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Ann. I. H. Poincaré – AN 26 (2009) 1953–1970

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Partial regularity results up to the boundary for harmonic maps into a Finsler manifold

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Received 8 July 2008; accepted 11 May 2009

Available online 27 May 2009

Abstract

We study the energy functional for maps from a Riemannian m-manifold M into a Finsler space $N = (\mathbb{R}^n, F)$. Under the restriction $2 \leqslant m \leqslant 4$, we prove the full Hölder regularity of weakly harmonic maps (i.e., weak solutions of its Euler–Lagrange equation) from M to N in the case that the Finsler structure $F(u, X)$ depends only on vectors X, and a partial Hölder regularity of energy minimizing maps in general cases.

Résumé

Nous étudions la fonctionnelle d'énergie pour les applications d'une variété riemannienne M dans un espace de Finsler $N = (\mathbb{R}^n, F)$. Sous la restriction $2 \leqslant m \leqslant 4$, nous prouvons la régularité de Hölder complète des applications faiblement harmoniques (i.e. solutions faibles de son équation d'Euler–Lagrange) de M à N dans le cas où la structure de Finsler $F(u, X)$ dépend seulement des vecteurs X, et nous prouvons une régularité de Hölder partielle des minimiseurs de l'énergie dans le cas général.

MSC: primary 49N60, 58E20; secondary 35B65, 53C60

Keywords: Harmonic map; Finsler manifold; Partial regularity

1. Introduction

In this paper we study regularity problems of energy minimizing and weakly harmonic maps from a Riemannian manifold into a Finsler manifold.

Since the pioneering work of J. Eells and J.H. Sampson [6] in 1964, *harmonic maps* between Riemannian manifolds have attracted great interest of nonlinear analysts as well as geometers. *Harmonic maps* between Riemannian manifolds are defined as follows. Let (M, g) and (N, h) be Riemannian m- and n-manifolds, respectively. The *energy density* of a map $u : (M, g) \to (N, h)$ is a function $e(u) : M \to \mathbb{R}$ defined by

$$e(u)(x) := \frac{1}{2}\left|du(x)\right|^2, \quad x \in M,$$

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2009.05.001

where $|du|$ denotes the Hilbert–Schmidt norm of $du(x) \in T_x^*M \otimes T_{u(x)}N$, namely, for an orthonormal basis $(e_1, \dots, e_m)$ of T_xM, $e(u)(x)$ can be written as

$$e(u)(x) = \sum_{\alpha=1}^{m} \big\| \big(du(x)\big)(e_\alpha) \big\|^2_{T_{u(x)}N}. \tag{1.1}$$

For a bounded domain Ω in M, *the energy* of u on Ω is defined by

$$E(u, \Omega) := \int_\Omega e(u)\, d\mu,$$

where $d\mu$ stands for the volume element on M. *Harmonic maps* are defined to be solutions of the Euler–Lagrange equation of the energy functional.

Finsler geometry is a natural generalization of Riemannian geometry, therefore it is very reasonable to expect to extend the notion of harmonic maps to Finsler geometry. Indeed, P. Centore [5] defined the energy and the notion of harmonicity for maps between Finsler manifolds. On the other hand, Y.-B. Shen and Y. Zhang [24] gave another definition of the energy by an integration on the sphere bundle over the source manifold. Both of these definitions of energy are extensions of the one for maps between Riemannian manifolds. We mention that their variational features are very similar at least for the case that the source manifold is Riemannian, and the results of this paper hold for both energies.

Concerning harmonic maps from Finsler manifolds into a Riemannian manifold, see, for example, X. Mo [18], X. Mo and Y. Yang [19] and recent work H. von der Mosel and S. Winklmann [26]. In [26] they obtain a priori estimates for harmonic maps with a Finsler source manifold and with small image in a Riemannian manifold.

In this paper, we discuss only the case where source manifolds are Riemannian. In the sequel, let us write shortly "Finsler case" when the target manifold is a Finsler manifold, and "Riemannian case" when the target manifold is Riemannian. Analytic features of the energy for the Finsler case are quite different from those for the Riemannian case. Indeed let (M, g) and (P, h) be Riemannian m- and n-manifolds, and let (N, F) be a Finsler n-manifold with a Finsler structure F. Using local coordinates, the energy functional for a map $u : (M, g) \to (P, h)$ can be written as

$$\int g^{\alpha\beta}(x) h_{ij}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}\, d\mu.$$

On the other hand, for a map $u : (M, g) \to (N, F)$, the energy should be expressed in the following form:

$$\int E_{ij}^{\alpha\beta}(x, u, Du) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}\, d\mu.$$

(See (2.6) below.) For the Finsler case the coefficients of the integrand depend also on the derivatives of u, whereas they depend only on x and u for the Riemannian case. Consequently, in the Finsler case, the nonlinearity of the Euler–Lagrange equation is much more subtle than that in the Riemannian case, and therefore it would be harder to obtain regularity results for *harmonic maps* and *energy minimizing maps*. One could also mention that with a Riemannian target the underlying pde decomposes into an elliptic system of diagonal form. This is another reason why life is so much harder in the Finsler setting.

For the Riemannian case, many results have been known on regularity of harmonic maps. Some of them are indeed best possible. See, for example, [15,8,11,22,23] and the references therein.

When investigating the regularity of harmonic maps into a Finsler manifold (N, F), the difficulty arises not only from the nonlinearity but also from the singularity of the Finsler structure $F(u, X)$. This singular character of Finsler metrics is inevitable, since it is known that if the second derivatives of $F(u, X)$ with respect to X are continuous at $X = 0$, then (N, F) should be nothing but a Riemannian manifold with the metric tensor

$$\frac{\partial^2 F^2}{\partial X^i \partial X^j}(u, 0).$$

In view of this, we cannot assume that $F(u, \cdot) \in C^2$, and therefore, many known regularity results for minimizers or critical maps of general variational integrals cannot be applied directly to harmonic maps into a Finsler manifold.

In [25], the author proves interior partial Hölder regularity of energy minimizing maps from the Euclidean space $\mathbb{R}^m$ into a Finsler space $(\mathbb{R}^n, F)$, under the restriction that $m \leqslant 4$, using Sobolev's imbedding theorem and the *direct approach*. Here, we call an approach getting (partial) regularity by a perturbation argument the *direct approach*. (See [7, Chapter VI].)

In the present paper, under the same restriction on the dimension m of the source manifold, we modify Campanato's method to prove the Hölder regularity of *weakly harmonic maps* into a Finsler space $(\mathbb{R}^n, F)$ where the given Finsler structure $F(u, X)$ depends only on the vector variable X. Using direct approach, we also obtain partial regularity of energy minimizing maps into a general Finsler space.

Recently, the importance of harmonic maps in Finsler geometry has been recognized and several authors investigated them. For example, in [20,21], S. Nishikawa introduced harmonic maps in complex Finsler geometry and pointed out their importance.

It will be worth to consider Finsler geometric variational problems from the viewpoint of applied mathematics as well. There are many variational and evolution problems which cannot be regarded as problems from Riemannian geometry, and some of them can be considered in the context of Finsler geometry. For example, many physical and biological applications of Finsler geometry are introduced by P.L. Antonelli, R.S. Ingarten and M. Matsumoto in [1]. We mention also that G. Bellettini and M. Paolini [2] studied a problem of anisotropic motion in the context of Finsler geometry.

2. Definitions and main results

Let N be an n-dimensional C^∞ manifold. Denote by TN the tangent bundle of N, and by T^*N its dual. We write each point in TN as (u, X) with $u \in N$ and $X \in T_uN$. The natural projection $\pi : TN \to N$ is given by $\pi(u, X) = u$.

We put

$$TN \setminus 0 := \{(u, X) \in TN;\ X \neq 0\}.$$

$TN \setminus 0$ is called the *slit tangent bundle* of N. The *projective sphere bundle* SN of N is defined by identifying positive multiples of a vector in each fiber of $TN \setminus 0$, that is,

$$SN = (TN \setminus 0)/\sim,$$

where $(u, X) \sim (v, Y)$ if and only if $u = v$ and $X = \lambda Y$ for some $\lambda > 0$. Given $(u, X) \in TN \setminus 0$, we denote its equivalence class by $(u, [X])$. The natural projection $p : SN \to N$ is given by $p(u, [X]) = u$.

In Finsler geometry, it is often convenient to handle the tensor calculus in bundles over TN, $TN \setminus 0$ or SN. Especially, the pull back bundles $p^*TN \to SM$, $\pi^*TN \to TN \setminus 0$, and their dual bundles $p^*T^*N \to SN$, $\pi^*T^*N \to TN \setminus 0$ are used quite frequently.

A Finsler structure of N is a function $F : TN \to [0, \infty)$ satisfying the following properties:

(i) *Regularity*: $F \in C^\infty(TN \setminus 0)$.
(ii) *Homogeneity*: $F(u, \lambda X) = \lambda F(u, X)$ for all $\lambda \geqslant 0$.
(iii) *Convexity*: The Hessian matrix of F^2 with respect to X

$$\big(h_{ij}(u, X)\big) = \left(\frac{1}{2}\frac{\partial^2 F^2(u, X)}{\partial X^i \partial X^j}\right)$$

is positive definite at every point $(u, X) \in TN \setminus 0$.

We call the pair (N, F) *a Finsler manifold*, and h_{ij} the *fundamental tensor* of (N, F). By virtue of positive homogeneity of F, we have

$$h_{ij}(x, \lambda X) = h_{ij}(u, X), \quad \lambda > 0, \tag{2.1}$$

and obtain by Euler's theorem that

$$F^2(u, X) = h_{ij}(u, X)X^i X^j. \tag{2.2}$$

Identities (2.1) and (2.2) imply that the fundamental tensor $h_{ij}(u,X)$ is well defined for $(u,[X]) \in SN$ and that it defines a bundle metric on the pull-back bundle p^*TN over SN, as well as on π^*TN over $TN \setminus 0$.

In [5], P. Centore defined the *energy* functional for maps between Finsler manifolds. According to his definition we define the energy density $e_C(u)$ of a map u from a Riemannian into a Finsler manifold as follows. Let (M,g) be a smooth Riemannian m-manifold and (N,F) a Finsler n-manifold. Let I_xM be the indicatrix of g at $x \in M$, namely,

$$I_xM := \{\xi \in T_xM;\ \|\xi\|_g \leqslant 1\}.$$

For a C^1-map $u: M \to N$ we define the *energy density* $e_C(u)(x)$ of u at $x \in M$ and the *energy functional* $E_C(u)$ by

$$e_C(u)(x) := \frac{\int_{I_xM} (u^*F)^2(\xi)\,d\xi}{\int_{I_xM} d\xi}, \tag{2.3}$$

$$E_C(u) := \int_M e_C(u)(x)\,d\mu, \tag{2.4}$$

where u^*F denotes the pull-back of F by u, and $d\mu$ the measure deduced from g.

Moreover, Y.-B. Shen and Y. Zhang [24] gave another definition of the energy of a map between Finsler manifolds by integration on the sphere bundle SM. They consider the differential du of a map $u: M \to N$ as a map from SM to SN and define the energy density of u at $(x,[\xi]) \in SM$ by

$$e_{SZ}(u)\big(x,[\xi]\big) := \frac{1}{2} g^{\alpha\beta}(x) h_{ij}\big(u(x),\big[du_x(\xi)\big]\big)\frac{\partial u^i}{\partial x^\alpha}(x)\frac{\partial u^j}{\partial x^\beta}(x).$$

The energy $E_{SZ}(u)$ of a map $u:(M,g) \to (N,h)$ is defined to be

$$E_{SZ}(u) := \frac{1}{\mathrm{vol}(S^{m-1})}\int_{SM} e_{SZ}(u)\,d\mu_{SM},$$

where μ_{SM} denotes the measure on SM deduced from the Sasaki metric on $TM \setminus 0$. See, for details, [21,24].

As in the Riemannian case, a (weak) solutions of the Euler–Lagrange equation of the energy functionals E_C or E_{SZ} is called a (*weakly*) *harmonic map*.

As pointed out by S. Nishikawa [21], using an orthonormal frame for the tangent bundle TM, we can write E_C and E_{SZ} in similar forms. Let us take an orthonormal frame $\{e_\alpha\}$ for the tangent bundle TM of M, given in local coordinates by

$$e_\alpha = \eta_\alpha^\kappa(x)\frac{\partial}{\partial x^\kappa}, \quad 1 \leqslant \alpha \leqslant m.$$

Using $\{e_\alpha\}$, we identify each fiber S_xM of SM and the indicatrix I_xM at $x \in M$ with the unit Euclidean $(m-1)$-sphere S^{m-1} and the unit Euclidean m-ball B^m, respectively. Then, by virtue of the identity

$$g^{\kappa\nu}(x) = \eta_\alpha^\kappa(x)\delta^{\alpha\beta}\eta_\beta^\nu(x),$$

we can write E_C and E_{SZ} as

$$E_C(u) = \int_M \left(\frac{1}{|B^m|}\int_{B^m} h_{ij}\big(u(x),du_x(\xi)\big)\xi^\kappa\xi^\nu\,d\xi\right)\eta_\kappa^\alpha\eta_\nu^\beta D_\alpha u^i D_\beta u^j\,d\mu,$$

$$E_{SZ}(u) = \int_M \left(\frac{1}{2|S^{m-1}|}\int_{S^{m-1}} h_{ij}\big(u(x),\big[du_x(\xi)\big]\big)\delta^{\kappa\nu}\,d\xi\right)\eta_\kappa^\alpha\eta_\nu^\beta D_\alpha u^i D_\beta u^j\,d\mu,$$

where $D_\alpha u^i = \partial u^i/\partial x^\alpha$. Although the terms in parentheses are not defined at points x where $du_x = 0$, we can define them to be arbitrary numbers without changing the values of the integrands $(\cdots)\eta_\kappa^\alpha\eta_\nu^\beta D_\alpha u^i D_\beta u^j$, because the integrands are equal to 0, being independent on the values of h_{ij} when $du_x = 0$. So, here and in the sequel, we regard $h_{ij}(u,X)$ and $h_{ij}(u,[X])$ as being defined also for $X = 0$.

Although E_{SZ} is very similar to E_C in appearance, there is a crucial difference between them. In general, when we consider regularity problems of minimizers for functionals defined as

$$\int G(x,u,Du)\,dx, \qquad \big(\Omega\subset\mathbb{R}^m,\ G(x,u,p):\Omega\times\mathbb{R}^n\times\mathbb{R}^{mn}\to\mathbb{R}\big),$$

the ellipticity of the Hessian $\partial^2 G(x,u,p)/\partial p^i_\alpha\partial p^j_\beta$ plays a very important role. For Centore's energy E_C we can deduce the ellipticity from the convexity condition (iii) on the Finsler structure. On the other hand, for E_{SZ} the condition (iii) does not imply the corresponding ellipticity. (See Lemma 3.1 and Remark 3.2.) For this technical reason we treat in this paper only Centore's energy.

For $(x,u,p)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{mn}$, let us define the quantities $E^{\alpha\beta}_{ij}(x,u,p)$ by

$$\left(\frac{1}{|B^m|}\int\limits_{B^m} h_{ij}(u,p\xi)\xi^\kappa\xi^\nu\,d\xi\right)\eta^\alpha_\kappa(x)\eta^\beta_\nu(x)\sqrt{\det\big(g_{\alpha\beta}(x)\big)}. \tag{2.5}$$

Since we are assuming smoothness of the manifolds, $E^{\alpha\beta}_{ij}(x,u,p)$ are smooth in $\mathbb{R}^m\times\mathbb{R}^n\times(\mathbb{R}^{mn}\setminus 0)$.

In order to consider boundary value problems on a bounded domain $\Omega\subset M$, let us define the energies on Ω, $E_C(u;\Omega)$ by

$$\int\limits_\Omega E^{\alpha\beta}_{ij}\big(x,u(x),Du(x)\big)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\,dx, \tag{2.6}$$

where $Du=(D_\alpha u^i)=(\partial u^i/\partial x^\alpha)$. In the sequel, by a "*weakly harmonic map*" or an "*energy minimizing map*" we mean the corresponding notions with respect to E_C only.

In this paper we treat only the cases in which the source manifold is a Riemannian m-manifold with $2\leqslant m\leqslant 4$ and the target manifold $(N,F)=(\mathbb{R}^n,F)$. Moreover, we assume that there exist positive constants $\lambda<\Lambda$ and a concave increasing function ω with $\lim_{t\to+0}\omega(t)=0$ such that

$$\lambda|\xi|^2\leqslant h_{ij}(u,X)\xi^i\xi^j=\frac{1}{2}\frac{\partial^2F^2(u,X)}{\partial X^i\partial X^j}\xi^i\xi^j\leqslant\Lambda|\xi|^2, \tag{2.7}$$

$$\big|F^2(u,X)-F^2(v,X)\big|\leqslant\omega\big(|u-v|^2\big)|X|^2 \tag{2.8}$$

hold for any $u,v\in\mathbb{R}^n$ and $(X,\xi)\in(\mathbb{R}^n\setminus 0)\times\mathbb{R}^n$.

Under these assumptions, the author proved interior partial $C^{0,\alpha}$-regularity in [25]. Namely, he proved that a minimizer of Centore's energy $E_C(u;\Omega)$ is Hölder continuous on an open subset $\Omega_0\subset\Omega$ which satisfies $\mathcal{H}^{m-2-\delta}(\Omega\setminus\Omega_0)=0$ for some $\delta>0$. Here and in the sequel $\mathcal{H}^q$ denote the q-dimensional Hausdorff measure. In this paper we prove everywhere Hölder regularity of weakly harmonic maps when $F(u,X)$ depends only on the vector X, and partial Hölder regularity of energy minimizing maps up to boundary for general cases. More precisely, we show the following results.

Theorem 2.1. *Let (M,g) be a Riemannian m-manifold of class C^3 and $\Omega\subset M$ a bounded domain with smooth boundary $\partial\Omega$ and $(\mathbb{R}^n,F)$ a Finsler space with the Finsler structure $F(u,X)$ which is independent on $u\in\mathbb{R}^n$ and satisfies* (2.7). *Suppose that $2\leqslant m\leqslant 4$ and $f\in H^{1,s}(\Omega,\mathbb{R}^n)$ for some $s>m$. Then every weakly harmonic map $v\in H^{1,2}(\Omega,\mathbb{R}^n)$ with boundary value f is in the class $C^{0,\alpha}$ up to the boundary for some $\alpha\in(0,1)$.*

Theorem 2.2. *Let (M,g) be a Riemannian m-manifold of class C^3 and $\Omega\subset M$ a bounded domain with smooth boundary $\partial\Omega$ and $(\mathbb{R}^n,F)$ a Finsler space with the Finsler structure F satisfying* (2.7) *and* (2.8). *Suppose that $2\leqslant m\leqslant 4$ and that $f\in H^{1,s}(\Omega,\mathbb{R}^n)$ for some $s>m$. Let $u\in H^{1,2}(\Omega,\mathbb{R}^n)$ be an energy minimizing map in the class*

$$H^{1,2}_f\big(\Omega,\mathbb{R}^n\big):=\big\{v\in H^{1,2}\big(\Omega,\mathbb{R}^n\big);\ v-f\in H^{1,2}_0\big(\Omega,\mathbb{R}^n\big)\big\}.$$

Then, there exists a relatively open subset $\Omega_0\subset\overline{\Omega}$ such that $C^{0,\alpha}(\Omega_0,\mathbb{R}^n)$ for some $\alpha\in(0,1)$. Moreover, $\mathcal{H}^{m-2-\delta}(\overline{\Omega}\setminus\Omega_0)=0$ for some $\delta>0$.

3. Preliminary results

If u is a weakly harmonic map or a minimizer of the energy functional on $\Omega \subset M$, then u has the same property on every coordinate neighborhood. On the other hand the regularity is a local property. So it suffices to study the problem on a domain $\Omega \subset \mathbb{R}^m$.

Let $E_{ij}^{\alpha\beta}$ be defined by (2.5). For $x \in \Omega$, $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^{mn}$ put

$$A(x,u,p) = E_{ij}^{\alpha\beta}(x,u,p)p_\alpha^i p_\beta^j. \tag{3.1}$$

Then, we can regard $A(x,u,p)$ as a function defined on $\mathcal{X} = \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$. Put $\mathcal{X}' = \Omega \times \mathbb{R}^n \times (\mathbb{R}^{mn} \setminus \{0\})$.

Lemma 3.1. *Let (M,g) is a Riemannian m-manifold of class C^3, (N,F) be a Finsler space with a Finsler structure which satisfies* (i)–(iii) *and let $A(x,u,p)$ be as above. Then $A(x,u,p)$ satisfies*

$$A(x,u,p) \in C^{1,1}(\mathcal{X}) \cap C^3(\mathcal{X}'). \tag{3.2}$$

Moreover, there exist positive constants $\lambda_0 < \Lambda_0$ such that

$$\lambda_0|p|^2 \leqslant A(x,u,p) \leqslant \Lambda_0|p|^2 \quad \text{for all } (x,u,p) \in \mathcal{X}, \tag{3.3}$$

$$\lambda_0|\xi|^2 \leqslant A_{p_\alpha^i p_\beta^j}(x,u,p)\xi_\alpha^i\xi_\beta^j \leqslant \Lambda_0|\xi|^2 \quad \text{for all } (x,u,p,\xi) \in \mathcal{X}' \times \mathbb{R}^{mn}. \tag{3.4}$$

Here and in the sequel, A_{x^γ}, A_{u^i}, $A_{p_\alpha^i}$, $A_{p_\alpha^i p_\beta^j}$, etc., denote partial derivatives.

Proof. Since (M,g) is assumed to be a Riemannian manifold of class C^3, by the assumption on F, we see that $A(x,u,p)$ is in the class $C^3(\mathcal{X}')$. Moreover, by virtue of the 2-homogeneity of F or, equivalently, 0-homogeneity of h implies that $A \in C^{1,1}$ by direct calculation on its partial derivatives.

The coercivity and the boundedness (3.3) is a direct consequence of (2.7).

In order to prove (3.4) we use the special structure of e_C. Let us choose a normal coordinate system centered at x. Then $g^{\kappa\nu}(x) = \delta^{\kappa\nu}$ and therefore $\eta_\alpha^\kappa = \delta_\alpha^\kappa$. So, we can see that

$$E_{ij}^{\alpha\beta}(x,u,p) = \left(\frac{1}{|B^m|}\int_{B^m} h_{ij}(u,p\xi)\xi^\kappa\xi^\nu\,d\xi\right)\delta_\kappa^\alpha\delta_\nu^\beta,$$

and that

$$A(x,u,p) = \left(\frac{1}{|B^m|}\int_{B^m} h_{ij}(u,p\xi)\xi^\alpha\xi^\beta\,d\xi\right)p_\alpha^i p_\beta^j. \tag{3.5}$$

Now, mention that the 0-homogeneity of h_{ij} implies the equality

$$X^k\frac{\partial h_{ij}(u,X)}{\partial X^k} = 0.$$

On the other hand, since $2h_{ij}(u,X) = \partial^2F^2(u,X)/\partial X^i\partial X^j$, we see that

$$\frac{\partial h_{ij}}{\partial X^k} = \frac{\partial h_{ik}}{\partial X^j} = \frac{\partial h_{kj}}{\partial X^i}.$$

So, we have

$$0 = \frac{\partial h_{ij}(u,X)}{\partial X^k}X^k = \frac{\partial h_{ik}(u,X)}{\partial X^j}X^k = \frac{\partial h_{kj}(u,X)}{\partial X^i}X^k. \tag{3.6}$$

From (3.6), we get the following important relation.

$$\frac{\partial h_{ij}}{\partial X^k}(u,p\xi)\xi^\alpha\xi^\beta p_\alpha^i p_\beta^j = 0. \tag{3.7}$$

Similarly, about second derivatives,

$$\frac{\partial^2 h_{ij}}{\partial X^k \partial X^l}(u, p\xi)\xi^\alpha \xi^\beta p^i_\alpha p^j_\beta = 0 \tag{3.8}$$

holds. By virtue of (3.7) and (3.8) when one calculates $A_{p^i_\alpha p^j_\beta}(x, u, p)$ all terms which include derivatives of $E^{\alpha\beta}_{ij}$ vanish. So, we obtain

$$A_{p^i_\alpha p^j_\beta}(x, u, p) = 2E^{\alpha\beta}_{ij}(x, u, p). \tag{3.9}$$

Now, (2.7) and (3.9) imply (3.4) immediately. □

Remark 3.2. For the energy E_{SZ}, because of lack of $\xi^\kappa \xi^\nu$ (there is $\delta^{\kappa\nu}$ instead of it), (3.7) and (3.8) do not imply the equality corresponding to (3.9). Namely, when we put

$$A_{SZ}(x, u, p) := \left(\frac{1}{2|S^{m-1}|} \int_{S^{m-1}} h_{ij}\big(u(x), \big[p(\xi)\big]\big)\delta^{\kappa\nu}\, d\xi\right)\eta^\alpha_\kappa \eta^\beta_\nu p^i_\alpha p^j_\beta,$$

and calculate $(A_{SZ})_{p^i_\alpha p^j_\beta}$ using a normal coordinate system centered at x as above, the terms including derivatives of

$$(E_{SZ})^{\alpha\beta}_{ij} := \left(\frac{1}{2|S^{m-1}|} \int_{S^{m-1}} h_{ij}\big(u(x), [p\xi]\big)\delta^{\alpha\beta}\, d\xi\right)$$

do not vanish. So, without considering further assumptions on the derivatives of h_{ij}, we cannot obtain the ellipticity.

Since we assume that F satisfies (2.8) and Ω is bounded, there exists a concave, nondecreasing function ω with $\lim_{t\to 0}\omega(t) = 0$ such that

$$\big|A(x, u, p) - A(y, v, p)\big| \leqslant \omega\big(|x-y|^2 + |u-v|^2\big)|p|^2. \tag{3.10}$$

When A does not depend on u, instead of (3.10), (2.8) and homogeneity of F imply that

$$\big|A_{p^i_\alpha}(x, p) - A_{p^i_\alpha}(y, p)\big| \leqslant \omega\big(|x-y|^2\big)|p|. \tag{3.11}$$

Using $A(x, u, p)$, we can express the energy functional as

$$E_C(u; \Omega) := \int_\Omega A(x, u, Du)\, dx. \tag{3.12}$$

In the sequel, we use the following notation:

$$A^{\alpha\beta}_{ij}(x, u, p) := A_{p^i_\alpha p^j_\beta}(x, u, p) \quad \text{for } p \neq 0, \tag{3.13}$$

$$\hat{A}^{\alpha\beta}_{ij}(x, u, p) := \begin{cases} A^{\alpha\beta}_{ij}(x, u, p) & \text{for } p \neq 0, \\ \delta^{\alpha\beta}\delta_{ij} & \text{for } p = 0, \end{cases} \tag{3.14}$$

$$\check{A}^{\alpha\beta}_{ij}(x, u, p) := \begin{cases} A^{\alpha\beta}_{ij}(x, u, p) & \text{for } p \neq 0, \\ 0 & \text{for } p = 0. \end{cases} \tag{3.15}$$

For some fixed $x_0 \in \Omega$ and $R > 0$ we will write

$$Q(x_0, R) = \big\{x \in \mathbb{R}^m;\ \big|x^\alpha - x^\alpha_0\big| < R,\ \alpha = 1, \ldots, m\big\},$$
$$\Omega(x_0, R) := Q(x_0, R) \cap \Omega,$$

and for a function w defined on Ω

$$w_R := \fint_{\Omega(x_0,R)} w\,dx = \frac{1}{\mathcal{L}^m(\Omega(x_0,R))} \int_{\Omega(x_0,R)} w\,dx,$$

where $\mathcal{L}^m$ is the m-dimensional Lebesgue measure.

Since $A_{p_i^\alpha}(x,u,p) \in C^{0,1}(\mathcal{X})$, we have the following lemma.

Lemma 3.3. *The following relation holds.*

$$A_{p_\alpha^i}(x,u,\eta) - A_{p_\alpha^i}(x,u,\xi) = \int_0^1 \hat{A}_{ij}^{\alpha\beta}\big(x,u,t\eta + (1-t)\xi\big)\big(\eta_\beta^j - \xi_\beta^j\big)\,dt. \tag{3.16}$$

By the chain rule for compositions of Nemitsky operators and Sobolev maps, we have the following.

Lemma 3.4. *For some fixed $(x_0,u_0) \in \Omega \times \mathbb{R}^n$, put $A^0(p) = A(x_0,u_0,p)$ and $(\check{A}^0)_{ij}^{\alpha\beta}(p) = (\check{A}^0)_{ij}^{\alpha\beta}(x_0,u_0,p)$. Then, for $v \in H^{2,2}(\Omega,\mathbb{R}^n)$, the derivatives*

$$D_\gamma A^0_{p_\alpha^i}\big(Dv(x)\big), \quad \gamma = 1,\dots,m,$$

are well defined for almost every $x \in \Omega$ and satisfy

$$D_\gamma A^0_{p_\alpha^i}\big(Dv(x)\big) = \big(\check{A}^0\big)_{ij}^{\alpha\beta}\big(Dv(x)\big) D_\gamma D_\beta v^j(x).$$

Proof. See Theorem 2.1 of [17]. □

Roughly speaking, by virtue of the above lemmata, we can proceed as in the standard theory established by S. Campanato [3,4], M. Giaquinta and E. Giusti [8–10], etc., and get regularity results.

4. A simple case

In this section we show some fundamental estimates for weak solutions of the Euler–Lagrange equation of the following simple functional defined for $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$.

$$\mathcal{A}^0(u;\Omega) = \int_\Omega A^0(Du)\,dx, \tag{4.1}$$

where A^0 is in the class $C^{1,1}(\mathbb{R}^{mn}) \cap C^3(\mathbb{R}^{mn} \setminus \{0\})$ and satisfies (3.3) and (3.4).

We mention that the assumption $m \leqslant 4$ is not used in this section. It will be used only after (5.10).

4.1. Interior estimates

Proposition 4.1. *Let $v \in H^{1,2}(\Omega,\mathbb{R}^n)$ be a solution of*

$$\int_\Omega A^0_{p_\alpha^i}(Dv) D_\alpha \varphi^i\,dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}\big(\Omega,\mathbb{R}^n\big). \tag{4.2}$$

Then there exists a positive constant ε_0 such that for any $q \in (2, 2+\varepsilon_0)$ we have $v \in H^{2,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. Moreover, for any $x \in \Omega$ and $r > 0$ with $Q(x,r) \Subset \Omega$ we have

$$\bigg(\fint_{Q(x,r/2)} \big|D^2 v\big|^q\,dy\bigg)^{1/q} \leqslant C\bigg(\fint_{Q(x,r)} \big|D^2 v\big|^2\,dy\bigg)^{1/2}. \tag{4.3}$$

Proof. Let $(e_1, \dots, e_m)$ be the standard basis of $\mathbb{R}^m$. For any $\varphi \in H^{1,2}(\Omega, \mathbb{R}^n)$ with $\operatorname{supp}\varphi \Subset \Omega$, choose $h > 0$ so that $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > h$. Then we see that

$$\int_\Omega \left\{A^0_{p^i_\alpha}\big(Dv(x + he_\gamma)\big) - A^0_{p^i_\alpha}\big(Dv(x)\big)\right\} D_\alpha \varphi^i \, dx = 0.$$

Writing

$$\tau^h_\gamma \psi(x) := \frac{1}{h}\left\{\psi(x + he_\gamma) - \psi(x)\right\},$$

and using Lemma 3.3, we have

$$\int_\Omega \left\{ \int_0^1 \big(\hat{A}^0\big)^{\alpha\beta}_{ij}\big((1-t)Dv(x) + tDv(x + he_\gamma)\big)\, dt \right\} \tau^h_\gamma D_\beta v^j(x) D_\alpha \varphi^i(x)\, dx = 0. \tag{4.4}$$

Now, fix a subdomain $D \Subset \Omega$ and a positive constant h_0 with $h_0 < \operatorname{dist}(D, \partial\Omega)/2$, and put

$$D' := \left\{x \in \Omega;\ \operatorname{dist}(x, D) < h_0\right\}.$$

Take a cut-off function $\eta \in C^\infty_0(D')$ so that $\eta \equiv 1$ on D, $0 \leqslant \eta \leqslant 1$ and $|D\eta| \leqslant 2/h_0$. For $h < h_0$, putting

$$\varphi(x) = \big(\tau^h_\gamma v(x)\big)\eta^2(x)$$

in (4.4), we get by (3.4)

$$\int_D \left|\tau^h_\gamma Dv\right|^2 dx \leqslant \frac{C}{h_0} \int_{D'} \left|\tau^h_\gamma v\right|^2 dx \leqslant \frac{C}{h_0} \int_\Omega |Dv|^2\, dx.$$

So, $D^2 v \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_D \left|D^2 v\right|^2 dx \leqslant C(D) \int_\Omega |Dv|^2\, dx.$$

Now, since we have shown that $Dv \in H^{1,2}_{\mathrm{loc}}(\Omega)$, by virtue of Lemma 3.4, we can deduce from (4.2) that

$$\int_\Omega \big(\check{A}^0\big)^{\alpha\beta}_{ij}(Dv) D_\gamma D_\beta v^i D_\alpha \varphi^j \, dx = 0 \quad \text{for } \gamma = 1, \dots, m,$$

for any $\varphi \in H^{1,2}(\Omega)$ with $\operatorname{supp}\varphi \Subset \Omega$. On the other hand, $D^2 v = 0$ for a.e. $x \in \{x \in \Omega;\ Dv(x) = 0\}$. So we see that

$$Dv(x) \neq 0 \quad \text{for a.e. } x \in \left\{x \in \Omega;\ D^2 v \neq 0\right\}.$$

This implies by definition of $\check{A}^0$ in (3.15) and (3.4) that

$$\sum_{\gamma=1}^m \big(\check{A}^0\big)^{\alpha\beta}_{ij}(Dv) D_\gamma D_\beta v^i D_\gamma D_\alpha v^j \geqslant \lambda_0 \left|D^2 v\right|^2 \quad \text{a.e. on } \Omega.$$

Thus we can see that a Caccioppoli inequality holds for Dv. Namely, we have

$$\int_{Q(x,r/2)} \left|D^2 v\right|^2(y)\, dy \leqslant \frac{C}{r^2} \int_{Q(x,r)} \left|Dv(y) - (Dv)_{x,r}\right|^2 dy \tag{4.5}$$

for any cube $Q(x, r) \Subset \Omega$. Combining (4.5) with the Sobolev–Poincaré inequality, we get

$$\left(\fint_{Q(x,r/2)} \left|D^2 v\right|^2 dy \right)^{1/2} \leqslant C \left(\fint_{Q(x,r)} \left|D^2 v\right|^{2_*} dy \right)^{1/2_*}, \tag{4.6}$$

where $2_* = 2m/(m+2)$. Now, using the reverse Hölder (or Gehring-type) inequality due to Giaquinta and Modica [13], we see that there exists a constant $\varepsilon_0 > 0$ such that, for any $q \in (2, 2+\varepsilon_0)$, $D^2 v \in L^q_{\mathrm{loc}}(\Omega)$ and

$$\left(\fint_{Q(x,r/2)} \left|D^2 v\right|^q dy \right)^{1/q} \leqslant C \left(\fint_{Q(x,r)} \left|D^2 v\right|^2 dy \right)^{1/2}$$

holds for any $Q(x,r) \Subset \Omega$. □

Remark 4.2. In general, it is hard to know the exact value of ε_0 in Proposition 4.1 and we cannot expect that it is a large number. So, in the following, we proceed assuming that it is very small.

Corollary 4.3. *Let v be as in Proposition* 4.1. *Then for any $x \in \Omega$ and ρ, r with $0 < \rho < r < \operatorname{dist}(x, \partial\Omega)/\sqrt{2}$, we have the following estimates for the case that $m \geqslant 3$.*

$$\int_{Q(x,\rho)} |Dv|^2 dy \leqslant C \left(\frac{\rho}{r} \right)^{2+m-2m/q} \int_{Q(x,r)} |Dv|^2 dy. \tag{4.7}$$

For $m = 2$ we have

$$\int_{Q(x,\rho)} |Dv|^2 dy \leqslant C \left(\frac{\rho}{r} \right)^{2-\varepsilon} \int_{Q(x,r)} |Dv|^2 dy \quad \textit{for any } \varepsilon > 0. \tag{4.8}$$

Here the constants C depend only on A, m and $\operatorname{dist}(x, \partial\Omega)$.

Proof. From (4.3) and Hölder's inequality, we see that

$$\int_{Q(x,\rho)} \left|D^2 v\right|^2 dy \leqslant C \left(\frac{\rho}{r} \right)^{m(1-2/q)} \int_{Q(x,r)} \left|D^2 v\right|^2 dy \tag{4.9}$$

for $0 < \rho < r < \operatorname{dist}(x, \partial\Omega)$. On the other hand, by Hölder and Poincaré inequalities, we have for $0 < \sigma < \operatorname{dist}(x, \partial\Omega)$ and $0 < t < \tau < 1$ that

$$\begin{aligned}
\int_{Q(x,t\sigma)} |Dv|^2 dy &\leqslant C \int_{Q(x,t\sigma)} \left|Dv - (Dv)_{x,\tau\sigma}\right|^2 dy + C(t\sigma)^m (Dv)^2_{x,\tau\sigma} \\
&\leqslant C(\tau\sigma)^2 \int_{Q(x,\tau\sigma)} \left|D^2 v\right|^2 dy + C \left(\frac{t}{\tau} \right)^m \int_{Q(x,\tau\sigma)} |Dv|^2 dy.
\end{aligned}$$

Combining the above inequality and (4.9), we get

$$\int_{Q(x,t\sigma)} |Dv|^2 dy \leqslant C \left(\frac{t}{\tau} \right)^m \int_{Q(x,\tau\sigma)} |Dv|^2 dy + C\tau^{2+m(1-2/q)}\sigma^2 \int_{Q(x,\sigma)} \left|D^2 v\right|^2 dy.$$

By the above inequality, using (4.5) with $r = 2\sigma$, we obtain

$$\int_{Q(x,t\sigma)} |Dv|^2 dy \leqslant C \left(\frac{t}{\tau} \right)^m \int_{Q(x,\tau\sigma)} |Dv|^2 dy + C\tau^{2+m-2m/q} \int_{Q(x,2\sigma)} |Dv|^2 dy. \tag{4.10}$$

If $m \geqslant 3$, then $2 + m - 2m/q < m$. So, using [12, Lemma 5.12], we have

$$\begin{aligned}\int\limits_{Q(x,t\sigma)} |Dv|^2\,dy &\leqslant C\left[\left(\frac{1}{\tau}\right)^{2+m-2m/q} \int\limits_{Q(x,\tau\sigma)} |Dv|^2\,dy + \int\limits_{Q(x,2\sigma)} |Dv|^2\,dy\right] t^{2+m-2m/q}\\ &\leqslant C\left[\left\{\left(\frac{1}{\tau}\right)^{2+m-2m/q} + 1\right\} \int\limits_{Q(x,2\sigma)} |Dv|^2\,dy\right] t^{2+m-2m/q}.\end{aligned}$$

Taking, for example, $\tau = 1/2$ we get

$$\int\limits_{Q(x,t\sigma)} |Dv|^2\,dy \leqslant Ct^{2+m-2m/q} \int\limits_{Q(x,2\sigma)} |Dv|^2\,dy$$

for $t \in (0, 1/2)$. Putting $2\sigma = r$, the above estimate implies (4.7) for $\rho < r/4$. On the other hand, for $\rho \in [r/4, r)$, (4.7) holds if we choose the constant C so that $C \geqslant 4^{2+2m-2m/q}$. Thus, taking C sufficiently large, we have (4.7) for any $\rho \in (0, r)$. When $m = 2$, since $\tau < 1$, (4.10) implies that for any $\varepsilon > 0$

$$\int\limits_{Q(x,t\sigma)} |Dv|^2\,dy \leqslant C\left(\frac{t}{\tau}\right)^2 \int\limits_{Q(x,\tau\sigma)} |Dv|^2\,dy + C\tau^{2-\varepsilon} \int\limits_{Q(x,2\sigma)} |Dv|^2\,dy.$$

Now, proceeding as above, we get (4.8). □

4.2. Boundary estimates

We can always reduce locally to the case of flat boundary, by means of a diffeomorphism which does not change the conditions on growth, convexity, etc., of the functional in question. Namely, for any fixed point $p_0 \in \partial\Omega$ and a sufficiently small $r > 0$, without loss of generality, we can assume that $p_0 = (0, \ldots, 0)$, $Q(p_0, r) \cap \Omega \subset \mathbb{R}^m_+ := \{x \in \mathbb{R}^m;\ x^m > 0\}$ and $Q(p_0, r) \cap \partial\Omega \subset \{x \in \mathbb{R}^m;\ x^m = 0\}$. In the sequel, we use the following notation:

$$\begin{aligned}&Q^+(x, r) := Q(x, r) \cap \mathbb{R}^m_+,\\ &\Gamma(x, r) := Q(x, r) \cap \big\{x \in \mathbb{R}^m;\ x^m = 0\big\},\\ &\Pi(x, r) := \partial Q^+(x, r) \setminus \Gamma(x, r).\end{aligned}$$

When $x = 0$, we write them simply as $Q^+(r)$, $\Gamma(r)$ and $\Pi(r)$.

By virtue of the above observation, near the boundary it is enough to consider the following problem.

$$\begin{cases} D_\alpha\big(A^0_{p^i_\alpha}(Dv)\big) = 0 & \text{in } Q^+(r),\\ v = f & \text{on } \Gamma(r).\end{cases} \tag{4.11}$$

Let us first investigate the case that $f = 0$.

Lemma 4.4. *Let $z \in H^{1,2}(Q^+(r), \mathbb{R}^m)$ be a solution of*

$$\begin{cases} \displaystyle\int\limits_{Q^+(r)} A^0_{p^i_\alpha}(Dz) D_\alpha\varphi^i\,dx = 0 & \textit{for all } \varphi \in H^{1,2}_0\big(Q^+(r), \mathbb{R}^n\big),\\ z = 0 & \textit{on } \Gamma(r).\end{cases} \tag{4.12}$$

Then, for any $r' \in (0, r)$, we have $z \in H^{2,2}(Q^+(r'), \mathbb{R}^m)$. Moreover, for any $x \in Q^+(r') \cup \Gamma(r')$ and ρ, σ with

$$0 < \rho < \sigma < h_0 := \frac{r - r'}{2},$$

we have

$$\int\limits_{\Omega(x,\rho)} \big|D^2 z\big|^2\,dy \leqslant \frac{C}{(\sigma - \rho)^2} \int\limits_{\Omega(x,\sigma)} \sum_{\gamma=1}^{m-1} |D_\gamma z|^2\,dy, \tag{4.13}$$

where $\Omega(x, \rho) := Q(x, \rho) \cap Q^+(r)$.

Proof. For $\gamma = 1, \dots, m-1$, $\eta \in C_0^\infty(Q(r))$ and $h < \operatorname{dist}(\operatorname{supp}\eta, \Pi(r))$, let $\varphi = \tau_\gamma^{-h}\{(\tau_\gamma^h z)\eta^2\}$ in (4.12). (Since $z = 0$ on $\Gamma(r)$, this choice of φ is admissible.) Then we see that

$$\int_{Q^+(r)} \tau_\gamma^h A^0_{p_\alpha^i}(Dz) D_\alpha\big((\tau_\gamma^h z^i)\eta^2\big)\,dx = 0.$$

So, putting

$$a_{ij}^{\alpha\beta}(x) := \int_0^1 \big(\hat{A}^0\big)_{ij}^{\alpha\beta}\big(tDz(x+he_\gamma) + (1-t)Dz(x)\big)\,dt,$$

and using (3.16), we get

$$\int_{Q^+(r)} a_{ij}^{\alpha\beta}(x)\tau_\gamma^h D_\beta z^j \big\{\tau_\gamma^h D_\alpha z^i \eta^2 + 2\tau_\gamma^h z^i \eta D_\alpha \eta\big\}\,dx = 0. \tag{4.14}$$

For any fixed $x \in Q^+(r') \cup \Gamma(r')$, let s and t be positive numbers satisfying

$$0 < s < t < r - \max\big\{|x^\gamma|;\ \gamma = 1, \dots, m\big\}.$$

Choosing η so that $\eta \in C_0^\infty(Q(x,t))$, $\eta \equiv 1$ on $Q(x,s)$ and that $|D\eta| \leqslant C/(t-s)$ in (4.14), we see with (3.4) that

$$\int_{\Omega(x,s)} \big|\tau_\gamma^h Dz\big|^2\,dy \leqslant \frac{C}{(t-s)^2}\int_{\Omega(x,t)} \big|\tau_\gamma^h z\big|^2\,dy, \quad \gamma = 1, \dots, m-1.$$

So, we obtain

$$\int_{\Omega(x,s)} |D_\gamma Dz|^2\,dy \leqslant \frac{C}{(t-s)^2}\int_{\Omega(x,t)} |D_\gamma z|^2\,dy, \quad \gamma = 1, \dots, m-1. \tag{4.15}$$

Moreover, Lemma 3.4 leads to

$$D_\gamma A^0_{p_\gamma^i}(Dz) = \big(\check{A}^0\big)_{ij}^{\gamma\beta}(Dz) D_\gamma D_\beta z^j \quad \text{a.e. on } Q^+(r),$$

for $\gamma = 1, \dots, m-1$. Now, we can proceed as in the proof of [4, Theorem 4.1] and obtain the assertion. □

Proposition 4.5. *Let $z \in H^{1,2}$ be as in Lemma 4.4. Then, there exists a constant $\varepsilon_0 > 0$ such that for every $q \in (2, 2+\varepsilon_0)$ and for any $r' \in (0,r)$, we have $a \in H^{2,q}(Q^+(r'))$ and*

$$\bigg(\fint_{\Omega(x,\sigma)} \big|D^2 z\big|^q\,dy\bigg)^{1/q} \leqslant C\bigg(\fint_{\Omega(x,2\sigma)} \big|D^2 z\big|^2\,dy\bigg)^{1/2}, \tag{4.16}$$

where $x \in Q^+(r')$ and $2\sigma < r - r'$.

Proof. Let x and σ be as above. If $Q(x,2\sigma) \cap \Gamma(r) = \emptyset$, then Proposition 4.1 implies (4.16). Assume that $Q(x,2\sigma) \cap \Gamma(r) \neq \emptyset$. Then

$$\mathcal{H}^{m-1}\big(Q(x,2\sigma) \cap \Gamma(r)\big) = (4\sigma)^{m-1}.$$

So, remarking that $z = 0$ on $\Gamma(r)$ (and therefore $D_\gamma z = 0$ on $\Gamma(r)$ for $\gamma = 1, \dots, m-1$), we can apply the Sobolev–Poincaré inequality for $D_\gamma z$ ($\gamma = 1, \dots, m-1$) to get

$$\bigg(\int_{\Omega(x,2\sigma)} \sum_{\gamma=1}^{m-1} |D_\gamma z|^2\,dy\bigg)^{1/2} \leqslant C\bigg(\int_{\Omega(x,2\sigma)} \big|D^2 z\big|^{2_*}\,dy\bigg)^{1/2_*}. \tag{4.17}$$

Combining (4.13) with (4.17), we get

$$\left(\fint_{\Omega(x,\sigma)} \left|D^2 z\right|^2 dy\right)^{1/2} \leqslant C\left(\fint_{\Omega(x,2\sigma)} \left|D^2 z\right|^{2_*} dy\right)^{1/2_*}. \tag{4.18}$$

Now, extending $D^2 z$ as 0 outside $Q^+(r')$, and using the interior estimate (4.6) we see that (4.18) holds for every $x \in Q^+(r)$ and $\sigma < (r - r')/2$. Thus, we can use the reverse Hölder inequality due to Giaquinta and Modica [13] to obtain (4.16). □

Now, we can proceed as in the proof of Corollary 4.3 and obtain the following.

Corollary 4.6. *Let z and $q > 2$ be as in Lemma 4.4 and Proposition 4.5. Then for any $x \in Q^+(r')$ and $0 < \rho < \sigma < (r - r')/2$, we have*

$$\int_{\Omega(x,\rho)} |Dz|^2 dy \leqslant C\left(\frac{\rho}{\sigma}\right)^{2+m-2m/q} \int_{\Omega(x,\sigma)} |Dz|^2 dy \tag{4.19}$$

for $m \geqslant 3$ and

$$\int_{\Omega(x,\rho)} |Dz|^2 dx \leqslant C\left(\frac{\rho}{\sigma}\right)^{2-\varepsilon} \int_{\Omega(x,\sigma)} |Dz|^2 dx \quad \text{for any } \varepsilon > 0, \tag{4.20}$$

for $m = 2$.

When the boundary conditions are general $H^{1,s}$-functions, using Corollary 4.6, we have the following.

Corollary 4.7. *Assume that $v \in H^{1,2}(Q^+(r))$ satisfies*

$$\begin{cases} \displaystyle\int_{Q^+(r)} A^0_{p^i_\alpha}(Dv) D_\alpha \varphi^i \, dx = 0 \quad \forall \varphi \in H^{1,2}_0\big(Q^+(r)\big), \\ v = f \quad \text{on } \Gamma(r), \end{cases} \tag{4.21}$$

where f is a given map in the class $H^{1,s}(Q^+(r))$ for some $s > m$. Let $r' < r$, $x \in \Gamma(r')$ and $0 < \rho < \sigma < (r - r')/2$.

If $m = 3, 4$, then, for some $\varepsilon_0 > 0$ for any $q \in (2, 2 + \varepsilon_0)$, we have

$$\int_{Q^+(x,\rho)} |Dv|^2 dx \leqslant C\left(\frac{\rho}{\sigma}\right)^{2+m-2m/q} \int_{Q^+(x,\sigma)} |Dv|^2 dx + C \int_{Q^+(x,\sigma)} |Df|^2 dx. \tag{4.22}$$

If $m = 2$, then for every $\varepsilon > 0$, we have

$$\int_{Q^+(x,\rho)} |Dv|^2 dx \leqslant C\left(\frac{\rho}{\sigma}\right)^{2-\varepsilon} \int_{Q^+(x,\sigma)} |Dv|^2 dx + C \int_{Q^+(x,\sigma)} |Df|^2 dx. \tag{4.23}$$

Proof. Let $w = v - f$. Then w satisfies

$$\begin{cases} \displaystyle\int_{Q^+(r)} A^0_{p^i_\alpha}(Dw + Df) D_\alpha \varphi^i \, dx = 0 \quad \forall \varphi \in H^{1,2}_0\big(Q^+(r)\big), \\ w = 0 \quad \text{on } \Gamma(r). \end{cases}$$

Let x, σ be as in the statement and let $z \in H^{1,2}(Q^+(x, \sigma))$ be a solution of

$$\begin{cases} \displaystyle\int_{Q^+(x,\sigma)} A^0(Dz)\, dy \to \text{minimum}, \\ z - w \in H^{1,2}_0\big(Q^+(x,\sigma)\big). \end{cases}$$

Then z satisfies the assumptions of Proposition 4.5, since $w = 0$ on $\Gamma(r)$. So, by Corollary 4.6, we see that z satisfies (4.19) for $m \geqslant 3$ or (4.20) for $m = 2$.

Put $b = z - w$. Then $b \in H_0^{1,2}(Q^+(x,\sigma))$ and

$$\int_{Q^+(x,\sigma)} A^0_{p^i_\alpha}(Db + Dw) D_\alpha \varphi^i \, dy = \int_{Q^+(x,\sigma)} A^0_{p^i_\alpha}(Dw + Df) D_\alpha \varphi^i \, dy. \tag{4.24}$$

Putting $\varphi = b$ in (4.24), subtracting

$$\int_{Q^+(x,\sigma)} A^0_{p^i_\alpha}(Dw) D_\alpha b^i \, dy$$

from both sides and using Lemma 3.3, we get

$$\int_{Q^+(x,\sigma)} \left(\int_0^1 (\hat{A}^0)^{\alpha\beta}_{ij}(t Db + Dw)\, dt \right) D_\alpha b^i D_\beta b^j \, dy = \int_{Q^+(x,\sigma)} \left(\int_0^1 (\hat{A}^0)^{\alpha\beta}_{ij}(t Df + Dw)\, dt \right) D_\alpha b^i D_\beta f^j \, dy.$$

By virtue of the coercivity of $\hat{A}^0$ and Young's inequality, from the above equality we get

$$\int_{Q^+(x,\sigma)} |Db|^2 \, dy \leqslant C \int_{Q^+(x,\sigma)} |Df|^2 \, dy. \tag{4.25}$$

Now, (4.19) and (4.25) imply that

$$\begin{aligned}
\int_{Q^+(x,\rho)} |Dw|^2 \, dy &\leqslant \int_{Q^+(x,\rho)} |Dz|^2 \, dy + \int_{Q^+(x,\rho)} |Db|^2 \, dy \\
&\leqslant C\left(\frac{\rho}{\sigma}\right)^{2+m-2m/q} \int_{Q^+(x,\sigma)} |Dz|^2 \, dy + C \int_{Q^+(x,\sigma)} |Df|^2 \, dy \\
&\leqslant C\left(\frac{\rho}{\sigma}\right)^{2+m-2m/q} \int_{Q^+(x,\sigma)} |Dw|^2 \, dy + C \int_{Q^+(x,\sigma)} |Df|^2 \, dy
\end{aligned}$$

for $m \geqslant 3$. Here, we used also the minimality of z. Similarly, for $m = 2$, (4.20) and (4.25) imply that

$$\int_{Q^+(x,\rho)} |Dw|^2 \, dy \leqslant C\left(\frac{\rho}{\sigma}\right)^{2-\varepsilon} \int_{Q^+(x,\sigma)} |Dw|^2 \, dy + C \int_{Q^+(x,\sigma)} |Df|^2 \, dy$$

for any $\varepsilon \in (0, 1)$. Since $v = w + f$ the assertion follows from the above estimates immediately. □

5. Proof of Theorems 2.1, 2.2

In order to prove Theorem 2.2, we use the results of the previous section and following 2 lemmata.

Lemma 5.1. *(See [9].) Let v be a minimizer of $\mathcal{A}^0(v, \Omega(x,r))$ defined by* (4.1) *in the class*

$$\left\{ w \in H^{1,2}\big(\Omega(x,r)\big);\ w - u \in H_0^{1,2}\big(\Omega(x,r)\big) \right\}$$

for a given $u \in H^{1,2}(\Omega(x,r))$. Then we have

$$\int_{\Omega(x,r)} |Du - Dv|^2 \, dy \leqslant C\left\{ \mathcal{A}^0\big(u, \Omega(x,r)\big) - \mathcal{A}^0\big(v, \Omega(x,r)\big) \right\}. \tag{5.1}$$

Proof. By virtue of Lemma 3.3, we can proceed as in the proof of [9, Lemma 2.1] to show the assertion. □

The following L^p-estimate is a direct consequence of [16, Lemma 1].

Lemma 5.2. *(See [16, Lemma 1].) Let u be a minimizer of $E_C(u)$ in the class*

$$X_f := \{v \in H^{1,2}(\Omega);\ u - f \in H_0^{1,2}(\Omega)\},$$

where f is a given map in the class $H^{1,p}(\Omega)$ for some $p > 2$. Then there exist positive numbers $q_0 \in (2, p)$, C_0 and r_0 such that for every $q \in (2, q_0)$, $x \in \overline{\Omega}$ and $r \in (0, r_0)$, we have $u \in H^{1,q}(\Omega)$ and

$$\left(\fint_{\Omega(x,r/2)} |Du|^q\,dy\right)^{1/q} \leqslant C_0\left\{\left(\fint_{\Omega(x,r)} |Du|^2\,dy\right)^{1/2} + \left(\fint_{\Omega(x,r)} |Df|^q\,dy\right)^{1/q}\right\}. \tag{5.2}$$

When $Q(x,r) \Subset \Omega$, the second term of the right-hand side of (5.2) *disappears.*

Remark 5.3. When Ω is bounded, (5.2) and the minimality of u imply

$$\left(\int_\Omega |Du|^q\,dx\right)^{1/q} \leqslant C\left(\int_\Omega |Df|^q\,dx\right)^{1/q}. \tag{5.3}$$

Proof of Theorem 2.2. Here, we give a proof only for the neighborhood of the boundary $\partial\Omega$. For the interior case, we can prove more easily by omitting the terms including the boundary data f.

We can always reduce locally to the case of flat boundary. So, we can assume that, for some $R > 0$, u satisfies

$$\begin{cases} E_C(u, Q^+(R)) \leqslant E_C(u+\varphi, Q^+(R)) & \text{for any } \varphi \in H_0^{1,2}(Q^+(R)), \\ u = f \quad \text{on } \Gamma(R). \end{cases} \tag{5.4}$$

Fix a point $x_0 \in \Gamma(R)$ and a positive number $r < (R - |x_0|)/2$ arbitrarily. Let $v \in H^{1,2}(Q^+(x_0, r))$ be a minimizer of

$$\mathcal{A}^0(v) := \int_{Q^+(x_0,r)} A^0(Dv)\,dx := \int_{Q^+(x_0,r)} A(x_0, u_r, Dv)\,dx \tag{5.5}$$

in the class

$$\{v \in H^{1,2}(Q^+(x_0,r));\ u - v \in H_0^{1,2}(Q^+(x_0,r))\},$$

where

$$u_r = \fint_{Q^+(x_0,r)} u\,dy.$$

By the assumption on f and Lemma 5.2, we see that $Du \in L^q(Q^+(x_0,r))$, and therefore, using (5.3) for v on $Q^+(x_0,r)$, we get

$$\fint_{Q^+(x_0,r)} |Dv|^q\,dx \leqslant \fint_{Q^+(x_0,r)} |Du|^q\,dx. \tag{5.6}$$

Now, let $w = u - v$, then we see that by Lemma 5.1 and by (3.10)

$$\begin{aligned} \int_{Q^+(x_0,r)} |Dw|^2\,dx &\leqslant C\left(\mathcal{A}^0(u) - \mathcal{A}^0(v)\right) \leqslant C\left(\mathcal{A}^0(u) - E_C(u) + E_C(v) - \mathcal{A}^0(v)\right) \\ &\leqslant C\Bigg[\int_{Q^+(x_0,r)} \omega\left(|x-x_0|^2 + |u_r - u|^2\right)|Du|^2\,dx \end{aligned}$$

$$+ \int_{Q^+(x_0,r)} \omega\big(|x-x_0|^2+|u_r-v|^2\big)|Dv|^2\,dx\Big]$$
$$=: C(I+II).$$

Since $Du \in L^q$ by Lemma 5.2, using (5.2), Hölder's inequality, boundedness of ω and Jensen's inequality, we can estimate I as follows.

$$I \leqslant C\tilde{\omega}\bigg(r^2+r^{2-m}\int_{Q^+(x_0,r)}|Du|^2\,dx\bigg)\int_{Q^+(x_0,2r)}|Du|^2\,dx + cr^{m(q-2)/q}\bigg(\int_{Q^+(x_0,2r)}|Df|^q\,dx\bigg)^{2/q},$$

where $\tilde{\omega}=\omega^{(q-2)/q}$. Noting (5.6), we can estimate II similarly. Thus, we have

$$\int_{Q^+(x_0,r)}|Dw|^2\,dx \leqslant C\tilde{\omega}\bigg(r^2+r^{2-m}\int_{Q^+(x_0,2r)}|Du|^2\,dx\bigg)\int_{Q^+(x_0,2r)}|Du|^2\,dx$$
$$+ Cr^{m(q-2)/q}\bigg(\int_{Q^+(x_0,2r)}|Df|^q\,dx\bigg)^{2/q}. \tag{5.7}$$

Let us write

$$\mu=\begin{cases} m+2-\frac{2m}{q} & (m\geqslant 3),\\ 2-\varepsilon & (m=2),\end{cases}$$

where we can take $\varepsilon>0$ to be arbitrarily small. By virtue of Corollary 4.7 and (5.7), we obtain for $0<\rho<r$

$$\int_{Q^+(x_0,\rho)}|Du|^2\,dx \leqslant C\bigg\{\bigg(\frac{\rho}{r}\bigg)^{\mu}+\tilde{\omega}\bigg(r^2+r^{2-m}\int_{Q^+(x_0,2r)}|Du|^2\,dx\bigg)\bigg\}\int_{Q^+(x_0,2r)}|Du|^2\,dx$$
$$+ C\int_{Q^+(x_0,2r)}|Df|^2\,dx + Cr^{m(q-2)/q}\bigg(\int_{Q^+(x_0,2r)}|Df|^q\,dx\bigg)^{2/q}. \tag{5.8}$$

On the other hand, when x_0 is an interior point and $Q(x_0,2r)\Subset\Omega$,

$$\int_{Q(x_0,\rho)}|Du|^2\,dx \leqslant C\bigg\{\bigg(\frac{\rho}{r}\bigg)^{\mu}+\tilde{\omega}\bigg(r^2+r^{2-m}\int_{Q(x_0,2r)}|Du|^2\,dx\bigg)\bigg\}\int_{Q(x_0,2r)}|Du|^2\,dx. \tag{5.9}$$

Let us estimate the terms containing f in (5.8). Since $s>m\geqslant 2$, noting that we can take $q>2$ sufficiently near to 2, we can assume that $s>q>2$. So, we have

$$\int_{Q^+(x_0,2r)}|Df|^2\,dx \leqslant Cr^{m(1-2/s)}\bigg(\int_{Q^+(x_0,2r)}|Df|^s\,dx\bigg)^{2/s},$$
$$r^{m(q-2)/q}\bigg(\int_{Q^+(x_0,2r)}|Df|^q\,dx\bigg)^{2/q} \leqslant Cr^{m(1-2/s)}\bigg(\int_{Q^+(x_0,2r)}|Df|^s\,dx\bigg)^{2/s}.$$

Thus, we obtain from (5.8) that

$$\int_{Q^+(x_0,\rho)}|Du|^2\,dx \leqslant C\bigg\{\bigg(\frac{\rho}{r}\bigg)^{\mu}+\tilde{\omega}\bigg(r^2+r^{2-m}\int_{Q^+(x_0,2r)}|Du|^2\,dx\bigg)\bigg\}\int_{Q^+(x_0,2r)}|Du|^2\,dx + C(f)r^{\beta}, \tag{5.10}$$

where $\beta=m(1-2/s)>m-2$. On the other hand, since we assume $m\leqslant 4$, we have $\mu>m-2$. If necessary, replacing the exponent β with a smaller one ν such that $m-2<\nu<\mu$, and proceeding as in [14, pp. 317–318], we can see that there exists a positive constant ε_1 with the following property: If

$$r_0^2 + r_0^{2-m} \int_{Q^+(x_0,2r_0)} |Du|^2\,dx < \varepsilon_1 \tag{5.11}$$

and $2r_0 < R - |x_0|$, then for $0 < \rho < r < r_0$,

$$\int_{Q^+(x_0,\rho)} |Du|^2\,dx \leqslant M_1\left(\frac{\rho}{r}\right)^{\nu} \int_{Q^+(x_0,r)} |Du|^2\,dx + M_2\rho^{\nu} \tag{5.12}$$

holds, where $M_1 = M_1(A, m, R)$ and $M_2 = M_2(A, m, R, f)$.

If $Q(x_0, 2r) \Subset Q^+(R)$, from (5.9) we easily see that there exists a positive number ε_1 with the same property with $M_2 = 0$. Namely, (5.11) leads us to the interior estimate

$$\int_{Q^+(x_0,\rho)} |Du|^2\,dx \leqslant M_1\left(\frac{\rho}{r}\right)^{\nu} \int_{Q^+(x_0,r)} |Du|^2\,dx. \tag{5.13}$$

Put

$$\nu = m - 2 + 2\alpha,$$
$$\Phi(x, r) = r^{2-m} \int_{\Omega(x,r)} \left|Du(y)\right|^2 dy.$$

Then (5.12) and (5.13) can be written as

$$\Phi(x_0, \rho) \leqslant M_1\left(\frac{\rho}{r}\right)^{2\alpha} \Phi(x_0, r) + M_2\rho^{2\alpha} \quad \text{for } x_0 \in \Gamma(R),$$
$$\Phi(x_0, \rho) \leqslant M_1\left(\frac{\rho}{r}\right)^{2\alpha} \Phi(x_0, r) \quad \text{for } Q(x_0, 2r) \Subset Q^+(R).$$

Now, we can proceed as in [14, pp. 318–319] to see that there exist a positive number ε_0 and C such that, for any $x_1 \in Q^+(R) \cup \Gamma(R)$, if $\Phi(x_1, r_0) < \varepsilon_0$ and $2r_0 < R - \max\{x^\beta;\ 1 \leqslant \beta \leqslant m\}$, then

$$\Phi(x_1, \rho) \leqslant C\rho^{2\alpha} \quad \text{for all } \rho \in (0, r_0). \tag{5.14}$$

From (5.14) we get the assertion using *Morrey's theorem on the growth of the Dirichlet-integral.* □

We can prove Theorem 2.1 by a slight modification of the above proof.

Proof of Theorem 2.1. In the proof of Theorem 2.2, we used the minimality of u only to obtain (5.7). For the case that the Finsler structure F does not depend on u, A defined by (3.1) depends only on x and p. We can get an estimate similar to (5.7) for weak solutions of the Euler–Lagrange equation of $\int A(x, Du)$ as follows.

Let $u \in H^{1,2}$ be a weakly harmonic map, namely, a solution of

$$\begin{cases} \displaystyle\int_\Omega A_{p^i_\alpha}(x, Du) D_\alpha \varphi^i\,dx = 0 \quad \text{for any } \varphi \in H^{1,2}\big(\Omega, \mathbb{R}^n\big), \\ u = f \quad \text{on } \partial\Omega, \end{cases}$$

and v be a minimizer of $\mathcal{A}_0$ defined in (5.5) as in the proof of Theorem 2.2. Then, using Lemma 3.3, (3.4) and (3.11), we have for $w = u - v$

$$\begin{aligned} \int_{\Omega(x_0,r)} |Dw|^2\,dx &\leqslant \frac{1}{\lambda_0} \int_{\Omega(x_0,r)} \big(A_{p^i_\alpha}(x, Du) - A_{p^i_\alpha}(x, Dv)\big)\big(D_\alpha u^i - D_\alpha v^i\big)\,dx \\ &= \frac{1}{\lambda_0} \int_{\Omega(x_0,r)} \big(A_{p^i_\alpha}(x_0, Dv) - A_{p^i_\alpha}(x, Dv)\big)\big(D_\alpha u^i - D_\alpha v^i\big)\,dx \\ &\leqslant C \sup_{\Omega(x_0,r)} \omega\big(|x_0 - x|\big) \cdot \int_{\Omega(x_0,r)} \big(|Du|^2 + |Dv|^2\big)\,dx. \end{aligned}$$

Now, we can proceed as in the proof of Theorem 2.2 and obtain the everywhere regularity of u mentioning that $\lim_{r\to 0}\sup_{\Omega(x_0,r)}\omega(|x_0-x|)=0$ for every x. $\square$

Acknowledgements

This paper is partly prepared while the author visited the *Centro di Ricerca Matematica Ennio De Giorgi* in September 2006. The hospitality of the center is greatly acknowledged. The author is grateful to S. Nishikawa for his useful advices on Finsler geometry. The author also acknowledges the anonymous referee for his very careful reading and many useful comments. Especially, by one of the referee's comments, the author could recognize that the results of this paper are valid only for Centore's energy.

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