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Stability of solitary waves for a system of nonlinear Schrödinger equations with three wave interaction

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Abstract

In this paper we consider a three components system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. We study the orbital stability of scalar solutions of the form $(e^{2i\omega t}\varphi, 0, 0)$, $(0, e^{2i\omega t}\varphi, 0)$, $(0, 0, e^{2i\omega t}\varphi)$, where φ is a ground state of the scalar nonlinear Schrödinger equation.

Résumé

Dans cet article, on s'intéresse à un système de trois équations de Schrödinger qui modélise le phénomène d'amplification Raman dans les plasmas. On étudie la stabilité orbitale de solutions scalaires de la forme $(e^{2i\omega t}\varphi, 0, 0)$, $(0, e^{2i\omega t}\varphi, 0)$, $(0, 0, e^{2i\omega t}\varphi)$, où φ est l'état fondamental d'une équation de Schrödinger scalaire.

MSC: 35Q55; 35A15; 35B35

Keywords: Solitary wave; Orbital stability; NLS

1. Introduction

In this paper, we investigate the stability properties of solitary waves for a Schrödinger-type system related to the Raman amplification in a plasma. The study of laser-plasma interactions is an active area of interest. The main goal is to simulate nuclear fusion in a laboratory. In order to simulate numerically these experiments we need some accurate models. The kinetic ones are the most relevant but very difficult to deal with for practical computations. The fluids ones like bifluid Euler–Maxwell system seem more convenient but still inoperative in practice because of the high frequency motion and the small wavelength involved in the problem. This is why we need some intermediate models which are reliable from a numerical point of view. In [5], a new set of equations describing nonlinear interaction between a laser beam and a plasma has been derived. This model describes the Raman process which is a nonlinear instability

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phenomenon. The physical situation is the following. When an incident laser field enters a plasma, it is backscattered by a Raman type process. These two waves interact to create an electronic plasma wave. The three waves combine to create a variation of the density of the ions which has itself an influence on the three preceedings waves. The system describing this phenomenon is composed by three Schrödinger equations coupled to a wave equation and reads in a suitable dimensionless form:

$$\big(i(\partial_t + v_C\partial_y) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\big)A_C = \frac{b^2}{2}nA_C - \gamma(\nabla\cdot E)A_R e^{-i\theta}, \tag{1.1}$$

$$\big(i(\partial_t + v_R\partial_y) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\big)A_R = \frac{bc}{2}nA_R - \gamma(\nabla\cdot E^*)A_C e^{i\theta}, \tag{1.2}$$

$$(i\partial_t + \delta_1\Delta)E = \frac{b}{2}nE + \gamma\nabla\big(A_R^* A_C e^{i\theta}\big), \tag{1.3}$$

$$(\partial_t^2 - v_s^2\Delta)n = a\Delta\big(|E|^2 + b|A_C|^2 + c|A_R|^2\big), \tag{1.4}$$

where A_0, A_R and E are complex vectors and are respectively the incident laser field, the backscattered Raman field and the electronic plasma-wave whereas n is the variation of the density of the ions and $\theta = (k_1 y - \omega_1 t)$ where $\omega_1 = k_1^2\delta_1$. For a complete description of this model as well as a precise description of the physical coefficients, we refer to [5] and [6]. In [5], it is proved that system (1.1)–(1.4) is locally well-posed in suitable Sobolev spaces. It is then natural to investigate the global well-posedness theory. The existence of global solutions is still an open problem since no suitable conservation laws have been yet derived to handle this question. From this point of view, solitary waves play a crucial role and the study of their dynamics represents an important step toward the global well-posedness. The study of their behaviour is the main motivation of this paper. Unfortunately, we are not able to perform such analysis on (1.1)–(1.4). We will study a subsystem that includes the spacial dynamics and that is obtained as follows.

Writing $E = Fe^{i\theta}$ and taking $n = 0$, (1.1)–(1.4) reads

$$\big(i(\partial_t + v_C\partial_y) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\big)A_C = -\gamma\nabla\cdot FA_R - ik_1\gamma FA_R, \tag{1.5}$$

$$\big(i(\partial_t + v_R\partial_y) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\big)A_R = \gamma ik_1F^*A_C - \gamma\nabla\cdot F^*A_C, \tag{1.6}$$

$$\big(i\partial_t + \omega_1 + \delta_1\Delta + 2ik_1\partial_y - \delta_1k_1^2\big)F = ik_1\gamma A_R^*A_C + \gamma\nabla\big(A_R^*A_C\big). \tag{1.7}$$

Now, in the right-hand side, we neglect the ∇ terms in front of ik_1 (it is an envelope approximation). In the left-hand side, we neglect the longitudinal dispersion terms ∂_y^2 in front of the transverse ones $\Delta_\perp$. We also use the dispersion relation $\omega_1 = k_1^2\delta_1$ in (1.7) and we study some stationary version $\partial_t = 0$. The system reads

$$(iv_C\partial_y + \alpha_2\Delta_\perp)A_C = -\gamma ik_1FA_R,$$
$$(iv_R\partial_y + \beta_2\Delta_\perp)A_R = \gamma ik_1F^*A_C,$$
$$(2ik_1\partial_y + \delta_1\Delta_\perp)F = ik_1\gamma A_R^*A_C.$$

Let us introduce $w_1 = A_C$, $w_2 = A_R^*$ and $w_3 = F$, letting the coefficients to 1 (note that $v_R < 0$ and $v_C > 0$) leads to

$$(i\partial_y + \Delta_\perp)w_1 = -i\gamma w_3w_2^*,$$
$$(i\partial_y + \Delta_\perp)w_2 = -i\gamma w_3w_1^*,$$
$$(i\partial_y + \Delta_\perp)w_3 = i\gamma w_1w_2.$$

Taking $v_j = iw_j$ for $j = 1, 2, 3$ gives

$$(i\partial_y + \Delta_\perp)v_1 = -\gamma v_3v_2^*,$$
$$(i\partial_y + \Delta_\perp)v_2 = -\gamma v_3v_1^*,$$
$$(i\partial_y + \Delta_\perp)v_3 = -\gamma v_1v_2.$$

In order to model nonlinear effects, we add some nonlinear terms and we switch to the usual notation using t as evolution variable instead of y. The system that we consider in this paper is the following simplified system of nonlinear Schrödinger equations:

$$i\partial_t u_1 = -\Delta u_1 - |u_1|^{p-1}u_1 - \gamma u_3\overline{u_2}, \tag{1.8}$$
$$i\partial_t u_2 = -\Delta u_2 - |u_2|^{p-1}u_2 - \gamma u_3\overline{u_1}, \tag{1.9}$$
$$i\partial_t u_3 = -\Delta u_3 - |u_3|^{p-1}u_3 - \gamma u_1 u_2, \tag{1.10}$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, where $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\gamma > 0$, and u_1, u_2 and u_3 are complex valued functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. We assume furthermore that $\gamma > 0$. Indeed, the case $\gamma < 0$ is obtained by replacing u_3 by $-u_3$ in system (1.8)–(1.10). Note that the values of N corresponding to physical cases are $N = 1$ or 2.

Here and hereafter, we put $\vec{u} = (u_1, u_2, u_3)$. We introduce the following quantities

$$E(\vec{u}) = \sum_{j=1}^{3}\left(\frac{1}{2}\|\nabla u_j\|_{L^2}^2 - \frac{1}{p+1}\|u_j\|_{L^{p+1}}^{p+1}\right) - \gamma\Re\int_{\mathbb{R}^N} u_1 u_2 \overline{u_3}\,dx,$$

$$Q_1(\vec{u}) = \|u_1\|_{L^2}^2 + \|u_3\|_{L^2}^2 \quad \text{and} \quad Q_2(\vec{u}) = \|u_2\|_{L^2}^2 + \|u_3\|_{L^2}^2,$$

where $\Re(z)$ denotes the real part of a complex number z. Note that (1.8)–(1.10) can be written as

$$\partial_t \vec{u}(t) = -iE'\big(\vec{u}(t)\big),$$

and that

$$E\big(e^{i\theta_1}u_1, e^{i\theta_2}u_2, e^{i(\theta_1+\theta_2)}u_3\big) = E(\vec{u}), \qquad E\big(\vec{u}(\cdot + y)\big) = E(\vec{u}),$$

for any $(\theta_1, \theta_2) \in \mathbb{R}^2$, $y \in \mathbb{R}^N$ and $\vec{u} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$. The Cauchy problem for (1.8)–(1.10) is globally well-posed in $H^1(\mathbb{R}^N, \mathbb{C}^3)$ (see [2]).

Proposition 1. *Let $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\gamma > 0$, and $\vec{u}_0 \in H^1(\mathbb{R}^N, \mathbb{C}^3)$. Then there exists a unique global solution $\vec{u} \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^N, \mathbb{C}^3))$ to system (1.8)–(1.10) satisfying $\vec{u}(0) = \vec{u}_0$. Furthermore, the solution $\vec{u}(t)$ satisfies the conservation laws*

$$E\big(\vec{u}(t)\big) = E(\vec{u}_0), \tag{1.11}$$
$$Q_1\big(\vec{u}(t)\big) = Q_1(\vec{u}_0), \qquad Q_2\big(\vec{u}(t)\big) = Q_2(\vec{u}_0) \tag{1.12}$$

for all $t \in \mathbb{R}$.

For $\omega > 0$, let $\varphi \in H^1(\mathbb{R}^N)$ be a positive radial solution of

$$-\Delta v + 2\omega v - |v|^{p-1}v = 0, \quad x \in \mathbb{R}^N. \tag{1.13}$$

It is known that φ is unique (see [15]). Then, we see that

$$\big(e^{2i\omega t}\varphi, 0, 0\big), \qquad \big(0, e^{2i\omega t}\varphi, 0\big), \qquad \big(0, 0, e^{2i\omega t}\varphi\big) \tag{1.14}$$

solve (1.8)–(1.10). Note that $u_\omega(t) = e^{2i\omega t}\varphi$ is a standing wave solution of the single nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.15}$$

and it is well known that for $\omega > 0$, $u_\omega(t)$ is orbitally stable if $1 < p < 4/N$. When $1 + 4/N \leqslant p < 1 + 4/(N-2)$, $u_\omega(t)$ is strongly unstable in the sense that for any $\lambda > 1$ the solution $w_\lambda(t)$ of (1.15) with initial data $w_\lambda(0) = \lambda\varphi$ blows up in finite time (see [1,3,18,21] and also [2,20,12,13]).

The purpose in this paper is to study the stability properties of the solitary wave solutions (1.14) for the coupled system (1.8)–(1.10). We first introduce the following definition.

Definition. We say that the solitary wave solution $(e^{2i\omega t}\varphi, 0, 0)$ of (1.8)–(1.10) is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\vec{u}_0 \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ and $\|\vec{u}_0 - (\varphi, 0, 0)\|_{H^1} < \delta$, then the solution $\vec{u}(t)$ of (1.8)–(1.10) with $\vec{u}(0) = \vec{u}_0$ satisfies

$$\sup_{t\geqslant 0}\inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^N}\big\|\vec{u}(t) - \big(e^{i\theta}\varphi(\cdot + y), 0, 0\big)\big\|_{H^1} < \varepsilon.$$

Otherwise, $(e^{2i\omega t}\varphi, 0, 0)$ is called *orbitally unstable*. The orbital stability and instability of $(0, e^{2i\omega t}\varphi, 0)$ and $(0, 0, e^{2i\omega t}\varphi)$ are defined analogously.

Here, we remark that when $1 + 4/N \leqslant p < 1 + 4/(N-2)$, the solitary wave solution $(e^{2i\omega t}\varphi, 0, 0)$ of (1.8)–(1.10) is strongly unstable, since for any $\lambda > 1$, $(w_\lambda(t), 0, 0)$ is a solution of (1.8)–(1.10) and blows up in finite time, where $w_\lambda(t)$ is the blowup solution of (1.15) with $w_\lambda(0) = \lambda\varphi$. By the same reason, the solitary wave solutions $(0, e^{2i\omega t}\varphi, 0)$ and $(0, 0, e^{2i\omega t}\varphi)$ of (1.8)–(1.10) are also strongly unstable when $1 + 4/N \leqslant p < 1 + 4/(N-2)$. Thus, in what follows, we consider the case $1 < p < 1 + 4/N$ only. The main results of this paper are the following. The first one is concerned with the stability of the first two solitary waves $(e^{2i\omega t}\varphi, 0, 0)$ and $(0, e^{2i\omega t}\varphi, 0)$, and reads as follows.

Theorem 2. *Let $1 \leqslant N \leqslant 3$, $1 < p < 1 + 4/N$, $\gamma > 0$, $\omega > 0$, and let φ be the positive radial solution of* (1.13). *Then, the solitary wave solutions $(e^{2i\omega t}\varphi, 0, 0)$ and $(0, e^{2i\omega t}\varphi, 0)$ of* (1.8)–(1.10) *are orbitally stable.*

The proof of Theorem 2 is very classical and follows the argument introduced in [3], the key point being the variational characterization of the ground states $(e^{2i\omega t}\phi, 0, 0)$ and $(0, e^{2i\omega t}\phi, 0)$ given in Lemma 4. The second result deals with the third wave $(0, 0, e^{2i\omega t}\varphi)$. For this case, the analysis is much more delicate. Indeed, it is proved that there exists a critical value γ^* of γ such that the solitary wave is stable for $0 < \gamma < \gamma^*$, whereas the wave is unstable for $\gamma > \gamma^*$. We notice that we do not prove any result for $\gamma = \gamma^*$, and we leave this question as an open problem.

Theorem 3. *Let $1 \leqslant N \leqslant 3$, $1 < p < 1 + 4/N$, $\omega > 0$, and let φ be the positive radial solution of* (1.13). *Then, there exists a positive constant γ^* satisfying the following.*

(i) *If $0 < \gamma < \gamma^*$, then the solitary wave solution $(0, 0, e^{2i\omega t}\varphi)$ of* (1.8)–(1.10) *is orbitally stable.*
(ii) *Assume further that $N \leqslant 2$ and $p > 2$. If $\gamma > \gamma^*$, then the solitary wave solution $(0, 0, e^{2i\omega t}\varphi)$ of* (1.8)–(1.10) *is orbitally unstable.*

We give the outline of the proof. First of all, the variational method used to prove Theorem 2 does not apply to the case of Theorem 3(i), because the conservation laws (1.11)–(1.12) are not the suitable ones. To prove Theorem 3(i), we first introduce the action S (see (3.1)) associated with system (1.8)–(1.10), so that the third solitary wave $(0, 0, e^{2i\omega t}\varphi)$ is a critical point of S. Following the general theory developed by Grillakis, Shatah and Strauss in [12], the proof is based on a carefull study of the linearized operator S''. The key point is to show that S'' is an elliptic operator on $H^1(\mathbb{R}^N, \mathbb{C}^3)$ which is a sufficient criteria to obtain the stability of $(0, 0, e^{2i\omega t}\varphi)$. For that purpose, we decompose S'' into two parts $\mathcal{B}_1$ and $\mathcal{B}_2$ (see (3.2)) where $\mathcal{B}_1$ depends on v_1 and v_2 whereas $\mathcal{B}_2$ depends only on v_3. The operator $\mathcal{B}_2$ is elliptic under some orthogonality conditions (see Lemma 5). In Lemma 7, we prove that $\mathcal{B}_1$ is elliptic if γ is less than a critical value γ^*, γ^* being closely related to a minimization problem (see Lemma 6). The end of the proof is classical and follows the arguments proposed in [12].

The proof of Theorem 3(ii) is more delicate. Since system (1.8)–(1.10) is symmetric with respect to u_1 and u_2, we first perform the change of variable (4.1) to obtain system (4.2). The main idea is to construct an unstable direction $\mathbf{z}_\gamma = (\zeta_\gamma, 0)$ that is if $\mathbf{v}_\delta$ is the solution of system (4.2) with the initial condition $\mathbf{v}_\delta(0) = \delta\mathbf{z}_\gamma$ where δ can be choosen arbitrarily small, then $\mathbf{v}_\delta$ does not stay in the orbit of the third solitary wave $(0, 0, e^{2i\omega t}\varphi)$. This contruction is based on a careful study of the spectrum of the two linearized operator $\mathcal{L}_1$ and $\mathcal{L}_2$ (see (4.3)–(4.4) for the definition of $\mathcal{L}_1$ and $\mathcal{L}_2$). More precisely, we show in Proposition 11 that if $\gamma > \gamma^*$, then the upper bounds of the real part of spectras of $\mathcal{L} = (\mathcal{L}_2, \mathcal{L}_2)$ and $\mathcal{L}_1$ are equal and that it is an eigenvalue of $\mathcal{L}$ and $\mathcal{L}_1$. Then the unstable direction is constructed from the corresponding eigenvector. We have to notice that the critical value γ^* is that of Theorem 3(i). This comes from the fact that γ^* is closely related to a minimization problem (see Lemma 7). For all $0 < \gamma < \gamma^*$, we have $\Lambda_\gamma + \omega > 0$ whereas if $\gamma > \gamma^*$ then $\Lambda_\gamma + \omega < 0$ (see Lemma 10). This explains why the behaviour of the third solitary wave $(0, 0, e^{2i\omega t}\phi)$ is different between part (i) and part (ii) of Theorem 3.

Remarks.

(1) The additional assumption $N \leqslant 2$ and $p > 2$ in Theorem 3(ii) is related to the regularity of the nonlinearity $|u|^{p-1}u$, and it is needed to estimate the nonlinear terms of the linearized equation (4.2) in Lemmas 12 and 13 essentially. Note that when $N \geqslant 2$, the function $z \mapsto |z|^{p-1}z$ is not so smooth under our assumption $p < 1 + 4/N \leqslant 3$. The nonlinear estimate for the case $N = 2$ and $2 < p < 3$ in Lemma 13 is due to Kenji Nakanishi and Tetsu Mizumachi.

(2) The study of the form and the orbital stability of the general standing waves of system (1.8)–(1.9) is a difficult question to deal with. Our aim is to proceed step by step and so to begin with the particular cases described here.

The paper is organized as follows. In Section 2, we prove Theorem 2, the orbital stability of the first two solitary waves, using the variational method by Cazenave and Lions [3]. In Section 3, we prove the first part of Theorem 3 concerning the stability of the third solitary wave for γ small, whereas in Section 4, the orbital instability of this solitary wave is established for large γ.

2. Proof of Theorem 2

In this section, we prove Theorem 2 using the variational method by Cazenave and Lions [3]. We put

$$E_0(u)=\frac{1}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

We recall the variational characterization of the positive radial solution φ to (1.13) (see [3]).

Lemma 4. *Let $N\geqslant 1$, $1<p<1+4/N$, $\omega>0$, and let φ be the positive radial solution of* (1.13). *Then,*

$$E_0(\varphi)=\inf\bigl\{E_0(v):\ v\in H^1\bigl(\mathbb{R}^N\bigr),\ \|v\|_{L^2}=\|\varphi\|_{L^2}\bigr\}.$$

Moreover, if $\{v_n\}\subset H^1(\mathbb{R}^N)$ satisfies

$$\|v_n\|_{L^2}\to\|\varphi\|_{L^2},\qquad E_0(v_n)\to E_0(\varphi),$$

then there exist a subsequence $\{v_{n_k}\}$ and a sequence $\{(\theta_k,y_k)\}$ in $\mathbb{R}\times\mathbb{R}^N$ such that $e^{i\theta_k}v_{n_k}(\cdot+y_k)\to\varphi$ in $H^1(\mathbb{R}^N)$.

Proof of Theorem 2. Suppose that $(e^{2i\omega t}\varphi,0,0)$ is not orbitally stable. Then, there exist a constant $\delta>0$, a sequence $\{\vec{u}_n(t)\}$ of solutions of (1.8)–(1.10) and a sequence $\{t_n\}$ in $(0,\infty)$ such that

$$\bigl\|\vec{u}_n(0)-(\varphi,0,0)\bigr\|_{H^1}\to 0, \tag{2.1}$$

$$\inf_{\theta\in\mathbb{R},\,y\in\mathbb{R}^N}\bigl\|\vec{u}_n(t_n)-\bigl(e^{i\theta}\varphi(\cdot+y),0,0\bigr)\bigr\|_{H^1}=\delta. \tag{2.2}$$

We denote $\vec{u}_n(t)=(u_{n1}(t),u_{n2}(t),u_{n3}(t))$. By (2.1) and the conservation laws (1.11) and (1.12), we have

$$\bigl\|u_{n2}(t_n)\bigr\|_{L^2}^2+\bigl\|u_{n3}(t_n)\bigr\|_{L^2}^2=\bigl\|u_{n2}(0)\bigr\|_{L^2}^2+\bigl\|u_{n3}(0)\bigr\|_{L^2}^2\to 0, \tag{2.3}$$

$$\bigl\|u_{n1}(t_n)\bigr\|_{L^2}^2+\bigl\|u_{n3}(t_n)\bigr\|_{L^2}^2=\bigl\|u_{n1}(0)\bigr\|_{L^2}^2+\bigl\|u_{n3}(0)\bigr\|_{L^2}^2\to \|\varphi\|_{L^2}^2, \tag{2.4}$$

$$E\bigl(\vec{u}_n(t_n)\bigr)=E\bigl(\vec{u}_n(0)\bigr)\to E(\varphi,0,0)=E_0(\varphi). \tag{2.5}$$

Since $\{\vec{u}_n(t_n)\}$ is bounded in $H^1(\mathbb{R}^N,\mathbb{C}^3)$, it follows from (2.3)–(2.5) that

$$\bigl\|u_{n2}(t_n)\bigr\|_{L^2}\to 0,\qquad \bigl\|u_{n3}(t_n)\bigr\|_{L^2}\to 0, \tag{2.6}$$

$$\bigl\|u_{n1}(t_n)\bigr\|_{L^2}\to\|\varphi\|_{L^2}, \tag{2.7}$$

$$E_0\bigl(u_{n1}(t_n)\bigr)+\frac{1}{2}\sum_{j=2}^{3}\bigl\|\nabla u_{nj}(t_n)\bigr\|_{L^2}^2\to E_0(\varphi). \tag{2.8}$$

Then, by (2.7), (2.8) and Lemma 4, we have

$$\begin{aligned}E_0(\varphi)&\leqslant\liminf_{n\to\infty}E_0\bigl(u_{n1}(t_n)\bigr)\leqslant\limsup_{n\to\infty}E_0\bigl(u_{n1}(t_n)\bigr)\\&\leqslant\lim_{n\to\infty}\left\{E_0\bigl(u_{n1}(t_n)\bigr)+\frac{1}{2}\sum_{j=2}^{3}\bigl\|\nabla u_{nj}(t_n)\bigr\|_{L^2}^2\right\}=E_0(\varphi),\end{aligned}$$

which implies

$$E_0\big(u_{n1}(t_n)\big) \to E_0(\varphi), \tag{2.9}$$

$$\big\|\nabla u_{n2}(t_n)\big\|_{L^2} \to 0, \qquad \big\|\nabla u_{n3}(t_n)\big\|_{L^2} \to 0. \tag{2.10}$$

By (2.7), (2.9) and Lemma 4, there exist a subsequence $\{u_{n_k 1}(t_{n_k})\}$ and a sequence $\{(\theta_k, y_k)\}$ in $\mathbb{R} \times \mathbb{R}^N$ such that

$$\big\|e^{i\theta_k} u_{n_k 1}(t_{n_k}, \cdot + y_k) - \varphi\big\|_{H^1} \to 0. \tag{2.11}$$

Moreover, by (2.6) and (2.10), we have

$$\big\|u_{n2}(t_n)\big\|_{H^1} + \big\|u_{n3}(t_n)\big\|_{H^1} \to 0. \tag{2.12}$$

By (2.11) and (2.12), we have

$$\inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^N} \big\|\vec{u}_{n_k}(t_{n_k}) - \big(e^{i\theta}\varphi(\cdot + y), 0, 0\big)\big\|_{H^1} \to 0,$$

which contradicts (2.2). Hence, $(e^{2i\omega t}\varphi, 0, 0)$ is orbitally stable. The stability of $(0, e^{2i\omega t}\varphi, 0)$ can be proved in the same way. □

3. Proof of Theorem 3(i)

In this section, we prove the first part of Theorem 3. We put $\Phi = (0, 0, \varphi)$. We regard Φ as a critical point of the functional S defined by

$$S(\vec{v}) = E(\vec{v}) + \omega Q(\vec{v}), \tag{3.1}$$
$$Q(\vec{v}) = Q_1(\vec{v}) + Q_2(\vec{v}) = \|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2 + 2\|v_3\|_{L^2}^2$$

for $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$. Then we have $S'(\Phi) = 0$ and a direct computation gives

$$\big\langle S''(\Phi)\vec{v}, \vec{v}\big\rangle = \mathcal{B}_1(v_1, v_2) + \mathcal{B}_2(v_3) \tag{3.2}$$

for $\vec{v} = (v_1, v_2, v_3) \in H^1(\mathbb{R}^N, \mathbb{C}^3)$, where

$$\mathcal{B}_1(v_1, v_2) = \sum_{j=1}^{2}\big(\|\nabla v_j\|_{L^2}^2 + \omega\|v_j\|_{L^2}^2\big) - 2\gamma\Re\int_{\mathbb{R}^N} \varphi v_1 v_2\, dx$$

for $(v_1, v_2) \in H^1(\mathbb{R}^N, \mathbb{C}^2)$, and

$$\mathcal{B}_2(v) = \|\nabla v\|_{L^2}^2 + 2\omega\|v\|_{L^2}^2 - p\int_{\mathbb{R}^N} \varphi^{p-1}(\Re v)^2\, dx - \int_{\mathbb{R}^N} \varphi^{p-1}(\Im v)^2\, dx$$

for $v = \Re v + i\Im v \in H^1(\mathbb{R}^N, \mathbb{C})$, where $\Im(v)$ denotes the imaginary part of v. The following positivity of $\mathcal{B}_2$ is well-known.

Lemma 5. *Let $N \geqslant 1$, $1 < p < 1 + 4/N$, $\omega > 0$, and let φ be the positive radial solution of* (1.13). *Then, there exists a constant $\delta_2 > 0$ such that $\mathcal{B}_2(v) \geqslant \delta_2\|v\|_{H^1}^2$ for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\Re(v, \varphi)_{L^2} = 0$, $\Re(v, i\varphi)_{L^2} = 0$ and $\Re(v, \nabla\varphi)_{L^2} = 0$.*

Proof. We put

$$L_+ = -\Delta + 2\omega - p\varphi^{p-1}, \qquad L_- = -\Delta + 2\omega - \varphi^{p-1}.$$

Then one has

$$\mathcal{B}_2(v) = \langle L_+\Re v, \Re v\rangle + \langle L_-\Im v, \Im v\rangle,$$

and the result follows from Proposition 1 in [14] and Lemma 4.2 in [10] (see also [22, Section 2], [12,13] and [9, Section 3]). □

In order to prove the positivity of $\mathcal{B}_1$, we first need the following.

Lemma 6. *Let $N \geqslant 1$, $1 < p < 1 + 4/(N-2)$, $\omega > 0$, and let φ be the positive radial solution of* (1.13). *For $\gamma > 0$, let*

$$\Lambda_\gamma = \inf\{B_\gamma(v):\ v \in H^1(\mathbb{R}^N, \mathbb{C}),\ \|v\|_{L^2} = 1\},$$
$$B_\gamma(v) = \|\nabla v\|_{L^2}^2 - \gamma \int_{\mathbb{R}^N} \varphi |v|^2\, dx.$$

Then we have

(i) $-\gamma\|\varphi\|_{L^\infty} \leqslant \Lambda_\gamma \leqslant (\|\nabla\varphi\|_{L^2}^2 - \gamma\|\varphi\|_{L^3}^3)/\|\varphi\|_{L^2}^2$ *for any* $\gamma > 0$.
(ii) *If $\Lambda_\gamma < 0$, then there exists $\chi_\gamma \in H^1(\mathbb{R}^N)$ such that $\|\chi_\gamma\|_{L^2} = 1$ and $-\Delta\chi_\gamma - \gamma\varphi\chi_\gamma = \Lambda_\gamma\chi_\gamma$.*
(iii) *If $0 < \gamma_1 < \gamma_2$ and $\Lambda_{\gamma_1} < 0$, then $\Lambda_{\gamma_1} > \Lambda_{\gamma_2}$.*

Proof. (i) Since $\varphi \in L^\infty(\mathbb{R}^N)$, we have

$$B_\gamma(v) \geqslant -\gamma \int_{\mathbb{R}^N} \varphi|v|^2\, dx \geqslant -\gamma\|\varphi\|_{L^\infty}\|v\|_{L^2}^2$$

for any $v \in H^1(\mathbb{R}^N)$, which shows $\Lambda_\gamma \geqslant -\gamma\|\varphi\|_{L^\infty}$. Moreover, since φ is positive, we have

$$\Lambda_\gamma \leqslant B_\gamma\left(\frac{\varphi}{\|\varphi\|_{L^2}}\right) \leqslant \frac{\|\nabla\varphi\|_{L^2}^2 - \gamma\|\varphi\|_{L^3}^3}{\|\varphi\|_{L^2}^2}.$$

(ii) See Lieb and Loss [16, Section 11.5].

(iii) Since $\Lambda_{\gamma_1} < 0$, by (ii) there exists $\chi_{\gamma_1} \in H^1(\mathbb{R}^N)$ such that $B_{\gamma_1}(\chi_{\gamma_1}) = \Lambda_{\gamma_1}$ and $\|\chi_{\gamma_1}\|_{L^2} = 1$. Then, since $\gamma_1 < \gamma_2$, we have $\Lambda_{\gamma_2} \leqslant B_{\gamma_2}(\chi_{\gamma_1}) < B_{\gamma_1}(\chi_{\gamma_1}) = \Lambda_{\gamma_1}$. □

We are now able to prove the positivity of $\mathcal{B}_1$ for small γ.

Lemma 7. *Under the same assumptions as in Lemma* 6, *let*

$$\gamma^* = \inf\{\gamma > 0:\ \Lambda_\gamma < -\omega\}. \tag{3.3}$$

Then, $0 < \gamma^* < \infty$. Moreover, if $0 < \gamma < \gamma^*$, then there exists a constant $\delta_1 > 0$ such that $\mathcal{B}_1(v_1, v_2) \geqslant \delta_1\|(v_1, v_2)\|_{H^1}^2$ for any $(v_1, v_2) \in H^1(\mathbb{R}^N, \mathbb{C}^2)$.

Proof. By Lemma 6, we have

$$0 < \frac{\omega}{\|\varphi\|_{L^\infty}} \leqslant \gamma^* \leqslant \frac{\|\nabla\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2}{\|\varphi\|_{L^3}^3},$$

which shows $0 < \gamma^* < \infty$. Moreover, if $0 < \gamma < \gamma^*$, then by Lemma 6 and (3.3), we see that $\Lambda_\gamma + \omega > 0$, and

$$B_\gamma(v) + \omega\|v\|_{L^2}^2 \geqslant (\Lambda_\gamma + \omega)\|v\|_{L^2}^2$$

for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$, from which it follows that there exists $\delta_1 > 0$ such that $B_\gamma(v) + \omega\|v\|_{L^2}^2 \geqslant \delta_1\|v\|_{H^1}^2$ for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$. Thus, we have

$$\begin{aligned}\mathcal{B}_1(v_1, v_2) &= \sum_{j=1}^{2}\{B_\gamma(v_j) + \omega\|v_j\|_{L^2}^2\} + \gamma\int_{\mathbb{R}^N}\varphi|v_1 - \overline{v_2}|^2\, dx \\ &\geqslant \delta_1\|(v_1, v_2)\|_{H^1}^2\end{aligned}$$

for any $(v_1, v_2) \in H^1(\mathbb{R}^N, \mathbb{C}^2)$. □

Using Lemmas 5 and 7, one can prove that under suitable restrictions, the linearized energy $S''(\Phi)$ controls the H^1-norm.

Proposition 8. *Let $1 \leqslant N \leqslant 3$, $1 < p < 1 + 4/N$, $\omega > 0$, and let $\Phi = (0, 0, \varphi)$, where φ is the positive radial solution of* (1.13). *Assume $0 < \gamma < \gamma^*$, where γ^* is the positive constant defined by* (3.3). *Then, there exists a constant $\delta > 0$ such that*

$$\langle S''(\Phi)\vec{v}, \vec{v}\rangle \geqslant \delta \|\vec{v}\|_{H^1}^2$$

for any $\vec{v} = (v_1, v_2, v_3) \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ satisfying $\Re(v_3, \varphi)_{L^2} = 0$, $\Re(v_3, i\varphi)_{L^2} = 0$ and $\Re(v_3, \nabla\varphi)_{L^2} = 0$.

As a consequence, we state the following lemma which is at the heart of Theorem 3(i).

Lemma 9. *Under the same assumption as in Proposition* 8, *there exist positive constants C and ε such that*

$$E(\vec{v}) - E(\Phi) \geqslant C d(\vec{v}, \Phi)^2$$

for any $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ satisfying $d(\vec{v}, \Phi) < \varepsilon$ and $Q(\vec{v}) = Q(\Phi)$, where we put

$$d(\vec{v}, \Phi) = \inf_{\theta \in \mathbb{R},\, y \in \mathbb{R}^N} \big\| \vec{v} - \big(0, 0, e^{i\theta}\varphi(\cdot + y)\big) \big\|_{H^1}.$$

Proof. This result follows from Proposition 8. Since it is classical, we refer to Theorem 3.4 in [12] for a complete proof (see also [14, Proposition 3] and [10, Lemma 2.1]). □

Proof of Theorem 3(i). We repeat briefly the argument of the proof of Theorem 3.5 in [12]. We use the notation in Lemma 9. Assume that the conclusion of Theorem 3(i) is not true, then there exist a sequence of initial data $\{\vec{u}_n(0)\}$ in H^1, $\mu \in (0, \varepsilon)$ and $\{t_n\}$ in $(0, +\infty)$ such that

$$\big\| \vec{u}_n(0) - \Phi \big\|_{H^1} \to 0, \tag{3.4}$$
$$d\big(\vec{u}_n(t_n), \Phi\big) = \mu. \tag{3.5}$$

By the conservation laws (1.11)–(1.12), and by (3.4) and the continuity of E and Q on H^1, we have

$$E\big(\vec{u}_n(t_n)\big) = E\big(\vec{u}_n(0)\big) \to E(\Phi), \tag{3.6}$$
$$Q\big(\vec{u}_n(t_n)\big) = Q\big(\vec{u}_n(0)\big) \to Q(\Phi). \tag{3.7}$$

Here we put $\vec{v}_n = (Q(\Phi)/Q(\vec{u}_n(t_n)))^{1/2}\vec{u}_n(t_n)$. Then, by (3.7), (3.5) and (3.6), we have $Q(\vec{v}_n) = Q(\Phi)$, $d(\vec{v}_n, \Phi) < \varepsilon$ for large n, and $E(\vec{v}_n) \to E(\Phi)$. We then apply Lemma 9 to obtain

$$C d(\vec{v}_n, \Phi)^2 \leqslant E(\vec{v}_n) - E(\Phi) \to 0.$$

Thus, we have $d(\vec{u}_n(t_n), \Phi) \to 0$, and this contradicts (3.5). □

4. Proof of Theorem 3(ii)

In this section, we prove the second part of Theorem 3. For that purpose, we make a special change of variables

$$\vec{u}(t) = \big(e^{i\omega t} v_1(t), e^{i\omega t} v_1(t), e^{2i\omega t}\big(\varphi + v_2(t)\big)\big) \tag{4.1}$$

in (1.8)–(1.10). Note that (1.8)–(1.10) is symmetric with respect to u_1 and u_2. Then, the equations for (v_1, v_2) read

$$\partial_t v_1 = \mathcal{L}_1 v_1 + F_1(v_1, v_2), \qquad \partial_t v_2 = \mathcal{L}_2 v_2 + F_2(v_1, v_2), \tag{4.2}$$

where the linear terms $\mathcal{L}_1 v_1$ and $\mathcal{L}_2 v_2$ are given by

$$\mathcal{L}_1 v = -i(-\Delta v + \omega v - \gamma\varphi\bar{v}), \tag{4.3}$$
$$\mathcal{L}_2 v = -i\left(-\Delta v + 2\omega v - \frac{p+1}{2}\varphi^{p-1} v - \frac{p-1}{2}\varphi^{p-1}\bar{v}\right), \tag{4.4}$$

and the nonlinear terms $F_1(v_1, v_2)$ and $F_2(v_1, v_2)$ are given by

$$F_1(v_1, v_2) = i\left(\gamma \overline{v_1} v_2 + |v_1|^{p-1} v_1\right),$$
$$F_2(v_1, v_2) = i\left\{\gamma v_1^2 + |\varphi + v_2|^{p-1}(\varphi + v_2) - \varphi^p - \frac{p+1}{2}\varphi^{p-1} v_2 - \frac{p-1}{2}\varphi^{p-1}\overline{v_2}\right\}.$$

We also write (4.2) as

$$\partial_t \mathbf{v} = \mathcal{L}\mathbf{v} + F(\mathbf{v}), \tag{4.5}$$

where $\mathbf{v} = (v_1, v_2)$, $\mathcal{L}\mathbf{v} = (\mathcal{L}_1 v_1, \mathcal{L}_2 v_2)$, and $F(\mathbf{v}) = (F_1(v_1, v_2), F_2(v_1, v_2))$. It is convenient to let

$$\mathcal{L}_1 v = (-\Delta + \omega + \gamma\varphi)\Im v - i(-\Delta + \omega - \gamma\varphi)\Re v,$$
$$\mathcal{L}_2 v = (-\Delta + 2\omega - \varphi^{p-1})\Im v - i(-\Delta + 2\omega - p\varphi^{p-1})\Re v.$$

We consider $\mathcal{L}_1$ and $\mathcal{L}_2$ as linear operators in $L^2(\mathbb{R}^N, \mathbb{C})$ with domains $D(\mathcal{L}_1) = D(\mathcal{L}_2) = H^2(\mathbb{R}^N, \mathbb{C})$. It is known that the spectrum $\sigma(\mathcal{L}_2)$ of $\mathcal{L}_2$ satisfies $\sigma(\mathcal{L}_2) \subset i\mathbb{R}$ if $p < 1 + 4/N$ (see, e.g., [4, Summary 2.5]). Furthermore, we define

$$Q = -\Delta + \omega - \gamma\varphi, \qquad P = -\Delta + \omega + \gamma\varphi.$$

Note that

$$\langle Qv, v\rangle = B_\gamma(v) + \omega\|v\|_{L^2}^2, \quad v \in H^1(\mathbb{R}^N),$$

and the operator $P : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is bounded, and there exist positive constants c_1 and c_2 such that

$$c_1\|v\|_{H^1}^2 \leqslant \langle Pv, v\rangle \leqslant c_2\|v\|_{H^1}^2, \quad v \in H^1(\mathbb{R}^N).$$

Thus, the inverse $P^{-1} : H^{-1}(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ exists and is bounded, and there exist positive constants c_3 and c_4 such that

$$c_3\|f\|_{H^{-1}}^2 \leqslant \langle f, P^{-1} f\rangle \leqslant c_4\|f\|_{H^{-1}}^2, \quad f \in H^{-1}(\mathbb{R}^N). \tag{4.6}$$

Lemma 10. *Let*

$$\mu_\gamma = \inf\left\{\frac{\langle Qv, v\rangle}{\langle P^{-1}v, v\rangle}:\ v \in H^1(\mathbb{R}^N) \setminus \{0\}\right\}, \tag{4.7}$$

and γ^* defined as in Lemma 7. If $\gamma > \gamma^*$, then $-\infty < \mu_\gamma < 0$, and there exists $\xi_\gamma \in H^2(\mathbb{R}^N) \setminus \{0\}$ such that

$$Q\xi_\gamma = \mu_\gamma P^{-1}\xi_\gamma.$$

Proof. By definition (3.3) of γ^* and Lemma 6(iii), we see that if $\gamma > \gamma^*$, then $\Lambda_\gamma < -\omega < 0$. Then, by Lemma 6(ii), there exists $\chi_\gamma \in H^1(\mathbb{R}^N)$ such that $\|\chi_\gamma\|_{L^2} = 1$ and $\langle Q\chi_\gamma, \chi_\gamma\rangle = B_\gamma(\chi_\gamma) + \omega\|\chi_\gamma\|_{L^2}^2 = \Lambda_\gamma + \omega < 0$. Thus, we have

$$\mu_\gamma \leqslant \frac{\langle Q\chi_\gamma, \chi_\gamma\rangle}{\langle P^{-1}\chi_\gamma, \chi_\gamma\rangle} < 0.$$

Moreover, for $v \in H^1(\mathbb{R}^N)$, by (4.6) we have

$$\begin{aligned}\gamma \int_{\mathbb{R}^N} \varphi|v|^2\,dx &\leqslant \gamma\|\varphi\|_{L^\infty}\|v\|_{L^2}^2 \leqslant \gamma\|\varphi\|_{L^\infty}\|v\|_{H^{-1}}\|v\|_{H^1} \\ &\leqslant \frac{1}{2}\left(\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2\right) + C\langle P^{-1}v, v\rangle,\end{aligned} \tag{4.8}$$

which implies

$$0 \leqslant \langle Qv, v\rangle + C\langle P^{-1}v, v\rangle,$$

and then $\mu_\gamma > -\infty$. Next, by the definition of μ_γ, there exists a sequence $\{v_n\}$ in $H^1(\mathbb{R}^N)$ such that

$$\langle P^{-1}v_n, v_n\rangle = 1, \qquad \langle Qv_n, v_n\rangle \to \mu_\gamma.$$

Then, by (4.8), we see that

$$\frac{1}{2}\big(\|\nabla v_n\|_{L^2}^2+\omega\|v_n\|_{L^2}^2\big)\leqslant\langle Qv_n,v_n\rangle+C\big\langle P^{-1}v_n,v_n\big\rangle,$$

and so $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, there exist a subsequence of $\{v_n\}$ (still denoted by v_n) and $w\in H^1(\mathbb{R}^N)$ such that $v_n\rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$. Then, we have

$$\int\limits_{\mathbb{R}^N}\varphi|v_n|^2\,dx\to\int\limits_{\mathbb{R}^N}\varphi|w|^2\,dx$$

(see [16, Section 11.4]), and

$$\langle Qw,w\rangle\leqslant\liminf_{n\to\infty}\langle Qv_n,v_n\rangle=\mu_\gamma<0, \tag{4.9}$$

$$\big\langle P^{-1}w,w\big\rangle\leqslant\liminf_{n\to\infty}\big\langle P^{-1}v_n,v_n\big\rangle=1. \tag{4.10}$$

By (4.9) and (4.10), we see that $w\neq 0$ and

$$\frac{\langle Qw,w\rangle}{\langle P^{-1}w,w\rangle}=\mu_\gamma.$$

Since $w\in H^1(\mathbb{R}^N)\setminus\{0\}$ attains the infimum in (4.7), it is easy to see that w satisfies $Qw=\mu_\gamma P^{-1}w$, and $w\in H^2(\mathbb{R}^N)$. □

In the next proposition, we study the upper bound of the real part of spectra of $\mathcal{L}_1$ and $\mathcal{L}$.

Proposition 11. *Let μ_γ be in Lemma* 10. *If $\gamma>\gamma^*$, then $\lambda_\gamma:=\sqrt{-\mu_\gamma}$ is a positive eigenvalue of $\mathcal{L}_1$ and $\mathcal{L}$, and*

$$\sup\big\{\Re\lambda\colon\ \lambda\in\sigma(\mathcal{L})\big\}=\sup\big\{\Re\lambda\colon\ \lambda\in\sigma(\mathcal{L}_1)\big\}=\lambda_\gamma.$$

Proof. We use the same notation as in Lemma 10. Since $\mu_\gamma<0$, we have $\lambda_\gamma=\sqrt{-\mu_\gamma}>0$. Let

$$\eta_\gamma=\lambda_\gamma P^{-1}\xi_\gamma,\qquad \zeta_\gamma=\xi_\gamma+i\eta_\gamma.$$

Then, we see that $P\eta_\gamma=\lambda_\gamma\xi_\gamma$, $Q\xi_\gamma=-\lambda_\gamma\eta_\gamma$, and $\zeta_\gamma\in H^2(\mathbb{R}^N,\mathbb{C})$, so that $\mathcal{L}_1\zeta_\gamma=P\eta_\gamma-iQ\xi_\gamma=\lambda_\gamma\zeta_\gamma$ and $\zeta_\gamma\neq 0$. Thus, λ_γ is a positive eigenvalue of $\mathcal{L}_1$ with eigenvector ζ_γ, and it is also an eigenvalue of $\mathcal{L}$ with eigenvector $(\zeta_\gamma,0)$. It is known that the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}_1)$ of $\mathcal{L}_1$ satisfies $\sigma_{\text{ess}}(\mathcal{L}_1)\subset i\mathbb{R}$, and that $\sigma(\mathcal{L}_1)\setminus\sigma_{\text{ess}}(\mathcal{L}_1)$ consists of finitely many eigenvalues. Let $\lambda\in\mathbb{C}\setminus i\mathbb{R}$ be an eigenvalue of $\mathcal{L}_1$ with eigenvector $w\in H^2(\mathbb{R}^N,\mathbb{C})$. Then, we have

$$-\lambda^2w=-\mathcal{L}_1^2w=PQ\Re w+iQP\Im w.$$

Since P and Q are self-adjoint operators in $L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}-\lambda^2\|w\|_{L^2}^2&=(PQ\Re w,\Re w)_{L^2}+(QP\Im w,\Im w)_{L^2}+i(QP\Im w,\Re w)_{L^2}-i(PQ\Re w.\Im w)_{L^2}\\&=(Q\Re w,P\Re w)_{L^2}+(P\Im w,Q\Im w)_{L^2}+i(P\Im w,Q\Re w)_{L^2}-i(Q\Re w,P\Im w)_{L^2}\\&=(Q\Re w,P\Re w)_{L^2}+(P\Im w,Q\Im w)_{L^2}.\end{aligned}$$

Since $w\neq 0$, we see that $\lambda^2\in\mathbb{R}$. Moreover, since $\lambda\notin i\mathbb{R}$, we have $\lambda\in\mathbb{R}\setminus\{0\}$. Thus, by $\mathcal{L}_1w=\lambda w$, we have $P\Im w=\lambda\Re w$ and $Q\Re w=-\lambda\Im w$. Then, we have $\Re w\neq 0$ and $PQ\Re w=-\lambda^2\Re w$. Since P is invertible, we have $Q\Re w=-\lambda^2P^{-1}\Re w$, and

$$-\lambda^2=\frac{(Q\Re w,\Re w)_{L^2}}{(P^{-1}\Re w,\Re w)_{L^2}}\geqslant\mu_\gamma.$$

Therefore, we have $\lambda\leqslant\sqrt{-\mu_\gamma}=\lambda_\gamma$, which shows $\sup\{\Re\lambda\colon\ \lambda\in\sigma(\mathcal{L}_1)\}=\lambda_\gamma$. Finally, since $\sigma(\mathcal{L}_2)\subset i\mathbb{R}$ if $p<1+4/N$, we see that $\sup\{\Re\lambda\colon\ \lambda\in\sigma(\mathcal{L})\}=\sup\{\Re\lambda\colon\ \lambda\in\sigma(\mathcal{L}_1)\}=\lambda_\gamma$. □

Next, we prove the orbital instability of $(0, 0, e^{2i\omega t}\varphi)$ using Proposition 11. The proof is based on the argument in Section 6 of Grillakis, Shatah and Strauss [13] (see also [7,8,17]).

For the nonlinear term $F(\mathbf{v})$ in (4.5), we have the following estimates. We remark that the additional assumption $N \leqslant 2$ and $p > 2$ in Theorem 3(ii) is needed here. Especially, for the case $N = 2$, we need a technical Lemma 13 below, which is due to Kenji Nakanishi and Tetsu Mizumachi.

Lemma 12. *Assume that $N \leqslant 2$ and $2 < p < 1 + 4/N$. Let*

$$\alpha = \min\left\{\frac{p-2}{4}, \frac{1}{2}\right\}, \qquad s = \begin{cases} 1 & \text{if } N = 1, \\ 1+\varepsilon & \text{if } N = 2, \end{cases} \tag{4.11}$$

where ε is a number such that

$$\frac{p-2}{2(p-1)} < \varepsilon < \frac{p-2}{2}. \tag{4.12}$$

Then, there exist positive constants C_0 and ρ_0 such that

$$\left\|F(\mathbf{v})\right\|_{H^s} \leqslant C_0 \|\mathbf{v}\|_{H^s}^{1+2\alpha} \tag{4.13}$$

for any $\mathbf{v} \in H^s(\mathbb{R}^N, \mathbb{C}^2)$ satisfying $\|\mathbf{v}\|_{H^s} \leqslant \rho_0$.

Proof. We put $f(z) := |z|^{p-1}z$ for $z \in \mathbb{C}$. Since $p > 2$, the function $f : \mathbb{C} \to \mathbb{C}$ is of class C^2 in the real sense. For $z \in \mathbb{C}$, the $\mathbb{R}$-linear map $f'(z) : \mathbb{C} \to \mathbb{C}$ is given by

$$f'(z)w = \frac{\partial f}{\partial z}(z)w + \frac{\partial f}{\partial \bar{z}}(z)\overline{w} = \frac{p+1}{2}|z|^{p-1}w + \frac{p-1}{2}|z|^{p-3}z^2\overline{w}$$

for $w \in \mathbb{C}$, and the $\mathbb{R}$-bilinear map $f''(z) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies

$$\left|f''(z) - f''(w)\right| \leqslant \begin{cases} C|z-w|^{p-2} & \text{if } 2 < p < 3, \\ C(|z|^{p-3} + |w|^{p-3})|z-w| & \text{if } p \geqslant 3 \end{cases}$$

for $z, w \in \mathbb{C}$. We also put

$$g(v) := f(\varphi + v) - f(\varphi) - f'(\varphi)v, \quad v \in H^s\big(\mathbb{R}^N, \mathbb{C}\big).$$

Then, by the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we see that

$$\left\|F_1(v_1, v_2)\right\|_{H^s} \leqslant C\big(\|v_1\|_{H^s}\|v_2\|_{H^s} + \|v_1\|_{H^s}^p\big), \tag{4.14}$$

$$\left\|F_2(v_1, v_2)\right\|_{H^s} \leqslant C\big(\|v_1\|_{H^s}^2 + \|g(v_2)\|_{H^s}\big) \tag{4.15}$$

for any $(v_1, v_2) \in H^s(\mathbb{R}^N, \mathbb{C}^2)$.

For the estimate of $\|g(v)\|_{H^s}$, we divide the proof into two cases $N = 1$ and $N = 2$. We first consider easier case $N = 1$. Since

$$g(v) = \int_0^1 \big\{f'(\varphi + \theta v) - f'(\varphi)\big\}v\,d\theta, = \int_0^1\int_0^1 f''(\varphi + \theta_1\theta_2 v)(v, v)\theta_1\,d\theta_1\,d\theta_2,$$

we have

$$\left\|g(v)\right\|_{L^2} \leqslant CM_\rho\|v\|_{H^1}^2 \tag{4.16}$$

for any $v \in H^1(\mathbb{R}, \mathbb{C})$ satisfying $\|v\|_{H^1} \leqslant \rho$, where we put

$$M_\rho = \sup\big\{\left|f''(z)\right|: |z| \leqslant \|\varphi\|_{L^\infty} + C_s\rho\big\},$$

and C_s is the best constant of the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Moreover, we have

$$\partial_x g(v) = \{f'(\varphi+v) - f'(\varphi)\}(\partial_x\varphi + \partial_x v) - f''(\varphi)(\partial_x\varphi, v)$$
$$= \int_0^1 \{f''(\varphi+\theta v) - f''(\varphi)\}(\partial_x\varphi, v)\,d\theta + \int_0^1 f''(\varphi+\theta v)(v, \partial_x v)\,d\theta,$$

and

$$\left\| \int_0^1 \{f''(\varphi+\theta v) - f''(\varphi)\}(\partial_x\varphi, v)\,d\theta \right\|_{L^2} \leqslant \begin{cases} C\|v\|_{H^1}^{p-1} & \text{if } 2 < p < 3, \\ C(\|v\|_{H^1}^2 + \|v\|_{H^1}^{p-1}) & \text{if } p \geqslant 3. \end{cases}$$

Thus, we have

$$\left\|\partial_x g(v)\right\|_{L^2} \leqslant C\left(\|v\|_{H^1}^{p-1} + \|v\|_{H^1}^2\right) + CM_\rho \|v\|_{H^1}^2 \tag{4.17}$$

for any $v \in H^1(\mathbb{R}, \mathbb{C})$ satisfying $\|v\|_{H^1} \leqslant \rho$. By (4.14), (4.15), (4.16) and (4.17), we obtain (4.13) for the case $N = 1$. Finally, the case $N = 2$ follows from the following Lemma 13, which is due to Kenji Nakanishi and Tetsu Mizumachi. $\square$

Lemma 13 *(due to Kenji Nakanishi and Tetsu Mizumachi). Let $N = 2$, $2 < p < 3$, $f(z) = |z|^{p-1}z$, and let $g(v) = f(\varphi+v) - f(\varphi) - f'(\varphi)v$. Then for all ε satisfying* (4.12)*, we have the following estimate*

$$\left\|g(v)\right\|_{H^{1+\varepsilon}} \lesssim \|v\|_{H^{1+\varepsilon}}^{p-1} + \|v\|_{H^{1+\varepsilon}}^{p/2},$$

where the implicit constant is determined by p, ε and $\|\varphi\|_{H^2}$.

Note that for $2 < p < 3$ we have

$$0 < \frac{p-2}{2(p-1)} < \frac{p-2}{2} < \frac{1}{2}, \quad 1 < \frac{p}{2} < p - 1 < 2.$$

Proof. Let $q \in (1, \infty)$ be given by $1/q = 1/2 - \varepsilon/2$. Then we have the Sobolev embedding $H^{1+\varepsilon} \subset H_q^1 \subset L^\infty$. We use the difference representation of the Besov norm:

$$\|v\|_{B_{p_1,p_2}^{1+\varepsilon}}^{p_2} \sim \|v\|_{L^{p_1}}^{p_2} + \sum_{k=1}^{N}\sum_{j=1}^{\infty} \left\|2^{\varepsilon j}\delta_{k,j}\nabla v\right\|_{L^{p_1}}^{p_2},$$

where $\delta_{k,j}$ denotes the difference operator defined by

$$\delta_{k,j} v(x) = v\left(x + 2^{-j}e_k\right) - v(x),$$

and e_k is the kth unit vector. Note that $H^{1+\varepsilon} = B_{2,2}^{1+\varepsilon}$. By the Taylor expansion we have

$$\nabla g(v) = \int_0^1 \left(f''(\varphi+\theta v) - f''(\varphi)\right)(\nabla\varphi, v)\,d\theta$$
$$+ \int_0^1\int_0^1 f'''(\varphi+\theta_1\theta_2 v)(v, \nabla v)\theta_1\,d\theta_1\,d\theta_2$$
$$+ \int_0^1 f''(\varphi+\theta v)(v, \nabla v)\theta\,d\theta. \tag{4.18}$$

We apply the above difference norm to the right-hand side, using the Leibniz rule for the difference operator and pulling the integration in θ_* out of the norms by Minkowski. We will neglect spatial translations involved in the

Leibniz rule because they will not affect when we apply the Hölder inequality. Hence the second term is estimated in the same way as for the last one. For the last term of (4.18), the difference hitting f'' is bounded pointwise by

$$|\delta\varphi|^{p-2}|v||\nabla v| + |\delta v|^{p-2}|v||\nabla v|. \tag{4.19}$$

For the second term we use that

$$\|\delta_{k,j} v\|_{L^q} \lesssim 2^{-j}\|\nabla v\|_{L^q} \lesssim 2^{-j}\|v\|_{H^{1+\varepsilon}}. \tag{4.20}$$

By Hölder we have

$$\big\| |\delta v|^{p-2} v \nabla v\big\|_{L^2} \leqslant \|\delta v\|_{L^q}^{p-2}\|\nabla v\|_{L^q}\|v\|_{L^s},$$

where $s \in (2,\infty)$ is determined by

$$\frac{1}{2} = \frac{p-1}{q} + \frac{1}{s}. \tag{4.21}$$

Note that $(p-1)/q < 1/2$ is equivalent to $\varepsilon > (p-2)/(2(p-1))$, which is one of the assumptions. Hence the contribution to the Besov $B^{1+\varepsilon}_{2,2}$ norm is bounded by

$$\big\|2^{(\varepsilon-p+2)j}\big\|_{\ell^q}\|v\|_{H^{1+\varepsilon}}^{p-1}\|v\|_{L^s} \lesssim \|v\|_{H^{1+\varepsilon}}^p,$$

where the sequence norm is finite since $\varepsilon < p-2$. The first term in (4.19) is estimated in the same way and thus bounded by

$$\|\varphi\|_{H^{1+\varepsilon}}\|v\|_{H^{1+\varepsilon}}^{p-1}.$$

The difference hitting v is bounded by

$$\big\| |\varphi+\theta v|^{p-2}\delta v \nabla v\big\|_{L^2} \lesssim \big(\|\varphi\|_{L^\infty} + \|v\|_{L^\infty}\big)^{p-2}\|\delta v\|_{L^{2/\varepsilon}}\|\nabla v\|_{L^q}, \tag{4.22}$$

and so its contribution to the Besov norm is bounded by

$$\big(\|\varphi\|_{H^2} + \|v\|_{H^{1+\varepsilon}}\big)^{p-2}\|v\|_{B^{\varepsilon}_{2/\varepsilon,2}}\|v\|_{H^{1+\varepsilon}},$$

where the second last norm is also bounded by the last norm, since $H^{1+\varepsilon} \subset B^{\varepsilon}_{s,2}$ for all $s \in [2,\infty]$. The difference hitting ∇v is bounded in L^2 similarly by

$$\big(\|\varphi\|_{L^\infty} + \|v\|_{L^\infty}\big)^{p-2}\|v\|_{L^\infty}\|\delta\nabla v\|_{L^2}, \tag{4.23}$$

and so the contribution to the Besov norm is bounded again by

$$\big(\|\varphi\|_{H^2} + \|v\|_{H^{1+\varepsilon}}\big)^{p-2}\|v\|_{H^{1+\varepsilon}}^2.$$

Next we estimate the first term on the right of (4.18). The difference hitting v or φ is bounded in the same way as for (4.22) or (4.23). For the difference hitting $f''(\varphi+\theta v) - f''(\varphi)$, we use interpolation of two estimates:

$$\begin{aligned}\big|\delta\big[f''(\varphi+\theta v) - f''(\varphi)\big]\big| &\lesssim \big|\delta f''(\varphi+\theta v)\big| + \big|\delta f''(\varphi)\big| \\ &\lesssim |\delta\varphi|^{p-2} + |\delta v|^{p-2},\end{aligned}$$

and

$$\begin{aligned}\big|\delta\big[f''(\varphi+\theta v) - f''(\varphi)\big]\big| &\lesssim \sum_{a=0}^{1}\big|\big[f''(\varphi+\theta v) - f''(\varphi)\big]\big(x+2^j a e_k\big)\big| \\ &\lesssim \sum_{a=0}^{1}|v|^{p-2}\big(x+2^j a e_k\big).\end{aligned}$$

By using the first bound and (4.20), we have

$$\big\|\delta\big[f''(\varphi+\theta v) - f''(\varphi)\big](\nabla\varphi, v)\big\|_{L^2} \lesssim 2^{(-p+2)j}\big[\|\varphi\|_{H^2} + \|v\|_{H^{1+\varepsilon}}\big]^{p-2}\|\nabla\varphi\|_{L^s}\|v\|_{L^\infty},$$

where $s \in (2, \infty)$ is determined by

$$\frac{1}{2} = \frac{p-2}{q} + \frac{1}{s}.$$

By using the second bound we have

$$\left\| \delta \left[f''(\varphi + \theta v) - f''(\varphi) \right](\nabla\varphi, v) \right\|_{L^2} \lesssim \|v\|_{L^\infty}^{p-1} \|\nabla\varphi\|_{L^2}.$$

Taking the geometric mean of these two estimates, we get

$$\left\| \delta \left[f''(\varphi + \theta v) - f''(\varphi) \right](\nabla\varphi, v) \right\|_{L^2} \lesssim 2^{(-p+2)j/2} \left[\|\varphi\|_{H^2} + \|v\|_{H^{1+\varepsilon}} \right]^{(p-2)/2} \|\varphi\|_{H^2} \|v\|_{H^{1+\varepsilon}}^{p/2},$$

and since $\varepsilon < (p-2)/2$, its contribution to the Besov norm is bounded. $\square$

Next, we estimate the growth rate of C^0-semigroup $e^{t\mathcal{L}}$.

Lemma 14. *Assume $N \leqslant 2$ and $2 < p < 1 + 4/N$. Let α and s are numbers defined by* (4.11). *Let λ_γ be the positive eigenvalue of $\mathcal{L}$ given in Proposition* 11. *Then, there exists $C_1 > 0$ such that*

$$\left\| e^{t\mathcal{L}} \mathbf{v} \right\|_{H^s} \leqslant C_1 e^{(1+\alpha)\lambda_\gamma t} \|\mathbf{v}\|_{H^s}$$

for all $t \geqslant 0$ and $\mathbf{v} \in H^s(\mathbb{R}^N, \mathbb{C}^2)$.

Proof. By the spectral mapping theorem by Gesztesy, Jones, Latushkin and Stanislavova [11], we have $\sigma(e^{\mathcal{L}}) = e^{\sigma(\mathcal{L})}$. Thus, by Proposition 11, the spectral radius of $e^{\mathcal{L}}$ is e^{λ_γ}, and by Lemma 3 of [19], there exists $C_2 > 0$ such that

$$\left\| e^{t\mathcal{L}} \mathbf{v} \right\|_{L^2} \leqslant C_2 e^{(1+\alpha)\lambda_\gamma t} \|\mathbf{v}\|_{L^2} \tag{4.24}$$

for all $t \geqslant 0$ and $\mathbf{v} \in L^2(\mathbb{R}^N, \mathbb{C}^2)$. Since $\|(\mathcal{L} - ia)\mathbf{v}\|_{L^2}$ is equivalent to $\|\mathbf{v}\|_{H^2}$ for a sufficiently large $a > 0$, it follows from (4.24) that there exists $C_3 > 0$ such that

$$\left\| e^{t\mathcal{L}} \mathbf{v} \right\|_{H^2} \leqslant C_3 e^{(1+\alpha)\lambda_\gamma t} \|\mathbf{v}\|_{H^2} \tag{4.25}$$

for all $t \geqslant 0$ and $\mathbf{v} \in H^2(\mathbb{R}^N, \mathbb{C}^2)$. By interpolating (4.24) and (4.25), we obtain the desired estimate in $H^s(\mathbb{R}^N)$. $\square$

Lemma 15. *Assume $N \leqslant 2$ and $2 < p < 1 + 4/N$. Let α and s are numbers defined by* (4.11). *Let λ_γ be the positive eigenvalue of $\mathcal{L}$ given in Proposition* 11, *and $\mathbf{z}_\gamma = (\zeta_\gamma, 0)$ be the corresponding eigenvector with $\|\mathbf{z}_\gamma\|_{H^s} = 1$. Let*

$$\varepsilon_0 = \min\left\{ \rho_0, \left(\frac{\alpha\lambda_\gamma \|\mathbf{z}_\gamma\|_{L^2}}{4C_0C_1} \right)^{1/2\alpha} \right\}. \tag{4.26}$$

For $\delta > 0$, let

$$T_\delta = \frac{1}{\lambda_\gamma} \log \frac{\varepsilon_0}{2\delta}, \tag{4.27}$$

and let $\mathbf{v}_\delta(t)$ be the solution of (4.5) *with $\mathbf{v}_\delta(0) = \delta\mathbf{z}_\gamma$. Then we have*

$$\left\| \mathbf{v}_\delta(t) \right\|_{H^s} \leqslant 2\delta e^{\lambda_\gamma t}, \tag{4.28}$$

$$\left\| \mathbf{v}_\delta(t) - \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma \right\|_{H^s} \leqslant \frac{\delta}{2} e^{\lambda_\gamma t} \|\mathbf{z}_\gamma\|_{L^2} \tag{4.29}$$

for all $0 \leqslant t \leqslant T_\delta$.

Proof. Note that $2\delta e^{\lambda_\gamma T_\delta} = \varepsilon_0$, and that $\mathbf{v}_\delta(t)$ satisfies the integral equation

$$\mathbf{v}_\delta(t) = \delta e^{\lambda_\gamma t} \mathbf{z}_\gamma + \int_0^t e^{(t-\tau)\mathcal{L}} F\left(\mathbf{v}_\delta(\tau)\right) d\tau. \tag{4.30}$$

Let $\tilde{T}$ be the supremum of T such that (4.28) holds for all $0 \leqslant t \leqslant T$. Suppose that $\tilde{T} < T_\delta$. Then, for $0 \leqslant t \leqslant \tilde{T}$, we have $\|\mathbf{v}_\delta(t)\|_{H^s} \leqslant \varepsilon_0 < \rho_0$, and by (4.13), (4.26) and Lemma 14, we have

$$\begin{aligned}\left\| \int_0^t e^{(t-\tau)\mathcal{L}} F\big(\mathbf{v}_\delta(\tau)\big)\, d\tau \right\|_{H^s} &\leqslant C_0 C_1 \int_0^t e^{(1+\alpha)\lambda_\gamma (t-\tau)} \big\|\mathbf{v}_\delta(\tau)\big\|_{H^s}^{1+2\alpha}\, d\tau \\ &\leqslant C_0 C_1 e^{(1+\alpha)\lambda_\gamma t} (2\delta)^{1+2\alpha} \int_0^t e^{\alpha\lambda_\gamma \tau}\, d\tau \\ &\leqslant \frac{2C_0C_1}{\alpha\lambda_\gamma} \big(2\delta e^{\lambda_\gamma t}\big)^{2\alpha} \delta e^{\lambda_\gamma t} \leqslant \frac{\delta}{2} e^{\lambda_\gamma t} \|\mathbf{z}_\gamma\|_{L^2}.\end{aligned}$$

Moreover, for $0 \leqslant t \leqslant \tilde{T}$, we have

$$\begin{aligned}\big\|\mathbf{v}_\delta(t)\big\|_{H^s} &\leqslant \big\|\delta e^{\lambda_\gamma t}\mathbf{z}_\gamma\big\|_{H^s} + \left\| \int_0^t e^{(t-\tau)\mathcal{L}} F\big(\mathbf{v}_\delta(\tau)\big)\, d\tau \right\|_{H^s} \\ &\leqslant \delta e^{\lambda_\gamma t}\|\mathbf{z}_\gamma\|_{H^s} + \frac{\delta}{2} e^{\lambda_\gamma t}\|\mathbf{z}_\gamma\|_{L^2} \leqslant \frac{3}{2}\delta e^{\lambda_\gamma t} < 2\delta e^{\lambda_\gamma t}.\end{aligned}$$

This contradicts the definition of $\tilde{T}$. Thus, we have $T_\delta \leqslant \tilde{T}$, and by the above estimates, we see that (4.28) and (4.29) hold for all $0 \leqslant t \leqslant T_\delta$. $\square$

Proof of Theorem 3(ii). We use the notation in Lemma 15. For $\delta > 0$, let $\mathbf{v}_\delta(t) = (v_{\delta,1}(t), v_{\delta,2}(t))$ be the solution of (4.5) with $\mathbf{v}_\delta(0) = \delta\mathbf{z}_\gamma = \delta(\zeta_\gamma, 0)$, and let

$$\vec{u}_\delta(t) = \big(e^{i\omega t} v_{\delta,1}(t), e^{i\omega t} v_{\delta,1}(t), e^{2i\omega t}\big(\varphi + v_{\delta,2}(t)\big)\big).$$

Then, $\vec{u}_\delta(t)$ is the solution of (1.8)–(1.10) with $\vec{u}_\delta(0) = (0,0,\varphi) + \delta(\zeta_\gamma, \zeta_\gamma, 0)$. By Lemma 15, we have

$$\big\|\mathbf{v}_\delta(T_\delta) - \delta e^{\lambda_\gamma T_\delta}\mathbf{z}_\gamma\big\|_{L^2}^2 \leqslant \frac{\delta^2}{4} e^{2\lambda_\gamma T_\delta} \|\mathbf{z}_\gamma\|_{L^2}^2,$$

which provides, by expanding $\|\mathbf{v}_\delta(T_\delta) - \delta e^{\lambda_\gamma T_\delta}\mathbf{z}_\gamma\|_{L^2}^2$

$$\Re\big(v_{\delta,1}(T_\delta), \zeta_\gamma\big)_{L^2} = \Re\big(\mathbf{v}_\delta(T_\delta), \mathbf{z}_\gamma\big)_{L^2} \geqslant \frac{3}{8}\delta^2 e^{2\lambda_\gamma T_\delta}\|\mathbf{z}_\gamma\|_{L^2}^2 = \frac{3}{32}\|\zeta_\gamma\|_{L^2}^2 \varepsilon_0^2,$$

and by the Schwarz inequality we have

$$\inf_{\theta\in\mathbb{R},\, y\in\mathbb{R}^N} \big\|\vec{u}_\delta(T_\delta) - \big(0,0,e^{i\theta}\varphi(\cdot + y)\big)\big\|_{H^1} \geqslant \big\|v_{\delta,1}(T_\delta)\big\|_{L^2} \geqslant \frac{3}{32}\|\zeta_\gamma\|_{L^2}\varepsilon_0^2.$$

Since $\|\vec{u}_\delta(0) - (0,0,\varphi)\|_{H^1} = \delta\|(\zeta_\gamma, \zeta_\gamma, 0)\|_{H^1} \to 0$ as $\delta \to 0$, this means that $(0,0,e^{2i\omega t}\varphi)$ is orbitally unstable. $\square$

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