\big(|\varphi_{4x\,|x=1}|^2 + \gamma_{|x=1}^2|\varphi_{xxx|x=1}|^2\big)\,dt\bigg), \tag{49}$$

where φ is the solution of (41).

Proof. We use the following energy estimate:

$$\|\varphi\|_{L^\infty(0,T/2;L^2(0,1))} \leqslant C\exp\big\{C\|a_k\|_{L^\infty(0,T;W^{k,\infty}(0,1))}\big\}\big(\|f\|_{L^2(0,3T/4;L^2(0,1))} + \|\varphi\|_{L^2(T/2,3T/4;L^2(0,1))}\big). \tag{50}$$

To get (50), introduce $\eta \in C^\infty([0,T];\mathbb{R}^+)$ such that $\eta = 1$ in $[0, T/2]$ and $\eta = 0$ in $[3T/4, T]$, multiply Eq. (41) by $\eta(t)(1+x)\varphi$, and perform several integration by parts as in (21).

Let us notice that the weight functions γ and $e^{-2s\gamma}$ are positive for $t \in [0, T/2]$. Hence there is a constant C such that

$$\big\|e^{-s\gamma}\gamma^{9/2}\varphi\big\|_{L^\infty(0,T/2;L^2(0,1))} \leqslant C\exp\big\{C\|a_k\|_{L^\infty(0,T;W^{k,\infty}(0,1))}\big\}$$
$$\times\big(\big\|e^{-s\gamma}f\big\|_{L^2(0,3T/4;L^2(0,1))} + \big\|e^{-s\gamma}\gamma^{9/2}\varphi\big\|_{L^2(T/2,3T/4;L^2(0,1))}\big). \tag{51}$$

Next, we use (45) and the choice of γ to deduce

$$\int_{\frac{T}{2}}^T \int_{(0,1)} e^{-2s\gamma}\gamma^9|\varphi|^2\,dx\,dt \leqslant C\bigg(\iint_Q e^{-2s\gamma}|f|^2\,dt\,dx$$
$$+ \int_0^T \gamma_{|x=1}e^{-2s\gamma_{|x=1}}\big(|\varphi_{4x\,|x=1}|^2 + \gamma_{|x=1}^2|\varphi_{xxx|x=1}|^2\big)\,dt\bigg), \tag{52}$$

for s large enough. Combining (51) and (52) we obtain (49). □

Let us consider s_0 as in Proposition 8. We introduce

$$\kappa_0 := \frac{s_0\sqrt{2}}{T^{1/4}}\max_{x\in[0,1]}\beta(x) \quad \text{and} \quad \kappa_1 := \frac{s_0}{T^{1/4}}\min_{x\in[0,1]}\beta(x). \tag{53}$$

Corollary 2. *Under the assumptions of Proposition* 8*, one has*

$$\iint_Q e^{-\frac{2\kappa_0}{(T-t)^{1/4}}}(T-t)^{-9/4}|\varphi|^2\,dx\,dt+\int_0^1\big|\varphi(0,x)\big|^2\,dx$$
$$\leqslant C\left(\iint_Q e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}|f|^2\,dt\,dx+\int_0^T(T-t)^{-1/4}e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}\big(|\varphi_{4x\,|x=1}|^2+(T-t)^{-1/2}|\varphi_{xxx|x=1}|^2\big)\,dt\right). \tag{54}$$

3.3. Controllability

We introduce the following space:

$$E_0=\big\{y\in L^2\big(0,T;L^2(0,1)\big)\ /\ e^{\frac{\kappa_1}{(T-t)^{1/4}}}y\in L^2\big(0,T;L^2(0,1)\big)\big\}. \tag{55}$$

We have the following controllability result.

Proposition 9. *Given h such that $(T-t)^{9/8}e^{\frac{\kappa_0}{(T-t)^{1/4}}}h\in L^2((0,T)\times(0,1))$ and $y_0\in L^2(0,1)$, there exist controls $v_2, v_4\in L^2(0,T)$ satisfying*

$$(T-t)^{1/8}e^{\frac{\kappa_1}{(T-t)^{1/4}}}v_2\in L^2(0,T)\quad\text{and}\quad(T-t)^{3/8}e^{\frac{\kappa_1}{(T-t)^{1/4}}}v_4\in L^2(0,T), \tag{56}$$

such that if we call y the solution of (38) *starting from y_0 with $v_1=v_3=v_5=0$, then y belongs to E_0. In particular, y, which belongs to $C^0([0,T];H^{-5}(0,1))$, satisfies*

$$y_{|t=T}=0\quad\text{on }(0,1). \tag{57}$$

Besides, there exists a constant $C>0$ such that

$$\big\|e^{\frac{\kappa_1}{(T-t)^{1/4}}}y\big\|_{L^2(0,T;L^2(0,1))}+\big\|(T-t)^{1/8}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\big(v_2,(T-t)^{1/4}v_4\big)\big\|_{L^2(0,T)}$$
$$\leqslant C\big(\|y_0\|_{L^2(0,1)}+\big\|(T-t)^{9/8}e^{\frac{\kappa_0}{(T-t)^{1/4}}}h\big\|_{L^2((0,T)\times(0,1))}\big). \tag{58}$$

Proof. The proof is inspired by Fursikov and Imanuvilov's approach [12]. Define L

$$Ly:=y_t+\alpha y_{5x}-\sum_{k=0}^3 a_k(t,x)\partial_x^k y, \tag{59}$$

and L^* its dual operator:

$$L^*\phi:=-\phi_t-\alpha\phi_{5x}-\sum_{k=0}^3(-1)^k\partial_x^k\big(a_k(t,x)\phi\big). \tag{60}$$

Let us set

$$F_0=\big\{\phi\in C^\infty\big([0,T]\times[0,1];\mathbb{R}\big)\ /\ \phi_{|x=0}=\phi_{|x=1}=\phi_{x|x=0}=\phi_{x|x=1}=\phi_{xx|x=1}=0\big\}.$$

Consider the bilinear form

$$a(\hat\phi,\phi)=\iint_Q e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}L^*\hat\phi L^*\phi\,dx\,dt$$
$$+\int_0^T e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}(T-t)^{-1/4}\big[\hat\phi_{4x|x=1}\phi_{4x|x=1}+(T-t)^{-1/2}\hat\phi_{xxx|x=1}\phi_{xxx|x=1}\big]\,dt,\quad\forall\hat\phi,\phi\in F_0.$$

We also introduce the linear form

$$\langle \ell, \phi\rangle = \iint_Q h\phi\, dt\, dx + \int_0^1 u_0 \phi_{|t=0}\, dx. \tag{61}$$

Introduce $\overline{F}_0$ the completion of F_0 for the norm $\phi \mapsto a(\phi,\phi)^{1/2}$ (it is a norm from Corollary 2).

The next step in this proof is to demonstrate that there exists exactly one $\hat{\phi}$ in the class $\overline{F}_0$ satisfying

$$a(\hat{\phi}, \phi) = l(\phi), \quad \forall \phi \in \overline{F}_0. \tag{62}$$

Now $\overline{F}_0$ is a Hilbert space for the scalar product $a(\cdot,\cdot)$, hence in order to get (62) it is sufficient to prove that ℓ is a continuous linear form on $\overline{F}_0$. From Cauchy–Schwarz inequality, we see that

$$\left| \iint_Q h\phi\, dt\, dx \right| \leqslant \left\| (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \right\|_{L^2(0,T;L^2(0,1))} \left\| (T-t)^{-9/8} e^{-\frac{\kappa_0}{(T-t)^{1/4}}} \phi \right\|_{L^2(0,T;L^2(0,1))}. \tag{63}$$

Using the assumption on h and Corollary 2, one sees that ℓ is indeed a continuous linear form on $\overline{F}_0$. Hence there exists a unique $\hat{\phi} \in \overline{F}_0$ satisfying (62).

Let us set

$$y = e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} L^* \hat{\phi}, \qquad v_2 = (T-t)^{-1/4} e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} \hat{\phi}_{4x|x=1} \quad \text{and} \quad v_4 = (T-t)^{-3/4} e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} \hat{\phi}_{xxx|x=1}. \tag{64}$$

Finally it is not difficult to see that $y \in E_0$, that (v_2, v_4) satisfies (56) and that y is a solution of (38) with $v_1 = v_3 = v_5 = 0$. This concludes the proof of Proposition 9. □

Now we define the space

$$\begin{aligned} E_1 = \Big\{ & y \in E_0 \,/\, (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y \in L^2\big(0,T; H^4(0,1)\big) \cap C^0\big([0,T]; H_0^2(0,1)\big), \\ & y_{|x=0} = y_{x|x=0} = y_{xx|x=0} = 0,\ (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} Ly \in L^2\big(0,T; L^2(0,1)\big) \Big\}. \end{aligned} \tag{65}$$

Proposition 10. *Given h such that $(T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \in L^2((0,T)\times(0,1))$ and $y_0 \in H_0^2(0,1)$, there exist controls $(v_2, v_4) \in L^2(0,T)^2$ such that the associated solution y of* (38) *with $v_1 = v_3 = v_5 = 0$ belongs to E_1 and moreover satisfies*

$$\begin{aligned} & \left\| (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y \right\|_{L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))} \\ & \quad \leqslant C \Big(\|y_0\|_{H_0^2(0,1)} + \left\| (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \right\|_{L^2((0,T)\times(0,1))} \Big), \end{aligned} \tag{66}$$

for some $C > 0$.

Proof. We extend the problem to the interval $[0,2]$. We extend y_0 (resp. h) by 0 in $[1,2]$ (resp. $[0,T]\times[1,2]$), we call $\tilde{y}_0$ (resp. $\tilde{h}$) the resulting function. We also extend a_k in $[0,T]\times[1,2]$ in a way that keeps the $L^\infty(0,T;W^{k,\infty})$ regularity (in a continuous way), and in such a way that

$$a_k(t,x) = 0 \quad \text{in } [0,T] \times \left[\frac{3}{2}, 2\right].$$

We now consider the following control problem:

$$\begin{cases} \tilde{y}_t + \alpha \tilde{y}_{5x} = \displaystyle\sum_{k=0}^{3} \tilde{a}_k(t,x) \partial_x^k \tilde{y} + \tilde{h} & \text{in } (0,T)\times(0,2), \\ \tilde{y}_{|x=0} = \tilde{y}_{x|x=0} = \tilde{y}_{xx|x=0} = 0 & \text{in } (0,T), \\ \tilde{y}_{|x=2} = \tilde{v}_2, \quad \tilde{y}_{x|x=2} = \tilde{v}_4 & \text{in } (0,T), \\ \tilde{y}_{|t=0} = \tilde{y}_0 & \text{in } (0,2). \end{cases} \tag{67}$$

According to Proposition 9, there exist $\tilde{v}_2, \tilde{v}_4$ fulfilling (56) such that the corresponding solution $\tilde{y}$ belongs to E_0 (adapted to the interval $[0,2]$ of course). Now we claim that the restriction of $\tilde{y}$ to $[0,T]\times[0,1]$ satisfies the required properties. We have to establish that

$$(T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}y \in L^2\big(0,T;H^4(0,1)\big)\cap C^0\big([0,T];H^2(0,1)\big).$$

For that, we introduce

$$y^*(t,x) := (T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\tilde{y}(t,x). \tag{68}$$

This function satisfies

$$\begin{cases} y^*_t + \alpha y^*_{5x} = \sum_{k=0}^{3}\tilde{a}_k(t,x)\partial_x^k y^* + h^* & \text{in } (0,T)\times(0,2),\\ y^*_{|x=0} = y^*_{x|x=0} = y^*_{xx|x=0} = 0 & \text{in } (0,T),\\ y^*_{|x=2} = v^*_2, \quad y^*_{x|x=2} = v^*_4 & \text{in } (0,T),\\ y^*_{|t=0} = y^*_0 & \text{in } (0,2), \end{cases} \tag{69}$$

where

$$y^*_0 = T^{5/4}e^{\frac{\kappa_1}{T^{1/4}}}\tilde{y}_0, \qquad \big(v^*_2, v^*_4\big) = (T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}(\tilde{v}_2,\tilde{v}_4) \quad \text{and}$$
$$h^* = (T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\tilde{h} + \frac{d}{dt}\big[(T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\big]\tilde{y}.$$

These data are in $H^2_0(0,2)$, in $L^2(0,T)^2$ and in $L^2(0,T;L^2(0,2))$ respectively, thanks to Proposition 9. We will use the following lemma, whose proof is postponed to Section 6.

Lemma 1. *For k large enough, one has $(2-x)^{k+\frac{1}{2}}y^* \in L^2(0,T;H^4(0,2))\cap C^0([0,T];H^2(0,2))$ with the estimate*

$$\begin{aligned}&\big\|(2-x)^{k+\frac{1}{2}}y^*\big\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4}\big\|(2-x)^{k+j-4}\partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))}\\ &\quad\leqslant C\big(\|h^*\|_{L^2(0,T;L^2(0,2))} + \|y^*_0\|_{H^2(0,2)} + \|v^*_2\|_{L^2(0,T)} + \|v^*_4\|_{L^2(0,T)}\big),\end{aligned} \tag{70}$$

for some positive constant C.

Now we use (70) and the continuity of the previous extensions from $(0,1)$ to $(0,2)$ to deduce

$$\begin{aligned}&\big\|(T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}y\big\|_{L^2(0,T;H^4(0,1))\cap C^0([0,T];H^2(0,1))}\\ &\quad\leqslant C\bigg(\|y_0\|_{H^2_0(0,1)} + \big\|(T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}h\big\|_{L^2((0,T)\times(0,1))} + \bigg\|\frac{d}{dt}\big[(T-t)^{5/4}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\big]y\bigg\|_{L^2((0,T)\times(0,1))}\\ &\qquad + \big\|(T-t)^{1/8}e^{\frac{\kappa_1}{(T-t)^{1/4}}}\big(v_2,(T-t)^{1/4}v_4\big)\big\|_{L^2(0,T)}\bigg),\end{aligned}$$

for some $C>0$. Finally, we use $\kappa_1<\kappa_0$ to estimate the second term in the right-hand side, and (58) to estimate the last two terms. We deduce (66). □

4. Nonlinear problem

4.1. Proof of Theorem 2

We use a fixed point scheme to prove local existence and uniqueness in $X := L^2(0,T;H^2(0,1))\cap C^0([0,T];L^2(0,1))$. Given $z\in X$, we introduce the solution of

$$\begin{cases} u_t + \alpha u_{5x} + \mu u_{xxx} + \beta z u_{xxx} + \delta z_x u_{xx} + P'(z) u_x = 0 & \text{in } (0,T)\times(0,1), \\ u_{|x=0} = v_1, \quad u_{|x=1} = v_2, \quad u_{x|x=0} = v_3, \quad u_{x|x=1} = v_4, \quad u_{xx|x=0} = v_5 & \text{in } (0,T), \\ u_{|t=0} = u_0 & \text{in } (0,1). \end{cases} \tag{71}$$

Call $\mathcal{T}$ the operator which maps z to u.

The existence and uniqueness of u is obtained as in Proposition 4: one associates to $\hat{\psi} \in X$ the solution of

$$\begin{cases} u_t + \alpha u_{5x} = -\mu \hat{\psi}_{xxx} - \beta z \hat{\psi}_{xxx} - \delta z_x \hat{\psi}_{xx} - P'(z)\hat{\psi}_x & \text{in } (0,T)\times(0,1), \\ u_{|x=0} = v_1, \quad u_{|x=1} = v_2, \quad u_{x|x=0} = v_3, \quad u_{x|x=1} = v_4, \quad u_{xx|x=0} = v_5 & \text{in } (0,T), \\ u_{|t=0} = u_0 & \text{in } (0,1). \end{cases} \tag{72}$$

Let us notice that $\hat{\psi}_{xxx} \in L^2(0,T;H^{-1}(0,1))$ while (by interpolation) $z \in L^4(0,T;H^1(0,1))$, and hence $z\hat{\psi}_{xxx} \in L^{4/3}(0,T;H^{-1}(0,1))$; on the other hand, one also sees that $z_x\hat{\psi}_{xx} \in L^{4/3}(0,T;H^{-1}(0,1))$. It follows then from Remark 3 and Corollary 1 that (72) defines a solution in $L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))$.

Now consider $\hat{\psi}_1$ and $\hat{\psi}_2$ in X, and their images u_1 and u_2 by the above mapping. Making the difference of the two equations, multiplying by $(2-x)(u_1-u_2)$ and performing the same operations as in Proposition 3, we infer

$$\begin{aligned} \|u_1-u_2\|_X^2 \leqslant C\bigg[&\iint_Q \big|(u_1-u_2)_x(\hat{\psi}_1-\hat{\psi}_2)_{xx}\big|\,dt\,dx + \iint_Q \big|(u_1-u_2)z_x(\hat{\psi}_1-\hat{\psi}_2)_{xx}\big|\,dt\,dx \\ &+ \iint_Q \big|z(u_1-u_2)_x(\hat{\psi}_1-\hat{\psi}_2)_{xx}\big|\,dt\,dx + \iint_Q \big|(u_1-u_2)(1+z^2)(\hat{\psi}_1-\hat{\psi}_2)_x\big|\,dt\,dx\bigg]. \end{aligned} \tag{73}$$

We deduce for small $\varepsilon > 0$,

$$\begin{aligned} \|u_1-u_2\|_X^2 \leqslant C\|\hat{\psi}_1-\hat{\psi}_2\|_{L^2(0,T;H^2(0,1))}\big[&\|u_1-u_2\|_{L^2(0,T;H^1(0,1))} \\ &+ \|z_x\|_{L^{\frac{8}{3+2\epsilon}}(0,T;L^\infty(0,1))}\|u_1-u_2\|_{L^{\frac{8}{1-2\varepsilon}}(0,T;L^2(0,1))} \\ &+ \|z\|_{L^{\frac{8}{1-2\varepsilon}}(0,T;L^2(0,1))}\|u_1-u_2\|_{L^{\frac{8}{3+2\epsilon}}(0,T;W^{1,\infty}(0,1))}\big] \\ &+ C\|\hat{\psi}_1-\hat{\psi}_2\|_{L^4(0,T;H^1(0,1))}\big[\|u_1-u_2\|_{L^{4/3}(0,T;L^2(0,1))} \\ &+ \|u_1-u_2\|_{L^4(0,T;L^6(0,1))}\|z\|^2_{L^4(0,T;L^6(0,1))}\big]. \end{aligned} \tag{74}$$

Note that by interpolation and Sobolev imbeddings we have

$$L^2\big(0,T;H^2(0,1)\big) \cap L^\infty\big(0,T;L^2(0,1)\big) \hookrightarrow L^{12}\big(0,T;L^6(0,1)\big) \cap L^{\frac{8}{3+2\varepsilon}}\big(0,T;W^{1,\infty}(0,1)\big). \tag{75}$$

We infer that (at least if $T \leqslant 1$)

$$\|u_1-u_2\|_X^2 \leqslant CT^{\frac{1-2\varepsilon}{8}}\big(1+\|z\|_X^2\big)\|\hat{\psi}_1-\hat{\psi}_2\|_X\|u_1-u_2\|_X.$$

Hence the operator is contractive for sufficiently small time T, which proves the local well-posedness of (71).

Now let us prove that $\mathcal{T}$ has a fixed point. We decompose $u = \hat{u} + \check{u}$ with

$$\begin{cases} \check{u}_t + \alpha \check{u}_{5x} = 0 & \text{in } (0,T)\times(0,1), \\ \check{u}_{|x=0} = v_1, \quad \check{u}_{|x=1} = v_2, \quad \check{u}_{x|x=0} = v_3, \quad \check{u}_{x|x=1} = v_4, \quad \check{u}_{xx|x=0} = v_5 & \text{in } (0,T), \\ \check{u}_{|t=0} = 0 & \text{in } (0,1), \end{cases}$$

and

$$\begin{cases} \hat{u}_t + \alpha \hat{u}_{5x} + \mu \hat{u}_{xxx} + \beta z \hat{u}_{xxx} + \delta z_x \hat{u}_{xx} + P'(z)\hat{u}_x = \check{h} & \text{in } (0,T)\times(0,1), \\ \hat{u}_{|x=0} = \hat{u}_{|x=1} = \hat{u}_{x|x=0} = \hat{u}_{x|x=1} = \hat{u}_{xx|x=0} = 0 & \text{in } (0,T), \\ \hat{u}_{|t=0} = u_0 & \text{in } (0,1), \end{cases} \tag{76}$$

with

$$\check{h} = -\big(\mu \check{u}_{xxx} + \beta z \check{u}_{xxx} + \delta z_x \check{u}_{xx} + P'(z)\check{u}_x\big).$$

Let us prove that for some constant $C > 0$ independent of $T \leqslant 1$ and $T^{1/16}\|z\|_X$ small enough, the solution u of (71) satisfies

$$\|u\|_X \leqslant C\exp\big(CT^{1/16}\big(1+\|z\|_X^2\big)\big)\big(\|u_0\|_{L^2(0,1)} + \|v_1\|_{H^{2/5}(0,T)} + \|v_2\|_{H^{2/5}(0,T)} + \|v_3\|_{H^{1/5}(0,T)} + \|v_4\|_{H^{1/5}(0,T)} + \|v_5\|_{L^2(0,T)}\big). \tag{77}$$

That $\check{u}$ satisfies (77) for some constant $C > 0$ is a consequence of Corollary 1. For what concerns $\hat{u}$, we multiply (76) by $(2-x)\hat{u}$; after some integration by parts, we can deduce

$$\frac{d}{dt}\|\hat{u}\|_{L^2}^2 + \frac{5\alpha}{2}\int_0^1 |\hat{u}_{xx}|^2\,dx \leqslant C\big(1+\big\|z_x(t,\cdot)\big\|_\infty^2 + \big\|z(t,\cdot)\big\|_\infty^2\big)\|\hat{u}\|_{L^2}^2 + \nu\int_0^1 |\hat{u}_{xx}|^2\,dx + 2\int_0^1 (2-x)\hat{u}\check{h}\,dx,$$

for arbitrarily small ν. We use (75) for z and deduce when putting $\varepsilon = 1/4$ that

$$\int_0^T \big(1+\big\|z_x(t,\cdot)\big\|_\infty^2 + \big\|z(t,\cdot)\big\|_\infty^2\big)\,dt \lesssim T^{1/16}\big(1+\|z\|_X^2\big).$$

We choose ν small enough, use a Gronwall argument and deduce for all $t\in[0,T]$ that

$$\big\|\hat{u}(t)\big\|_{L^2}^2 + 2\alpha\int_0^t\int_0^1 |\hat{u}_{xx}|^2\,dx\,d\tau \leqslant \exp\big\{CT^{1/16}\big(1+\|z\|_X^2\big)\big\}\left(\|u_0\|_{L^2}^2 + 2\int_0^T\int_0^1 \big|(2-x)\hat{u}\check{h}\big|\,dx\,dt\right).$$

Now we take the supremum in t. The last term is treated as follows:

$$\begin{aligned}\int_0^T\int_0^1 \big|(2-x)\hat{u}z_x\check{u}_{xx}\big|\,dx\,dt &\lesssim T^{\frac{1-2\varepsilon}{8}}\left\|\int_0^1 |\hat{u}z_x\check{u}_{xx}|\,dx\right\|_{L^{\frac{8}{7+2\varepsilon}}(0,T)}\\ &\lesssim T^{\frac{1-2\varepsilon}{8}}\big\|\|\hat{u}\|_{L^2(0,1)}\|z_x\|_{L^\infty(0,1)}\|\check{u}_{xx}\|_{L^2(0,1)}\big\|_{L^{\frac{8}{7+2\varepsilon}}(0,T)}\\ &\lesssim T^{\frac{1-2\varepsilon}{8}}\|\hat{u}\|_X\|z\|_X\|\check{u}\|_X,\end{aligned}$$

using again (75). The other trilinear terms can be estimated analogously (up to an integration by parts). Then taking again $\varepsilon = 1/4$ and imposing $T^{1/16}\|z\|_X \ll 1$ we obtain (77).

Now let us show that $\mathcal{T}$ is contractive on

$$B := \big\{u\in X \big/ \|u\|_X \leqslant (\overline{C}+1)\big(\|u_0\|_{L^2(0,1)} + \|v_1\|_{H^{2/5}(0,T)} + \|v_2\|_{H^{2/5}(0,T)} + \|v_3\|_{H^{1/5}(0,T)} + \|v_4\|_{H^{1/5}(0,T)} + \|v_5\|_{L^2(0,T)}\big)\big\},$$

for sufficiently small T, where $\overline{C}$ is the constant in (77).

From (77), we see that, provided that T is suitably small, B is stable by $\mathcal{T}$. We now consider that this is the case. Now consider $z_1, z_2 \in B$ and denote $u_1 := \mathcal{T}z_1$, $u_2 := \mathcal{T}z_2$, $z := z_1 - z_2$, $u := u_1 - u_2$. We have

$$u_t + \alpha u_{5x} + \mu u_{xxx} + \beta z_1 u_{xxx} + \beta z u_{2,xxx} + \delta z_{1,x}u_{xx} + \delta z_x u_{2,xx} + P'(z_1)u_x + \big[P'(z_1)-P'(z_2)\big]u_{2,x} = 0 \quad \text{in } (0,T)\times(0,1). \tag{78}$$

We multiply (78) by $(2-x)u$, integrate in both time and space and perform the same reasoning as in (73)–(74). After lengthy but straightforward computations, we deduce (if $T\leqslant 1$)

$$\|u\|_X^2 \leqslant CT^{1/16}\big[\|u_2\|_X\|u\|_X\|z\|_X\big(1+\|z_1\|_X+\|z_2\|_X\big) + \big(1+\|z_1\|_X^2\big)\|u\|_X^2\big]. \tag{79}$$

Using that both u_2 and z_1 belong to B, this establishes that $\mathcal{T}$ is contractive on B for sufficiently small time T.

4.2. Proof of Theorem 3

We consider a solution $u \in L^2(0,T;H_0^2(0,1)) \cap C^0([0,T];L^2(0,1))$ of (1) with homogeneous boundary conditions. We introduce $\varepsilon_0 \in (0,1)$ arbitrarily small and $\eta \in C^\infty([0,T];\mathbb{R})$ such that $\eta = 0$ in $[0, \varepsilon_0 T/4]$ and $\eta = 1$ in $[\varepsilon_0 T/2, T]$. We consider the equation satisfied by ηu

$$(\eta u)_t + \alpha(\eta u)_{5x} = -\mu\eta u_{xxx} - \beta\eta u u_{xxx} - \delta\eta u_x u_{xx} - \eta P'(u) u_x + \eta' u \quad \text{in } (0,T)\times(0,1).$$

Now let us look at the regularity of the right-hand side. It is not difficult to see that the less regular terms are the second and third one. Concerning the term $u_x u_{xx}$, we have by interpolation $u \in L^4(0,T;H^1(0,1))$, so that $(u_x)^2 \in L^2(0,T;L^1(0,1))$, and hence $u_x u_{xx} \in L^2(0,T;W^{-1,1}(0,1)) \hookrightarrow L^2(0,T;H^{-\frac{3}{2}-\varepsilon}(0,1))$.

For what concerns uu_{xxx}, we use that $uu_{xxx} = (uu_{xx})_x - u_x u_{xx}$. Now we use that by interpolation $u \in L^{\frac{8}{1+2\varepsilon}}(0,T;H^{\frac{1}{2}+\varepsilon}(0,1))$ and $u_{xx} \in L^{\frac{8}{3-2\varepsilon}}(0,T;H^{-\frac{1}{2}-\varepsilon}(0,1))$, so that the product is in $L^2(0,T;H^{-\frac{1}{2}-\varepsilon}(0,1))$. As a conclusion, the term uu_{xxx} has the same regularity as $u_x u_{xx}$.

Now we use Proposition 3, and infer that for arbitrary $\varepsilon > 0$, one has $\eta u \in L^2(0,T;H^{\frac{5}{2}-\varepsilon}(0,1)) \cap C^0([0,T];H^{\frac{1}{2}-\varepsilon}(0,1))$. Then repeating the above steps we can show by a bootstrap argument that the solution u becomes C^∞ in time and space in arbitrary small time.

4.3. Proof of Remark 2

We start from a solution $u \in L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))$. Then proceeding as in the proof of Theorem 3 we can prove that the right-hand side is in $L^2(0,T;H^{-\frac{3}{2}-\varepsilon}(0,1))$.

Now using Lemmata 2 and 3 posed in $[0,1]$ rather than in $[0,2]$ (see in Section 6 below) and interpolation arguments, we deduce that $(1-x)^7 u \in L^2(0,T;H^{\frac{5}{2}-\varepsilon}(0,1))$. Then restricting to the space interval $[0, 1-\nu]$ for $\nu > 0$ suitably small and applying a bootstrap procedure, we see that u is regular at positive distance of the right endpoint.

4.4. Proof of Theorem 1

We consider a trajectory $\bar{u}$ as indicated in the statement; then $u = \bar{u} + y$ satisfies (1) if and only if y satisfies

$$\begin{aligned} &y_t + y_{5x} + \mu y_{3x} + \beta\big((\bar{u}+y)y_{3x} + \bar{u}_{3x} y\big) + \delta\big((\bar{u}+y)_x y_{xx} + \bar{u}_{xx} y_x\big) \\ &\quad + p y_x + 2q\big((\bar{u}+y)y_x + \bar{u}_x y\big) + 3r\big(y(2\bar{u}+y)(\bar{u}+y)_x + \bar{u}^2 y_x\big) = 0. \end{aligned} \tag{80}$$

Conspicuously, the controllability of (1) to the trajectory $\bar{u}$ is equivalent to the null controllability of (80).

Now we have the following result for (80).

Proposition 11. *Given $y_0 \in L^2(0,1)$ and $\bar{u} \in L^\infty(0,T;W^{3,\infty}(0,1))$, there exists $T > 0$ such that the nonlinear problem* (80) *with homogeneous boundary conditions* (3) *($v_1 = v_2 = v_3 = v_4 = v_5 = 0$) admits a unique solution $y \in L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))$, which regularizes in the sense that for any $\tau \in (0,T]$, $u \in L^2([\tau,T];H^4(0,1)) \cap C^0([\tau,T];H^2(0,1))$, with moreover*

$$\|y\|_{C^0([\tau,T];H^2(0,1))} \leqslant C(\tau,\bar{u})\|u_0\|_{L^2(0,1)}. \tag{81}$$

The proof of Proposition 11 follows the steps of the proof of Theorems 2 and 3; all the computations are justified thanks to $\bar{u} \in L^\infty(0,T;W^{3,\infty}(0,1))$. We omit the details.

We now turn to the proof of Theorem 1. The solution of the controllability problem is obtained in two successive steps. In a first step, we set the controls (v_2, v_4) to $(0,0)$. According to Proposition 11, this regularizes the state of the system, so that we may consider that the initial state y_0 belongs to $H_0^2(0,1)$ and is small (see (81)). From now, we consider that this is the case, and proceed to the proof of the null-controllability of system (80) with such an initial state, by using the inverse mapping theorem.

We introduce the coefficients a_k as follows

$$\begin{aligned} a_0 &= \beta\bar{u}_{3x} + 2q\bar{u}_x + 6r\bar{u}\bar{u}_x,\\ a_1 &= \delta\bar{u}_{xx} + p + 2q\bar{u} + 3r\bar{u}^2,\\ a_2 &= \delta\bar{u}_x,\\ a_3 &= \mu + \beta\bar{u}. \end{aligned}$$

Recall that L is expressed by (59). Define

$$Y_1 := \left\{ f \in L^2\big(0,T;L^2(0,1)\big) \,/\, (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} f \in L^2\big(0,T;L^2(0,1)\big)\right\}, \tag{82}$$

equipped with the clear corresponding norm. We consider the following map:

$$\Lambda: \quad \begin{cases} E_1 \to H_0^2(0,1)\times Y_1 \\ y \mapsto \big(y(0), Ly + \beta y y_{xxx} + \delta y_x y_{xx} + (2q+6r\bar{u}) y y_x + 3r\bar{u} y^2 + 3r y^2 y_x\big) \end{cases}. \tag{83}$$

Recall that the definition of E_1 was given in (65). Note that the mapping Λ is well defined and C^1. Indeed, from $y \in E_1$, we find out that

$$\beta(T-t)^{5/2} e^{\frac{2\kappa_1}{(T-t)^{1/4}}} y y_{xxx} \in L^2\big(0,T;L^2(0,1)\big).$$

Then, thanks to (46) and (53), we have that

$$\beta(T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} y y_{xxx} \in L^2\big(0,T;L^2(0,1)\big).$$

The same can be done for all the other terms (since they are bilinear or trilinear). Now using Proposition 10, we see that $\Lambda'(0)$ is a surjective map. Hence there exists a neighborhood of $(0,0)$ in $H_0^2(0,1)\times Y_1$ on which Λ is onto. This gives the desired result.

5. Proof of Proposition 7

Let $\psi := e^{-s\alpha}\varphi$, where α is given by (42) and φ fulfills system (41). We deduce that

$$L_1\psi + L_2\psi = L_3\psi,$$

with

$$L_1\psi = \psi_t + \psi_{5x} + 10s^2\alpha_x^2\psi_{xxx} + 5s^4\alpha_x^4\psi_x, \tag{84}$$

$$L_2\psi = 5s\alpha_x\psi_{4x} + 10s^3\alpha_x^3\psi_{xx} + s^5\alpha_x^5\psi + s\alpha_t\psi + 10s\alpha_{xx}\psi_{xxx} + 30s^3\alpha_x^2\alpha_{xx}\psi_x, \tag{85}$$

and

$$\begin{aligned} L_3\psi = {} & -e^{-s\alpha}\Big\{\big[e^{s\alpha}\big(3s\alpha_{xx}\psi_x + 3s^2\alpha_x\alpha_{xx}\psi\big)\big]_{xx} + \big[e^{s\alpha}\big(3s\alpha_{xx}\psi_{xx} + 6s^2\alpha_x\alpha_{xx}\psi_x\big)\big]_x\Big\}\\ & - \big\{-6s\alpha_{xx}\psi_{xxx} + 12s^2\alpha_x\alpha_{xx}\psi_{xx} - 15s^3\alpha_x^2\alpha_{xx}\psi_x + 7s^4\alpha_x^3\alpha_{xx}\psi + 6s^3\alpha_x\alpha_{xx}^2\psi\big\}\\ & + e^{-s\alpha}\left\{ f + \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k\big(a_k(t,x)e^{s\alpha}\psi\big)\right\}. \end{aligned} \tag{86}$$

(We recall that $\alpha_{xxx}=0$.) Then, we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2\iint_Q L_1\psi\, L_2\psi\,dx\,dt = \|L_3\psi\|^2_{L^2(Q)}. \tag{87}$$

The main part of what follows consists in evaluating the double product term. We will denote by $(L_i\psi)_j$ ($1\leqslant i\leqslant 4$, $1\leqslant j\leqslant 6$) the jth term in the expression of $L_i\psi$. We recall that $\alpha>0$, $\alpha_x<0$, $\alpha_{xx}<0$ and $\alpha_{xxx}=0$. In the sequel we will repeatedly use that $\psi_{|x=0,1} = \psi_{x|x=0,1} = 0$.

• First, integrating by parts with respect to x and t, we have

$$\begin{aligned}\big((L_1\psi)_1,(L_2\psi)_1\big)_{L^2(Q)} &= -5s\iint_Q \alpha_x\psi_{tx}\psi_{xxx}\,dt\,dx - 5s\iint_Q \alpha_{xx}\psi_t\psi_{xxx}\,dt\,dx\\ &= \frac{5}{2}s\iint_Q \alpha_x\big(|\psi_{xx}|^2\big)_t\,dt\,dx + 10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx + 5s\iint_Q \alpha_{xxx}\psi_t\psi_{xx}\,dt\,dx\\ &\geqslant -CsT\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx + 10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx. \qquad (88)\end{aligned}$$

For the second term, we get

$$\begin{aligned}\big((L_1\psi)_1,(L_2\psi)_2\big)_{L^2(Q)} &= -5s^3\iint_Q \alpha_x^3\big(|\psi_x|^2\big)_t\,dt\,dx - 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_t\psi_x\,dt\,dx\\ &\geqslant -Cs^3T\iint_Q \alpha^7|\psi_x|^2\,dt\,dx - 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_t\psi_x\,dt\,dx. \qquad (89)\end{aligned}$$

For the third term, we obtain

$$\begin{aligned}\big((L_1\psi)_1,(L_2\psi)_3\big)_{L^2(Q)} &= \frac{1}{2}s^5\iint_Q \alpha_x^5\big(\psi^2\big)_t\,dt\,dx\\ &\geqslant -Cs^5T\iint_Q \alpha^9\psi^2\,dt\,dx. \qquad (90)\end{aligned}$$

We consider now the fourth term of $L_2\psi$ and using (44) we readily get

$$\begin{aligned}\big((L_1\psi)_1,(L_2\psi)_4\big)_{L^2(Q)} &= \frac{s}{2}\iint_Q \alpha_t\big(\psi^2\big)_t\,dt\,dx\\ &\geqslant -CsT^2\iint_Q \alpha^9\psi^2\,dt\,dx. \qquad (91)\end{aligned}$$

The next term gives

$$\big((L_1\psi)_1,(L_2\psi)_5\big)_{L^2(Q)} = -10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx. \qquad (92)$$

The last term gives

$$\big((L_1\psi)_1,(L_2\psi)_6\big)_{L^2(Q)} = 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_x\psi_t\,dt\,dx. \qquad (93)$$

All these computations ((88)–(93)) show that

$$\begin{aligned}\big((L_1\psi)_1,(L_2\psi)\big)_{L^2(Q)} &\geqslant -CsT\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - Cs^3T\iint_Q \alpha^7|\psi_x|^2\,dt\,dx\\ &\quad - C\big(s^5T + sT^2\big)\iint_Q \alpha^9\psi^2\,dt\,dx\\ &\geqslant -\epsilon s^5\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - \epsilon s^7\iint_Q \alpha^7|\psi_x|^2\,dt\,dx - \epsilon s^9\iint_Q \alpha^9\psi^2\,dt\,dx, \qquad (94)\end{aligned}$$

for any $\epsilon > 0$, provided that $s \geqslant CT^{1/4}$, where C depends on ϵ.

• Now we consider the second term of L_1. The product with the first term of L_2 gives

$$\big((L_1\psi)_2,(L_2\psi)_1\big)_{L^2(Q)} = -\frac{5}{2}s\iint\limits_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + \frac{5}{2}s\int\limits_0^T \alpha_{x|x=1}|\psi_{4x|x=1}|^2\,dt - \frac{5}{2}s\int\limits_0^T \alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt. \tag{95}$$

Similar computations give the following for the second term:

$$\begin{aligned}\big((L_1\psi)_2,(L_2\psi)_2\big)_{L^2(Q)} &= -5s^3\iint\limits_Q \alpha_x^3\big(|\psi_{xxx}|^2\big)_x\,dt\,dx - 10s^3\int\limits_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt\\ &\quad - 30s^3\iint\limits_Q \alpha_x^2\alpha_{xx}\psi_{4x}\psi_{xx}\,dt\,dx\\ &\geqslant 45s^3\iint\limits_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx - 5s^3\int\limits_0^T \alpha^3_{x|x=1}|\psi_{xxx|x=1}|^2\,dt\\ &\quad + 5s^3\int\limits_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt - 10s^3\int\limits_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt\\ &\quad + 30s^3\int\limits_0^T \alpha^2_{x|x=0}\alpha_{xx|x=0}\psi_{xxx|x=0}\psi_{xx|x=0}\,dt\\ &\quad - Cs^3\iint\limits_Q \alpha^3|\psi_{xxx}||\psi_{xx}|\,dt\,dx.\end{aligned}\tag{96}$$

For the third one we have

$$\begin{aligned}\big((L_1\psi)_2,(L_2\psi)_3\big)_{L^2(Q)} &= -s^5\iint\limits_Q \alpha_x^5\psi_{4x}\psi_x\,dt\,dx - 5s^5\iint\limits_Q \alpha_x^4\alpha_{xx}\psi_{4x}\psi\,dt\,dx\\ &= \frac{s^5}{2}\iint\limits_Q \alpha_x^5\big(|\psi_{xx}|^2\big)_x\,dt\,dx + 10s^5\iint\limits_Q \alpha_x^4\alpha_{xx}\psi_{xxx}\psi_x\,dt\,dx\\ &\quad + 5s^5\iint\limits_Q \big(\alpha_x^4\alpha_{xx}\big)_x\psi_{xxx}\psi\,dt\,dx\\ &\geqslant -\frac{s^5}{2}\int\limits_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt - \frac{25}{2}s^5\iint\limits_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx\\ &\quad - Cs^5\iint\limits_Q \alpha^5\big(|\psi_{xxx}||\psi| + |\psi_{xx}||\psi_x|\big)\,dt\,dx.\end{aligned}\tag{97}$$

Then, we see that

$$\big((L_1\psi)_2,(L_2\psi)_4\big)_{L^2(Q)} = -s\iint\limits_Q \alpha_t\psi_{4x}\psi_x\,dt\,dx - s\iint\limits_Q \alpha_{tx}\psi_{4x}\psi\,dt\,dx$$

$$
\begin{aligned}
&= \frac{s}{2}\iint\limits_Q \alpha_t\big(|\psi_{xx}|^2\big)_x\,dt\,dx + 2s\iint\limits_Q \alpha_{tx}\psi_{3x}\psi_x\,dt\,dx + s\iint\limits_Q \alpha_{txx}\psi_{xxx}\psi\,dt\,dx\\
&\geqslant -CsT\iint\limits_Q \alpha^5\big(|\psi_{xx}|^2 + |\psi_{xxx}|\big(|\psi| + |\psi_x|\big)\big)\,dt\,dx\\
&\quad - CsT\int_0^T \alpha^5_{|x=0}|\psi_{xx|x=0}|^2\,dt.
\end{aligned}
\tag{98}
$$

Next,

$$
\begin{aligned}
\big((L_1\psi)_2,(L_2\psi)_5\big)_{L^2(Q)} &= -10s\iint\limits_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + 10s\int_0^T \alpha_{xx|x=1}\psi_{4x|x=1}\psi_{xxx|x=1}\,dt\\
&\quad - 10s\int_0^T \alpha_{xx|x=0}\psi_{4x|x=0}\psi_{xxx|x=0}\,dt.
\end{aligned}
\tag{99}
$$

We used that $\alpha_{xxx} = 0$.

Finally,

$$
\begin{aligned}
\big((L_1\psi)_2,(L_2\psi)_6\big)_{L^2(Q)} &= -30s^3\iint\limits_Q \alpha_x^2\alpha_{xx}\psi_{4x}\psi_{xx}\,dt\,dx - 30s^3\iint\limits_Q \big(\alpha_x^2\alpha_{xx}\big)_x\psi_{4x}\psi_x\,dt\,dx\\
&= 30s^3\iint\limits_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx + 30s^3\int_0^T \big(\alpha_x^2\alpha_{xx}\big)_{|x=0}\psi_{xxx|x=0}\psi_{xx|x=0}\,dt\\
&\quad - Cs^3\iint\limits_Q \alpha^3\big(|\psi_{4x}||\psi_x| + |\psi_{xxx}||\psi_{xx}|\big)\,dt\,dx.
\end{aligned}
\tag{100}
$$

Putting together all the computations concerning the second term of $L_1\psi$ ((95)–(100)), we obtain

$$
\begin{aligned}
\big((L_1\psi)_2, L_2\psi\big)_{L^2(Q)} &\geqslant -\frac{25}{2}s\iint\limits_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + 75s^3\iint\limits_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx\\
&\quad - \frac{25}{2}s^5\iint\limits_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - \frac{5}{2}s\int_0^T \alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt\\
&\quad - \frac{s^5}{2}\int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt - 10s^3\int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt\\
&\quad + 5s^3\int_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt - \epsilon s^9\iint\limits_Q \alpha^9\psi^2\,dt\,dx\\
&\quad - \epsilon s^7\iint\limits_Q \alpha^7|\psi_x|^2\,dt\,dx - \epsilon s^5\iint\limits_Q \alpha^5|\psi_{xx}|^2\,dt\,dx\\
&\quad - \epsilon s^3\iint\limits_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx - \epsilon s\iint\limits_Q \alpha|\psi_{4x}|^2\,dt\,dx
\end{aligned}
$$

$$- \epsilon s^5 \int_0^T \alpha^5_{|x=0} |\psi_{xx|x=0}|^2 \, dt - \epsilon s^3 \int_0^T \alpha^3_{|x=0} |\psi_{xxx|x=0}|^2 \, dt$$
$$- \epsilon s \int_0^T \alpha_{|x=0} |\psi_{4x|x=0}|^2 \, dt - Cs \int_0^T \alpha_{|x=1} |\psi_{4x|x=1}|^2 \, dt$$
$$- Cs^3 \int_0^T \alpha^3_{|x=1} |\psi_{xxx|x=1}|^2 \, dt, \tag{101}$$

for any $\epsilon > 0$, provided that $s \geqslant C(T^{1/4} + T^{1/2})$, where C depends on ϵ. (We used that $s \geqslant C(\epsilon)T^{1/2}$ for appropriate $C(\epsilon)$ and $\alpha \leqslant CT\alpha^3$.)

• We consider now the products concerning the third term of $L_1\psi$. First, we have

$$\big((L_1\psi)_3, (L_2\psi)_1\big)_{L^2(Q)} = -75s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xxx}|^2 \, dt \, dx + 25s^3 \int_0^T \alpha^3_{x|x=1} |\psi_{xxx|x=1}|^2 \, dt$$
$$- 25s^3 \int_0^T \alpha^3_{x|x=0} |\psi_{xxx|x=0}|^2 \, dt. \tag{102}$$

Secondly

$$\big((L_1\psi)_3, (L_2\psi)_2\big)_{L^2(Q)} = -250s^5 \iint_Q \alpha_x^4 \alpha_{xx} |\psi_{xx}|^2 \, dt \, dx - 50s^5 \int_0^T \alpha^5_{x|x=0} |\psi_{xx|x=0}|^2 \, dt. \tag{103}$$

Third,

$$\big((L_1\psi)_3, (L_2\psi)_3\big)_{L^2(Q)} = -5s^7 \iint_Q \alpha_x^7 \big(|\psi_x|^2\big)_x \, dt \, dx - 70s^7 \iint_Q \alpha_x^6 \alpha_{xx} \psi_{xx} \psi \, dt \, dx$$
$$\geqslant 105s^7 \iint_Q \alpha_x^6 \alpha_{xx} |\psi_x|^2 \, dt \, dx - Cs^7 \iint_Q \alpha^7 |\psi_x| |\psi| \, dt \, dx. \tag{104}$$

For the fourth term, we have

$$\big((L_1\psi)_3, (L_2\psi)_4\big)_{L^2(Q)} = -10s^3 \iint_Q \alpha_x^2 \alpha_t \psi_{xx} \psi_x \, dt \, dx - 10s^3 \iint_Q \big(\alpha_x^2 \alpha_t\big)_x \psi_{xx} \psi \, dt \, dx$$
$$\geqslant -Cs^3 T \iint_Q \alpha^7 \big(|\psi_x|^2 + |\psi| |\psi_{xx}|\big) \, dt \, dx. \tag{105}$$

We obtain the following for the fifth term:

$$\big((L_1\psi)_3, (L_2\psi)_5\big)_{L^2(Q)} = 100s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xxx}|^2 \, dt \, dx. \tag{106}$$

Finally,

$$\big((L_1\psi)_3, (L_2\psi)_6\big)_{L^2(Q)} = -300s^5 \iint_Q \alpha_x^4 \alpha_{xx} |\psi_{xx}|^2 \, dx \, dt - Cs^5 \iint_Q \alpha^5 |\psi_{xx}| |\psi_x| \, dt \, dx. \tag{107}$$

Consequently, we get the following for the third term of $L_1\psi$ ((102)–(107)):

$$\begin{aligned}\big((L_1\psi)_3, L_2\psi\big)_{L^2(Q)} \geqslant{} & 25s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx - 550s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx \\ & + 105s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx - 25s^3 \int_0^T \alpha_{x|x=0}^3|\psi_{xxx|x=0}|^2\,dt \\ & - 50s^5 \int_0^T \alpha_{x|x=0}^5|\psi_{xx|x=0}|^2\,dt - \epsilon s^9 \iint_Q \alpha^9\psi^2\,dt\,dx \\ & - \epsilon s^7 \iint_Q \alpha^7|\psi_x|^2\,dt\,dx - \epsilon s^5 \iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx \\ & - Cs^3 \int_0^T \alpha_{|x=1}^3|\psi_{xxx|x=1}|^2\,dt, \end{aligned} \tag{108}$$

for any $\epsilon > 0$, where again $s \geqslant C(T^{1/4} + T^{1/2})$ and C depends on ϵ.

• Now, we compute the fourth term. First, we have

$$\begin{aligned}\big((L_1\psi)_4, (L_2\psi)_1\big)_{L^2(Q)} ={} & -\frac{25}{2}s^5 \iint_Q \alpha_x^5\big(|\psi_{xx}|^2\big)_x\,dx\,dt - 125s^5 \iint_Q \alpha_x^4\alpha_{xx}\psi_{xxx}\psi_x\,dx\,dt \\ \geqslant{} & \frac{375}{2}s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx + \frac{25}{2}s^5 \int_0^T \alpha_{x|x=0}^5|\psi_{xx|x=0}|^2\,dt \\ & - Cs^5 \iint_Q \alpha^5|\psi_x||\psi_{xx}|\,dx\,dt. \end{aligned} \tag{109}$$

Next, we obtain

$$\big((L_1\psi)_4, (L_2\psi)_2\big)_{L^2(Q)} = -175s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx. \tag{110}$$

For the third term, we get

$$\big((L_1\psi)_4, (L_2\psi)_3\big)_{L^2(Q)} = -\frac{45}{2}s^9 \iint_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx. \tag{111}$$

Then,

$$\big((L_1\psi)_4, (L_2\psi)_4\big)_{L^2(Q)} \geqslant -Cs^5T \iint_Q \alpha^9|\psi|^2\,dt\,dx. \tag{112}$$

The fifth term gives

$$\big((L_1\psi)_4, (L_2\psi)_5\big)_{L^2(Q)} = -50s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - Cs^5 \iint_Q \alpha^5|\psi_{xx}||\psi_x|\,dt\,dx. \tag{113}$$

Direct computations for the last term provide

$$\big((L_1\psi)_4, (L_2\psi)_6\big)_{L^2(Q)} = 150s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx. \tag{114}$$

All these computations ((109)–(114)) give

$$\begin{aligned}\big((L_1\psi)_4, L_2\psi\big)_{L^2(Q)} \geqslant{}& \frac{275}{2}s^5 \iint\limits_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - 25s^7 \iint\limits_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx\\ &- \frac{45}{2}s^9 \iint\limits_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx + \frac{25}{2}s^5 \int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt\\ &- \epsilon s^9 \iint\limits_Q \alpha^9\psi^2\,dt\,dx - \epsilon s^7 \iint\limits_Q \alpha^7|\psi_x|^2\,dt\,dx\\ &- \epsilon s^5 \iint\limits_Q \alpha^5|\psi_{xx}|^2\,dt\,dx, \end{aligned} \tag{115}$$

for any $\epsilon > 0$, where again $s \geqslant C(T^{1/4} + T^{1/2})$ and C depends on ϵ.

Let us now gather all the product $(L_1\psi, L_2\psi)_{L^2(Q)}$ coming from (94), (101), (108) and (115):

$$\begin{aligned}(L_1\psi, L_2\psi)_{L^2(Q)} \geqslant{}& -\frac{25}{2}s \iint\limits_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + 100s^3 \iint\limits_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx\\ &- 425s^5 \iint\limits_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx + 80s^7 \iint\limits_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx\\ &- \frac{45}{2}s^9 \iint\limits_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx - \frac{5}{2}s \int_0^T \alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt\\ &- 20s^3 \int_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt - 38s^5 \int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt\\ &- 10s^3 \int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt\\ &- \epsilon s^9 \iint\limits_Q \alpha^9\psi^2\,dt\,dx - \epsilon s^7 \iint\limits_Q \alpha^7|\psi_x|^2\,dt\,dx\\ &- \epsilon s^5 \iint\limits_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - \epsilon s^3 \iint\limits_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx\\ &- \epsilon s \iint\limits_Q \alpha|\psi_{4x}|^2\,dt\,dx - \epsilon s^5 \int_0^T \alpha^5_{|x=0}|\psi_{xx|x=0}|^2\,dt\\ &- \epsilon s^3 \int_0^T \alpha^3_{|x=0}|\psi_{xxx|x=0}|^2\,dt - \epsilon s \int_0^T \alpha_{|x=0}|\psi_{4x|x=0}|^2\,dt\\ &- Cs \int_0^T \alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt - Cs^3 \int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt, \end{aligned} \tag{116}$$

for $s \geqslant C(T^{1/4} + T^{1/2})$.

Let us explain how we handle the wrongly signed terms in $|\psi_{xxx}|^2$ and $|\psi_x|^2$. After integration by parts, we get

$$100s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xxx}|^2 \, dt \, dx \geqslant -100s^3 \iint_Q \alpha_x^2 \alpha_{xx} \psi_{xx} \psi_{4x} \, dt \, dx - \epsilon s^5 \iint_Q \alpha^5 |\psi_{xx}|^2 \, dt \, dx - \epsilon s^3 \int_0^T \alpha^3_{|x=0} |\psi_{xxx|x=0}|^2 \, dt - \epsilon s^5 \int_0^T \alpha^5_{|x=0} |\psi_{xx|x=0}|^2 \, dt, \tag{117}$$

by taking $s \geqslant CT^{1/2}$. The last two terms in the right-hand side are already in (116), while the first one is estimated as follows, by using Cauchy–Schwarz's inequality:

$$100s^3 \left| \iint_Q \alpha_x^2 |\alpha_{xx}| \psi_{xx} \psi_{4x} \, dt \, dx \right| \leqslant 12s \iint_Q \alpha_{xx} |\psi_{4x}|^2 \, dt \, dx + \frac{625}{3} s^5 \iint_Q \alpha_x^4 |\alpha_{xx}| |\psi_{xx}|^2 \, dt \, dx. \tag{118}$$

On the other hand, by integration by parts and Cauchy–Schwarz's inequality, we have for $s \geqslant CT^{1/2}$:

$$80s^7 \left| \iint_Q \alpha_x^6 \alpha_{xx} |\psi_x|^2 \, dt \, dx \right| \leqslant (22+\epsilon) s^9 \iint_Q \alpha_x^8 |\alpha_{xx}| |\psi|^2 \, dt \, dx + \frac{800}{11} s^5 \iint_Q \alpha_x^4 |\alpha_{xx}| |\psi_{xx}|^2 \, dt \, dx. \tag{119}$$

Observe that

$$\frac{625}{3} + \frac{800}{11} < 425. \tag{120}$$

Finally, we have

$$10s^3 \left| \int_0^T \alpha^3_{x|x=0} \psi_{4x|x=0} \psi_{xx|x=0} \, dt \right| \leqslant 2s \int_0^T |\alpha_{x|x=0}| |\psi_{4x|x=0}|^2 \, dt + \frac{25}{2} s^5 \int_0^T |\alpha_{x|x=0}|^5 |\psi_{xx|x=0}|^2 \, dt. \tag{121}$$

Now we observe that thanks to (43) we can absorb all the "ϵ terms" in (116) provided that $s \geqslant CT^{1/2}$. Finally, using again (43) we deduce from (87) the following inequality for ψ:

$$s \iint_Q \alpha |\psi_{4x}|^2 \, dt \, dx + s^3 \iint_Q \alpha^3 |\psi_{xxx}|^2 \, dt \, dx + s^5 \iint_Q \alpha^5 |\psi_{xx}|^2 \, dt \, dx + s^7 \iint_Q \alpha^7 |\psi_x|^2 \, dt \, dx + s^9 \iint_Q \alpha^9 |\psi|^2 \, dt \, dx \leqslant C \left(\|L_3 \psi\|^2_{L^2(Q)} + s \int_0^T \alpha_{|x=1} |\psi_{4x|x=1}|^2 \, dt + s^3 \int_0^T \alpha^3_{|x=1} |\psi_{xxx|x=1}|^2 \, dt \right). \tag{122}$$

Now it is not difficult to see that all the terms in $L_3 \psi$ yield an L^2-norm estimated by

$$\|L_3 \psi\|^2_{L^2(Q)} \leqslant C \left(s^2 \iint_Q \alpha^2 |\psi_{xxx}|^2 \, dt \, dx + s^4 \iint_Q \alpha^4 |\psi_{xx}|^2 \, dt \, dx + s^6 \iint_Q \alpha^6 |\psi_x|^2 \, dt \, dx + s^8 \iint_Q \alpha^8 |\psi|^2 \, dt \, dx + \iint_Q e^{-2s\alpha} |f|^2 \, dt \, dx \right), \tag{123}$$

for $s \geqslant CT^{1/2}$. Here we have used that $a_k \in L^\infty(0,T; W^{k,\infty}(0,1))$. Hence they can be absorbed by the left-hand side of (122) provided that $s \geqslant CT^{1/2}$. We deduce the Carleman inequality for ψ

$$s\iint_Q \alpha|\psi_{4x}|^2\,dt\,dx + s^3\iint_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx + s^5\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx$$
$$+ s^7\iint_Q \alpha^7|\psi_x|^2\,dt\,dx + s^9\iint_Q \alpha^9|\psi|^2\,dt\,dx$$
$$\leqslant C\left(s\int_0^T \alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt + s^3\int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt + \iint_Q e^{-2s\alpha}|f|^2\,dt\,dx\right). \tag{124}$$

It remains to replace ψ by φ, to use (43) and $s \geqslant CT^{1/2}$ in order to deduce (45).

6. Proof of Lemma 1

We first establish two lemmas before turning to the core of the proof.

Lemma 2. *Let p satisfy*

$$\begin{cases} p_t + \alpha p_{5x} = g & \text{in } (0,T)\times(0,2),\\ p_{|x=0} = p_{x|x=0} = p_{xx|x=0} = 0 & \text{in } (0,T),\\ p_{|x=2} = \hat{v}_2, \quad p_{x|x=2} = \hat{v}_4 & \text{in } (0,T),\\ p_{|t=0} = p_0 & \text{in } (0,2). \end{cases} \tag{125}$$

Then for $k \geqslant 2$, one has

$$\left\|(2-x)^{k+\frac12}p\right\|_{L^\infty(0,T;L^2(0,2))} + \left\|(2-x)^k p_{xx}\right\|_{L^2(0,T;L^2(0,2))} + \left\|(2-x)^{k-1}p_x\right\|_{L^2(0,T;L^2(0,2))}$$
$$\lesssim \left\|(2-x)^{k+1}g\right\|_{L^2(0,T;H^{-2}(0,2))} + \left\|(2-x)^{k-2}p\right\|_{L^2(0,T;L^2(0,2))} + \left\|(2-x)^{k+\frac12}p_0\right\|_{L^2(0,2)}. \tag{126}$$

Proof. As previously, we multiply by $(2-x)^{2k+1}p$; we get

$$\frac12\frac{d}{dt}\int_0^2 (2-x)^{2k+1}|p|^2\,dx + 5\frac{2k+1}{2}\alpha\int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx$$
$$= \int_0^2 (2-x)^{2k+1}pg\,dx + 6k(2k+1)\alpha\int_0^2 (2-x)^{2k-1}p_x p_{xx}\,dx$$
$$- 2k(2k+1)(2k-1)\alpha\int_0^2 (2-x)^{2k-2}pp_{xx}\,dx.$$

We utilize Young's inequality:

$$\left|\int_0^2 (2-x)^{2k-2}pp_{xx}\,dx\right| \leqslant \epsilon\int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx + \frac{1}{\epsilon}\int_0^2 (2-x)^{2k-4}|p|^2\,dx,$$
$$\left|\int_0^2 (2-x)^{2k-1}p_x p_{xx}\,dx\right| \leqslant \epsilon\int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx + \frac{1}{\epsilon}\int_0^2 (2-x)^{2k-2}|p_x|^2\,dx.$$

Integrate by parts in the last term, we deduce (126). □

Lemma 3. *Let p satisfy* (125). *Then for $k \geqslant 7$, one has*

$$\begin{aligned}&\big\|(2-x)^{k+\frac{1}{2}} p\big\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4}\big\|(2-x)^{k+j-4}\partial_x^j p\big\|_{L^2(0,T;L^2(0,2))}\\&\quad\lesssim \|g\|_{L^2(0,T;L^2(0,2))} + \|p_0\|_{H^2(0,2)} + \|\hat{v}_2\|_{L^2(0,T)} + \|\hat{v}_4\|_{L^2(0,T)}.\end{aligned} \tag{127}$$

Proof. First step. *Higher-order estimates.* Let $g \in L^2(0,T;H_0^3(0,2))$. We apply Lemma 2 to p_{5x} (which satisfies the boundary conditions), and get

$$\begin{aligned}&\big\|(2-x)^{k+\frac{1}{2}} p_{5x}\big\|_{L^\infty(0,T;L^2(0,2))} + \big\|(2-x)^{k} p_{5x}\big\|_{L^2(0,T;H^2(0,2))}\\&\quad\lesssim \big\|(2-x)^{k+1} g_{5x}\big\|_{L^2(0,T;H^{-2}(0,2))} + \big\|(2-x)^{k-2} p_{5x}\big\|_{L^2(0,T;L^2(0,2))} + \big\|(2-x)^{k+\frac{1}{2}} p_{0,5x}\big\|_{L^2(0,2)}.\end{aligned} \tag{128}$$

By an integration by parts, this inequality yields

$$\begin{aligned}&\big\|(2-x)^{k+\frac{1}{2}} p_{5x}\big\|_{L^\infty(0,T;L^2(0,2))} + \big\|(2-x)^{k} p_{7x}\big\|_{L^2(0,T;L^2(0,2))} + \big\|(2-x)^{k-1} p_{6x}\big\|_{L^2(0,T;L^2(0,2))}\\&\quad\lesssim \big\|(2-x)^{k+1} g_{5x}\big\|_{L^2(0,T;H^{-2}(0,2))} + \big\|(2-x)^{k-2} p_{5x}\big\|_{L^2(0,T;L^2(0,2))} + \big\|(2-x)^{k+\frac{1}{2}} p_{0,5x}\big\|_{L^2(0,2)}.\end{aligned} \tag{129}$$

Now in order to estimate the term concerning p_{5x} in the right-hand side, we observe that

$$\begin{aligned}&\int_0^T\int_0^2 (2-x)^{2k-4} p_{5x} p_{5x}\,dx\,dt\\&\quad= -\int_0^T\int_0^2 (2-x)^{2k-4} p_{6x} p_{4x}\,dx\,dt + (k-2)(2k-5)\int_0^T\int_0^2 (2-x)^{2k-6} |p_{4x}|^2\,dx\,dt,\end{aligned} \tag{130}$$

$$\begin{aligned}&\int_0^T\int_0^2 (2-x)^{2k-2} p_{6x} p_{6x}\,dx\,dt\\&\quad= -\int_0^T\int_0^2 (2-x)^{2k-2} p_{5x} p_{7x}\,dx\,dt + (k-1)(2k-3)\int_0^T\int_0^2 (2-x)^{2k-4} p_{5x}^2\,dx\,dt.\end{aligned} \tag{131}$$

The identity (131) may be used to estimate the first integral in the right-hand side of (130) (with $ab \leqslant \epsilon a^2 + b^2/\epsilon$). Now injecting in (129), we obtain

$$\begin{aligned}&\big\|(2-x)^{k+\frac{1}{2}} p_{5x}\big\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{2}\big\|(2-x)^{k-j}\partial_x^{7-j} p\big\|_{L^2(0,T;L^2(0,2))}\\&\quad\lesssim \big\|(2-x)^{k+1} g_{5x}\big\|_{L^2(0,T;H^{-2}(0,2))} + \big\|(2-x)^{k-3} p_{4x}\big\|_{L^2(0,T;L^2(0,2))} + \big\|(2-x)^{k+\frac{1}{2}} p_{0,5x}\big\|_{L^2(0,2)}.\end{aligned} \tag{132}$$

Now to absorb the term concerning p_{4x} in the right-hand side, we operate in the same way, but here a boundary term appears:

$$\int_0^T\int_0^2 (2-x)^{2k-6} p_{4x} p_{4x}\,dx\,dt$$
$$= -\int_0^T\int_0^2 (2-x)^{2k-6} p_{5x} p_{3x}\,dx\,dt + (k-3)(2k-7)\int_0^T\int_0^2 (2-x)^{2k-8}|p_{3x}|^2\,dx\,dt$$
$$+ 2^{2k-6}\int_0^T p_{3x|x=0} p_{4x|x=0}\,dt. \tag{133}$$

This latter term is treated as follows:

$$\left|\int_0^T p_{3x|x=0} p_{4x|x=0}\,dt\right| \leqslant \epsilon \int_0^T |p_{4x|x=0}|^2\,dt + \frac{1}{\epsilon}\int_0^T |p_{3x|x=0}|^2\,dt.$$

Now

$$\int_0^T |p_{4x|x=0}|^2\,dt = -\frac{1}{2^{2k-6}}\int_0^T\int_0^2 (2-x)^{2k-5} p_{4x} p_{5x}\,dt\,dx - \frac{(2k-5)}{2^{2k-5}}\int_0^T\int_0^2 (2-x)^{2k-6}|p_{4x}|^2\,dt\,dx,$$

which can be treated as above, while

$$\int_0^T |p_{3x|x=0}|^2\,dt = -\frac{1}{2^{2k-8}}\int_0^T\int_0^2 (2-x)^{2k-7} p_{3x} p_{4x}\,dt\,dx - \frac{(2k-7)}{2^{2k-7}}\int_0^T\int_0^2 (2-x)^{2k-8}|p_{3x}|^2\,dt\,dx,$$

which leads us to

$$\left\|(2-x)^{k+\frac12} p_{5x}\right\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{3}\left\|(2-x)^{k-j}\partial_x^{7-j} p\right\|_{L^2(0,T;L^2(0,2))}$$
$$\lesssim \left\|(2-x)^{k+1} g_{5x}\right\|_{L^2(0,T;H^{-2}(0,2))} + \left\|(2-x)^{k-4} p_{3x}\right\|_{L^2(0,T;L^2(0,2))} + \left\|(2-x)^{k+\frac12} p_{0,5x}\right\|_{L^2(0,2)}. \tag{134}$$

Then one follows the same steps as previously (note that $p_{|x=0} = p_{x|x=0} = p_{xx|x=0} = 0$) and finally gets

$$\left\|(2-x)^{k+\frac12} p_{5x}\right\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{6}\left\|(2-x)^{k-7+j}\partial_x^{j} p\right\|_{L^2(0,T;L^2(0,2))}$$
$$\lesssim \left\|(2-x)^{k+1} g_{5x}\right\|_{L^2(0,T;H^{-2}(0,2))} + \left\|(2-x)^{k-7} p\right\|_{L^2(0,T;L^2(0,2))} + \left\|(2-x)^{k+\frac12} p_{0,5x}\right\|_{L^2(0,2)}, \tag{135}$$

and consequently, using Proposition 6,

$$\left\|(2-x)^{k+\frac12} p_{5x}\right\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{6}\left\|(2-x)^{k-7+j}\partial_x^{j} p\right\|_{L^2(0,T;L^2(0,2))}$$
$$\lesssim \|g\|_{L^2(0,T;H^3(0,2))} + \left\|(2-x)^{k+\frac12} p_{0,5x}\right\|_{L^2(0,2)} + \|\hat v_2\|_{L^2(0,T)} + \|\hat v_4\|_{L^2(0,T)}. \tag{136}$$

Second step. *Interpolation.* Now we consider the operator which maps $(p_0, g, \hat v_2, \hat v_4)$ to $(2-x)^{k+1}p$: it is continuous from

$$L^2(0,2)\times L^2\big(0,T;H^{-2}(0,2)\big)\times L^2(0,T)^2 \quad \text{to} \quad L^2\big(0,T;H^2(0,2)\big)\cap C^0\big([0,T];L^2(0,2)\big),$$

respectively

$$H_0^5(0,2)\times L^2\big(0,T;H_0^3(0,2)\big)\times L^2(0,T)^2 \quad \text{to} \quad L^2\big(0,T;H^7(0,2)\big)\cap C^0\big([0,T];H^5(0,2)\big).$$

By interpolation, it is hence continuous from

$$H_0^2(0,2) \times L^2\big(0,T;L^2(0,2)\big) \times L^2(0,T)^2 \quad \text{to} \quad L^2\big(0,T;H^4(0,2)\big) \cap C^0\big([0,T];H^2(0,2)\big).$$

This concludes the proof of Lemma 3. □

Proof of Lemma 1. We apply Lemma 3 with $p = y^*$ and $g = \sum_{j=0}^{3} \tilde{a}_j(t,x)\partial_x^j y^* + h^*$. We infer

$$\begin{aligned}
&\big\|(2-x)^{k+\frac{1}{2}} y^*\big\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4} \big\|(2-x)^{k+j-4}\partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))} \\
&\quad \lesssim \big\|h^*\big\|_{L^2(0,T;L^2(0,2))} + \sum_{j=0}^{3} \big\|\tilde{a}_j \partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))} + \big\|y_0^*\big\|_{H^2(0,2)} + \big\|v_2^*\big\|_{L^2(0,T)} + \big\|v_4^*\big\|_{L^2(0,T)}.
\end{aligned} \tag{137}$$

Now, using that the supports of $\tilde{a}_j$ are away from 2, we can estimate the terms $\sum_{j=0}^{3} \|\tilde{a}_j \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))}$ as follows

$$\sum_{j=0}^{3} \big\|\tilde{a}_j \partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))} \leqslant \epsilon \sum_{j=0}^{4} \big\|(2-x)^{k+j-4}\partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))} + C\big\|y^*\big\|_{L^2(0,T;L^2(0,2))},$$

exactly as in Lemma 3. We get

$$\begin{aligned}
&\big\|(2-x)^{k+\frac{1}{2}} y^*\big\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4} \big\|(2-x)^{k+j-4}\partial_x^j y^*\big\|_{L^2(0,T;L^2(0,2))} \\
&\quad \lesssim \big\|h^*\big\|_{L^2(0,T;L^2(0,2))} + \big\|y^*\big\|_{L^2(0,T;L^2(0,2))} + \big\|y_0^*\big\|_{H^2(0,2)} + \big\|v_2^*\big\|_{L^2(0,T)} + \big\|v_4^*\big\|_{L^2(0,T)}.
\end{aligned}$$

Using again Proposition 6, and thanks to (58), this gives (70), hence completing the argument. □

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