Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 27 (2010) 57–72
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

The limiting behavior of the value-function for variational problems arising in continuum mechanics

Alexander J. Zaslavski

Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel

Received 17 May 2009; received in revised form 16 July 2009; accepted 16 July 2009

Available online 5 August 2009

Abstract

In this paper we study the limiting behavior of the value-function for one-dimensional second order variational problems arising in continuum mechanics. The study of this behavior is based on the relation between variational problems on bounded large intervals and a limiting problem on $[0, \infty)$.

MSC: 49J99

Keywords: Good function; Infinite horizon; Minimal long-run average cost growth rate; Variational problem

1. Introduction

The study of properties of solutions of optimal control problems and variational problems defined on infinite domains and on sufficiently large domains has recently been a rapidly growing area of research. See, for example, [3,5,6,15–19,21–24] and the references mentioned therein. These problems arise in engineering [8], in models of economic growth [10,25], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [2,20] and in the theory of thermodynamical equilibrium for materials [7,9,11–14]. In this paper we study the limiting behavior of the value-function for variational problems arising in continuum mechanics which were considered in [7,9,11–14,21–24]. The study of this behavior is based on the relation between variational problems on bounded large intervals and a limiting problem on $[0, \infty)$.

In this paper we consider the variational problems

$$\int_0^T f\big(w(t), w'(t), w''(t)\big)\, dt \to \min, \quad w \in W^{2,1}\big([0, T]\big),$$

$$\big(w(0), w'(0)\big) = x \quad \text{and} \quad \big(w(T), w'(T)\big) = y, \tag{P}$$

E-mail address: ajzasl@tx.technion.ac.il.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2009.07.005

where $T > 0$, $x, y \in R^2$, $W^{2,1}([0,T]) \subset C^1([0,T])$ is the Sobolev space of functions possessing an integrable second derivative [1] and f belongs to a space of functions to be described below. The interest in variational problems of the form (P) and the related problem on the half line:

$$\liminf_{T\to\infty} T^{-1} \int_0^T f\big(w(t), w'(t), w''(t)\big)\,dt \to \min, \quad w \in W^{2,1}_{loc}\big([0,\infty)\big) \tag{P_∞}$$

stems from the theory of thermodynamical equilibrium for second-order materials developed in [7,9,11–14]. Here $W^{2,1}_{loc}([0,\infty)) \subset C^1([0,\infty))$ denotes the Sobolev space of functions possessing a locally integrable second derivative [1] and f belongs to a space of functions to be described below.

We are interested in properties of the valued-function for the problem (P) which are independent of the length of the interval, for all sufficiently large intervals.

Let $a = (a_1, a_2, a_3, a_4) \in R^4$, $a_i > 0$, $i = 1, 2, 3, 4$ and let α, β, γ be positive numbers such that $1 \leqslant \beta < \alpha$, $\beta \leqslant \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha, \beta, \gamma, a)$ the set of all functions $f : R^3 \to R^1$ such that:

$$f(w, p, r) \geqslant a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4 \quad \text{for all } (w, p, r) \in R^3; \tag{1.1}$$

$$f, \partial f/\partial p \in C^2, \qquad \partial f/\partial r \in C^3, \qquad \partial^2 f/\partial r^2(w, p, r) > 0 \quad \text{for all } (w, p, r) \in R^3; \tag{1.2}$$

there is a monotone increasing function $M_f : [0,\infty) \to [0,\infty)$ such that for every $(w, p, r) \in R^3$

$$\max\big\{f(w, p, r), \big|\partial f/\partial w(w, p, r)\big|, \big|\partial f/\partial p(w, p, r)\big|, \big|\partial f/\partial r(w, p, r)\big|\big\} \\ \leqslant M_f\big(|w| + |p|\big)\big(1 + |r|^\gamma\big). \tag{1.3}$$

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\left\{\liminf_{T\to\infty} T^{-1} \int_0^T f\big(w(t), w'(t), w''(t)\big)\,dt\colon\ w \in A_x\right\}, \tag{1.4}$$

where

$$A_x = \big\{v \in W^{2,1}_{loc}\big([0,\infty)\big)\colon\ \big(v(0), v'(0)\big) = x\big\}.$$

It was shown in [9] that $\mu(f) \in R^1$ is well defined and is independent of the initial vector x. A function $w \in W^{2,1}_{loc}([0,\infty))$ is called an (f)-*good function* if the function

$$\phi^f_w : T \to \int_0^T \big[f\big(w(t), w'(t), w''(t)\big) - \mu(f)\big]\,dt, \quad T \in (0,\infty)$$

is bounded. For every $w \in W^{2,1}_{loc}([0,\infty))$ the function ϕ^f_w is either bounded or diverges to ∞ as $T \to \infty$ and moreover, if ϕ^f_w is a bounded function, then

$$\sup\big\{\big|\big(w(t), w'(t)\big)\big|\colon\ t \in [0,\infty)\big\} < \infty$$

[22, Proposition 3.5]. Leizarowitz and Mizel [9] established that for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfying $\mu(f) < \inf\{f(w, 0, s)\colon\ (w, s) \in R^2\}$ there exists a periodic (f)-good function. In [21] it was shown that a periodic (f)-good function exists for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$.

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. For each $T > 0$ define a function $U^f_T : R^2 \times R^2 \to R^1$ by

$$U^f_T(x, y) = \inf\left\{\int_0^T f\big(w(t), w'(t), w''(t)\big)\,dt\colon\ w \in W^{2,1}\big([0,T]\big), \right. \\ \left. \big(w(0), w'(0)\big) = x \text{ and } \big(w(T), w'(T)\big) = y\right\}. \tag{1.5}$$

In [9], analyzing problem (P_∞) Leizarowitz and Mizel studied the function $U_T^f : R^2 \times R^2 \to R^1$, $T > 0$ and established the following representation formula

$$U_T^f(x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x, y), \quad x, y \in R^2,\ T > 0, \tag{1.6}$$

where $\pi^f : R^2 \to R^1$ and $(T, x, y) \to \theta_T^f(x, y)$ and $(T, x, y) \to U_T^f(x, y)$, $x, y \in R^2$, $T > 0$ are continuous functions,

$$\pi^f(x) = \inf\left\{ \liminf_{T\to\infty} \int_0^T \left[f\big(w(t), w'(t), w''(t)\big) - \mu(f)\right] dt:\right.$$
$$\left. w \in W^{2,1}_{loc}\big([0,\infty)\big) \text{ and } \big(w(0), w'(0)\big) = x \right\}, \quad x \in R^2, \tag{1.7}$$

$\theta_T^f(x, y) \geqslant 0$ for each $T > 0$, and each $x, y \in R^2$, and for every $T > 0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta_T^f(x, y) = 0$.

Denote by $|\cdot|$ the Euclidean norm in R^n. For every $x \in R^n$ and every nonempty set $\Omega \subset R^n$ set

$$d(x, \Omega) = \inf\big\{|x - y|:\ y \in \Omega\big\}.$$

For each function $g : X \to R^1 \cup \{\infty\}$, where the set X is nonempty, put

$$\inf(g) = \inf\big\{g(z):\ z \in X\big\}.$$

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. It is easy to see that

$$\mu(f) \leqslant \inf\big\{f(t, 0, 0):\ t \in R^1\big\}.$$

If $\mu(f) = \inf\{f(t, 0, 0):\ t \in R^1\}$, then there is an (f)-good function which is a constant function. If $\mu(f) < \inf\{f(t, 0, 0):\ t \in R^1\}$, then there exists a periodic (f)-good function which is not a constant function. It was shown in [14] that in this case the extremals of (P_∞) have interesting asymptotic properties. In [26] we equipped the space $\mathfrak{M}(\alpha, \beta, \gamma, a)$ with a natural topology and showed that there exists an open everywhere dense subset $\mathcal{F}$ of this topological space such that for every $f \in \mathcal{F}$,

$$\mu(f) < \inf\big\{f(t, 0, 0):\ t \in R^1\big\}.$$

In other words, the inequality above holds for a typical integrand $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$.

In the present paper for an integrand $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfying

$$\mu(f) < \inf\big\{f(t, 0, 0):\ t \in R^1\big\}$$

we study the limiting behavior of the value-function U_T^f as $T \to \infty$ and establish the following two results.

Theorem 1.1. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfy $\mu(f) < \inf\{f(t, 0, 0):\ t \in R^1\}$. Then for each $x, y \in R^2$ there exists*

$$U_\infty^f(x, y) := \lim_{T\to\infty} \big(U_T^f(x, y) - T\mu(f)\big).$$

Moreover, $U_T^f(x, y) - T\mu(f) \to U_\infty^f(x, y)$ as $T \to \infty$ uniformly on bounded subsets of $R^2 \times R^2$.

Theorem 1.2. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfy $\mu(f) < \inf\{f(t, 0, 0):\ t \in R^1\}$. Then there exists a nonempty compact set $E_\infty \subset R^2 \times R^2$ such that*

$$E_\infty = \big\{(x, y) \in R^2 \times R^2:\ U_\infty^f(x, y) = \inf\big(U_\infty^f\big)\big\}.$$

Moreover, for any $\epsilon > 0$ there exist $\delta > 0$ and $\bar{T} > 0$ such that if $T \geqslant \bar{T}$ and if $x, y \in R^2$ satisfy $U_T^f(x, y) \leqslant \inf(U_T^f) + \delta$, then $d((x, y), E_\infty) \leqslant \epsilon$.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3 we prove several auxiliary results. Theorems 1.1 and 1.2 are proved in Sections 4 and 5 respectively.

2. Preliminaries

For $\tau > 0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v : [0,\tau] \to R^2$ as follows:

$$X_v(t) = \big(v(t), v'(t)\big), \quad t \in [0,\tau].$$

We also use this definition for $v \in W^{2,1}_{loc}([0,\infty))$ and $v \in W^{2,1}_{loc}(R^1)$.

Put

$$\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, a).$$

We consider functionals of the form

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f\big(v(t), v'(t), v''(t)\big)\,dt, \tag{2.1}$$

$$\Gamma^f(T_1, T_2, v) = I^f(T_1, T_2, v) - (T_2 - T_1)\mu(f) - \pi^f\big(X_v(T_1)\big) + \pi^f\big(X_v(T_2)\big), \tag{2.2}$$

where $-\infty < T_1 < T_2 < +\infty$, $v \in W^{2,1}([T_1, T_2])$ and $f \in \mathfrak{M}$.

If $v \in W^{2,1}_{loc}([0,\infty))$ satisfies

$$\sup\big\{\big|X_v(t)\big|:\ t \in [0,\infty)\big\} < \infty,$$

then the set of limiting points of $X_v(t)$ as $t \to \infty$ is denoted by $\Omega(v)$.

For each $f \in \mathfrak{M}$ denote by $\mathcal{A}(f)$ the set of all $w \in W^{2,1}_{loc}([0,\infty))$ which have the following property:

There is $T_w > 0$ such that

$$w(t + T_w) = w(t) \quad \text{for all } t \in [0,\infty) \quad \text{and} \quad I^f(0, T_w, w) = \mu(f)T_w.$$

In other words $\mathcal{A}(f)$ is the set of all periodic (f)-good functions. By a result of [21], $\mathcal{A}(f) \neq \emptyset$ for all $f \in \mathfrak{M}$.

The following result established in [13, Lemma 3.1] describes the structure of periodic (f)-good functions.

Proposition 2.1. *Let $f \in \mathfrak{M}$. Assume that $w \in \mathcal{A}(f)$,*

$$w(0) = \inf\big\{w(t):\ t \in [0,\infty)\big\}$$

and $w'(t) \neq 0$ for some $t \in [0,\infty)$. Then there exist $\tau_1(w) > 0$ and $\tau(w) > \tau_1(w)$ such that the function w is strictly increasing on $[0, \tau_1(w)]$, w is strictly decreasing on $[\tau_1(w), \tau(w)]$,

$$w\big(\tau_1(w)\big) = \sup\big\{w(t):\ t \in [0,\infty)\big\} \quad \text{and} \quad w\big(t + \tau(w)\big) = w(t) \quad \text{for all } t \in [0,\infty).$$

In [24, Theorem 3.15] we established the following result.

Proposition 2.2. *Let $f \in \mathfrak{M}$. Assume that $w \in \mathcal{A}(f)$ and $w'(t) \neq 0$ for some $t \in [0,\infty)$. Then there exists $\tau > 0$ such that*

$$w(t + \tau) = w(t), \quad t \in [0,\infty) \quad \text{and} \quad X_w(T_1) \neq X_w(T_2)$$

for each $T_1 \in [0,\infty)$ and each $T_2 \in (T_1, T_1 + \tau)$.

In the sequel we use the following result of [23, Proposition 5.1].

Proposition 2.3. *Let $f \in \mathfrak{M}$. Then there exists a number $S > 0$ such that for every (f)-good function v,*

$$\big|X_v(t)\big| \leqslant S \quad \text{for all large enough } t.$$

The following result was proved in [13, Lemma 3.2].

Proposition 2.4. *Let $f \in \mathfrak{M}$ satisfy*

$$\mu(f) < \inf\{f(t,0,0):\ t \in R^1\}.$$

Then no element of $\mathcal{A}(f)$ is a constant and $\sup\{\tau(w):\ w \in \mathcal{A}(f)\} < \infty$.

Proposition 2.5. *Let $f \in \mathfrak{M}$ and let M_1, M_2, c be positive numbers. Then there exists $S > 0$ such that the following assertion holds*:

If $T_1 \geqslant 0$, $T_2 \geqslant T_1 + c$ and if $v \in W^{2,1}([T_1,T_2])$ satisfies

$$|X_v(T_1)|, |X_v(T_2)| \leqslant M_1 \quad \textit{and} \quad I^f(T_1,T_2,v) \leqslant U^f_{T_2-T_1}(X_v(T_1), X_v(T_2)) + M_2,$$

then

$$|X_v(t)| \leqslant S \quad \textit{for all } t \in [T_1,T_2].$$

For this result we refer the reader to [9] (see the proof of Proposition 4.4).

The following result was established in [14, Theorem 1.2].

Proposition 2.6. *Let $f \in \mathfrak{M}$ satisfy*

$$\mu(f) < \inf\{f(t,0,0):\ t \in R^1\}$$

and let $v \in W^{2,1}_{loc}([0,\infty))$ be such that

$$\sup\{|X_v(t)|:\ t \in [0,\infty)\} < \infty, \qquad I^f(0,T,v) = U^f_T(X_v(0), X_v(T)) \quad \textit{for all } T > 0.$$

Then there exists a periodic (f)-good function w such that $\Omega(v) = \Omega(w)$ and the following assertion holds:

Let $T > 0$ be a period of w. Then for every $\epsilon > 0$ there exists $\tau(\epsilon) > 0$ such that for every $\tau \geqslant \tau(\epsilon)$ there exists $s \in [0,T)$ such that

$$|(v(t+\tau), v'(t+\tau)) - (w(s+t), w'(s+t))| \leqslant \epsilon, \quad t \in [0,T].$$

The next useful result was proved in [13, Lemma 2.6].

Proposition 2.7. *Let $f \in \mathfrak{M}$. Then for every compact set $E \subset R^2$ there exists a constant $M > 0$ such that for every $T \geqslant 1$*

$$U^f_T(x,y) \leqslant T\mu(f) + M \quad \textit{for all } x,y \in E.$$

The next important ingredient of our proofs is established in [13, Lemma B5] which is an extension of [23, Lemma 3.7].

Proposition 2.8. *Let $f \in \mathfrak{M}$, $w \in \mathcal{A}(f)$ and $\epsilon > 0$. Then there exist $\delta, q > 0$ such that for each $T \geqslant q$ and each $x,y \in R^2$ satisfying $d(x,\Omega(w)) \leqslant \delta$, $d(y,\Omega(w)) \leqslant \delta$, there exists $v \in W^{2,1}([0,\tau])$ which satisfies*

$$X_v(0) = x, \qquad X_v(\tau) = y, \qquad \Gamma^f(0,\tau,v) \leqslant \epsilon.$$

We also need the following auxiliary result of [21, Proposition 2.3].

Proposition 2.9. *Let $f \in \mathfrak{M}$. Then for every $T > 0$*

$$U^f_T(x,y) \to \infty \quad \textit{as } |x| + |y| \to \infty.$$

Proposition 2.10. *(See [12, Lemma 3.1].) Let $f \in \mathfrak{M}$ and δ, τ are positive numbers. Then there exists $M > 0$ such that for every $T \geqslant \tau$ and every $v \in W^{2,1}([0,T])$ satisfying*

$$I^f(0,T,v) \leqslant \inf\{U^f_T(x,y):\ x,y \in R^2\} + \delta$$

the following inequality holds:

$$|X_v(t)| \leqslant M \quad \textit{for all } t \in [0,T].$$

3. Auxiliary results

Let $f \in \mathfrak{M}$. By Proposition 2.2 for each $w \in \mathcal{A}(f)$ which is not a constant there exists $\tau(w) > 0$ such that

$$w\big(t+\tau(w)\big) = w(t), \quad t \in [0,\infty), \qquad X_w(T_1) \neq X_w(T_2) \quad \text{for each } T_1 \in [0,\infty) \text{ and each } T_2 \in \big(T_1, T_1+\tau(w)\big). \tag{3.1}$$

By Proposition 2.3 there exists a number $\bar{M} > 0$ such that

$$\sup\big\{\big|X_v(t)\big|: t \in [0,\infty)\big\} < \bar{M} \quad \text{for all } v \in \mathcal{A}(f). \tag{3.2}$$

Proposition 3.1. *Suppose that $\mu(f) < \inf\{f(t,0,0): t \in R^1\}$. Then*

$$\inf\big\{\tau(w): w \in \mathcal{A}(f)\big\} > 0.$$

Proof. Let us assume the contrary. Then there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathcal{A}(f)$ such that $\lim_{n\to\infty} \tau(w_n) = 0$. It follows from (3.2), the definition of $\tau(w)$, $w \in \mathcal{A}(f)$ and the equality above that for $n = 1, 2, \ldots,$

$$\sup\big\{\big|w_n(t) - w_n(s)\big|: t, s \in [0,\infty)\big\} \leqslant \bar{M}\tau(w_n) \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Since $\{w_n\}_{n=1}^{\infty} \subset \mathcal{A}(f)$ it follows from (3.2) and the continuity of the functions U_T^f, $T > 0$ that for any natural number k the sequence $\{I^f(0,k,w_n)\}_{n=1}^{\infty}$ is bounded. Combined with (3.2) and the growth condition (1.1) this implies that for any integer $k \geqslant 1$ the sequence $\{\int_0^k |w_n''(t)|^{\gamma}\,dt\}_{k=1}^{\infty}$ is bounded. Since this fact holds for any natural number k it follows from (3.2) that the sequence $\{w_n\}_{n=1}^{\infty}$ is bounded in $W^{2,\gamma}([0,k])$ for any natural number k and it possesses a weakly convergent subsequence in this space. By using a diagonal process we obtain that there exist a subsequence $\{w_{n_i}\}_{i=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ and $w_* \in W_{loc}^{2,1}([0,\infty))$ such that for each natural number k

$$\big(w_{n_i}, w_{n_i}'\big) \to \big(w_*, w_*'\big) \quad \text{as } i \to \infty \text{ uniformly on } [0,k], \tag{3.4}$$

$$w_{n_i}'' \to w_*'' \quad \text{as } i \to \infty \text{ weakly in } L^{\gamma}[0,k]. \tag{3.5}$$

By (3.4), (3.5) and the lower semicontinuity of integral functionals [4] for each natural number k,

$$I^f(0,k,w_*) \leqslant \liminf_{i\to\infty} I^f(0,k,w_{n_i}).$$

Combined with (3.4) and (2.2), the continuity of π^f and the inclusion $w_n \in \mathcal{A}(f)$, $n = 1, 2, \ldots,$ this inequality implies that for any natural number k

$$\Gamma^f(0,k,w_*) \leqslant \liminf_{i\to\infty} \Gamma^f(0,k,w_{n_i}) = 0.$$

In view of (3.3) and (3.4), w_* is a constant function. Together with the relation above and (2.2) this implies that

$$\mu(f) = f\big(u_*(0),0,0\big) = \inf\big\{f(t,0,0): t \in R^1\big\}.$$

The contradiction we have reached proves Proposition 3.1. □

Proposition 3.2. *Suppose that*

$$\mu(f) < \inf\big\{f(t,0,0): t \in R^1\big\}. \tag{3.6}$$

Let $M, l, \epsilon > 0$. Then there exist $\delta > 0$ and $L > l$ such that for each $T \geqslant L$ and each $v \in W^{2,1}([0,T])$ satisfying

$$\big|X_v(0)\big|, \big|X_v(T)\big| \leqslant M, \qquad \Gamma^f(0,T,v) \leqslant \delta, \tag{3.7}$$

there exist $s \in [0, T-l]$ and $w \in \mathcal{A}(f)$ such that

$$\big|X_v(s+t) - X_w(t)\big| \leqslant \epsilon, \quad t \in [0,l].$$

Proof. Assume the contrary. Then there exists a sequence $v_i \in W^{2,1}([0, T_i])$, $i = 1, 2, \ldots,$ such that

$$T_i \geqslant l, \quad i = 1, 2, \ldots,$$
$$T_i \to \infty \quad \text{as } i \to \infty, \qquad \Gamma^f(0, T_i, v_i) \to 0 \quad \text{as } i \to \infty, \tag{3.8}$$
$$\left|X_{v_i}(0)\right|, \left|X_{v_i}(T_i)\right| \leqslant M, \quad i = 1, 2, \ldots, \tag{3.9}$$

and that for each natural number i the following property holds:

$$\sup\left\{\left|X_{v_i}(s+t) - X_w(t)\right|: t \in [0, l]\right\} > \epsilon \quad \text{for each } s \in [0, T - l] \text{ and each } w \in \mathcal{A}(f). \tag{3.10}$$

We may assume without loss of generality that

$$\Gamma^f(0, T_i, v_i) \leqslant 1, \quad i = 1, 2, \ldots. \tag{3.11}$$

It follows from (2.2), (3.11), (1.6) and (1.5) that for each integer $i \geqslant 1$

$$\begin{aligned} I^f(0, T_i, v_i) &= \pi^f\left(X_{v_i}(0)\right) - \pi^f\left(X_{v_i}(T_i)\right) + T_i\mu(f) + \Gamma^f(0, T_i, v_i) \\ &\leqslant 1 + \pi^f\left(X_{v_i}(0)\right) - \pi^f\left(X_{v_i}(T_i)\right) + T_i\mu(f) \\ &\leqslant 1 + U^f_{T_i}\left(X_{v_i}(0), X_{v_i}(T_i)\right). \end{aligned} \tag{3.12}$$

By (3.12), (3.9), (3.8) and Proposition 2.5 there exists a constant $M_1 > 0$ such that

$$\left|X_{v_i}(t)\right| \leqslant M_1, \quad t \in [0, T_i],\ i = 1, 2, \ldots \tag{3.13}$$

By (3.13), (3.12) and the continuity of U^f_T, $T > 0$, for each natural number n, the sequence $\{I^f(0, n, v_i)\}_{i=i(n)}^\infty$ is bounded, where $i(n)$ is a natural number such that $T_i > n$ for all integers $i \geqslant i(n)$ (see (3.8)). Together with (3.13) and (1.1) this implies that for any natural number n the sequence $\{\int_0^n |v_i''(t)|^\gamma\, dt\}_{i=i(n)}^\infty$ is bounded. Since this fact holds for any natural number n it follows from (3.13) that the sequence $\{v_i\}_{i=i(n)}^\infty$ is bounded in $W^{2,\gamma}([0, n])$ for any natural number n and it possesses a weakly convergent subsequence in this space. By using a diagonal process we obtain that there exist a subsequence $\{v_{i_k}\}_{k=1}^\infty$ of $\{v_i\}_{i=1}^\infty$ and $u \in W^{2,1}_{loc}([0, \infty))$ such that for each natural number n

$$\left(v_{i_k}, v'_{i_k}\right) \to (u, u') \quad \text{as } k \to \infty \text{ uniformly on } [0, n], \tag{3.14}$$
$$v''_{i_k} \to u'' \quad \text{as } k \to \infty \text{ weakly in } L^\gamma[0, k]. \tag{3.15}$$

In view of (3.14) and (3.13),

$$\left|X_u(t)\right| \leqslant M_1 \quad \text{for all } t \geqslant 0. \tag{3.16}$$

It follows from (3.14), (3.15), (3.13) and the lower semicontinuity of integral functionals [4] for each natural number n

$$I^f(0, n, u) \leqslant \liminf_{k\to\infty} I^f(0, n, v_{i_k}).$$

Combined with (3.14), (3.13), (2.2), (1.6), the continuity of π^f and (3.8) the inequality above implies that for any natural number n

$$\Gamma^f(0, n, u) \leqslant \liminf_{k\to\infty} \Gamma^f(0, n, v_{i_k}) = 0.$$

Thus

$$\Gamma^f(0, T, u) = 0 \quad \text{for all } T > 0. \tag{3.17}$$

By (3.16), (3.17) and Proposition 2.6 there exists $w \in \mathcal{A}(f)$ such that $\Omega(w) = \Omega(u)$ and the following assertion holds:

(A1) Let T_w be a period of w (not necessarily minimal). Then for each $\gamma > 0$ there exists $\tau(\gamma) > 0$ such that for each $\tau \geqslant \tau(\gamma)$ there is $s \in [0, T_w)$ such that

$$\left|X_u(t+\tau) - X_w(s+t)\right| \leqslant \gamma, \quad t \in [0, T_w].$$

We may assume without loss of generality that a period T_w of w satisfies $T_w > l$. Assumption (A1) implies that there exist $\tau > 0$ and $\tilde{w} \in \mathcal{A}(f)$ such that

$$\left|X_u(\tau + t) - X_{\tilde{w}}(t)\right| \leqslant \epsilon/4, \quad t \in [0, l].$$

Combined with (3.14) this implies that for all sufficiently large natural numbers k

$$\left|X_{v_{i_k}}(\tau + t) - X_{\tilde{w}}(t)\right| \leqslant \epsilon/2, \quad t \in [0, l].$$

This contradicts (3.10). The contradiction we have reached proves Proposition 3.2. □

Proposition 3.3. *Let $M > 0$ and $\delta > 0$. Then there exists a natural number n such that for each number $T \geqslant 1$ and each $v \in W^{2,1}([0, T])$ satisfying*

$$\left|X_v(0)\right|, \left|X_v(T)\right| \leqslant M, \quad I^f(0, T, v) \leqslant U_T^f\big(X_v(0), X_v(T)\big) + 1 \tag{3.18}$$

the following property holds:

There exists a sequence $\{t_i\}_{i=0}^m$ with $m \leqslant n$ such that

$$0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_m = T,$$
$$\Gamma^f(t_i, t_{i+1}, v) = \delta \quad \text{for any integer } i \text{ satisfying } 0 \leqslant i < m - 1, \qquad \Gamma^f(t_{m-1}, t_m, v) \leqslant \delta. \tag{3.19}$$

Proof. By Proposition 2.7 there exists a constant $M_1 > 0$ such that

$$U_T^f(x, y) \leqslant T\mu(f) + M_1 \quad \text{for each } T \geqslant 1 \quad \text{and each} \quad x, y \in R^2 \quad \text{satisfying } |x|, |y| \leqslant M. \tag{3.20}$$

Together with (2.2) and (3.20) this implies that if $T \geqslant 1$ and if $v \in W^{2,1}([0, T])$ satisfies (3.18), then

$$\Gamma^f(0, T, v) \leqslant U_T^f\big(X_v(0), X_v(T)\big) + 1 - T\mu(f), \qquad -\pi^f\big(X_v(0)\big) + \pi^f\big(X_v(T)\big) \leqslant M_1 + 1 + 2M_2, \tag{3.21}$$

where

$$M_2 = \sup\big\{\left|\pi^f(z)\right|: z \in R^2 \text{ and } |z| \leqslant M\big\}. \tag{3.22}$$

Choose a natural number $n > 4$ such that

$$(n - 2)\delta > 2(M_2 + M_1 + 1). \tag{3.23}$$

Assume now that $T \geqslant 1$ and that $v \in W^{2,1}([0, T])$ satisfies (3.18). Then by (3.21) and (3.22),

$$\Gamma^f(0, T, v) \leqslant M_1 + 1 + 2M_2. \tag{3.24}$$

Clearly for each $\tau \in [0, T)$, $\lim_{s \to \tau^+} \Gamma^f(\tau, s, v) = 0$ and one of the following cases holds:

$\Gamma^f(\tau, T, v) \leqslant \delta$; there exists $\bar{\tau} \in (\tau, T)$ such that $\Gamma^f(\tau, \bar{\tau}, v) = \delta$.

This implies that there exist a natural number m and a sequence $\{t_i\}_{i=0}^m$ such that (3.19) is true. In order to complete the proof of the proposition it is sufficient to show that $m \leqslant n$. By (3.24), (3.19) and (3.23),

$$2M_2 + 1 + M_1 \geqslant \Gamma^f(0, T, v) \geqslant (m - 1)\delta$$

and

$$m \leqslant 1 + \delta^{-1}(2M_2 + 1 + M_1) < n.$$

Proposition 3.3 is proved. □

The following proposition is a result on the uniform equicontinuity of the family $(U_T^f)_{T \geqslant \tau}$ on bounded sets.

Proposition 3.4. *Let $M > 0$ and $\tau > 0$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $T \geqslant \tau$ and each $x, y, \bar{x}, \bar{y} \in R^2$ satisfying*

$$|x|, |y|, |\bar{x}|, |\bar{y}| \leqslant M, \qquad |x - \bar{x}|, |y - \bar{y}| \leqslant \delta \tag{3.25}$$

the following inequality holds:

$$\left|U_T^f(x, y) - U_T^f(\bar{x}, \bar{y})\right| \leqslant \epsilon. \tag{3.26}$$

Proof. Let $\epsilon > 0$. By Proposition 2.5 there exists a constant $M_1 > M$ such that for each $T \geqslant \tau$ and each $v \in W^{2,1}([0,T])$ satisfying

$$|X_v(0)|, |X_v(T)| \leqslant M, \qquad I^f(0,T,v) \leqslant U_T^f(X_v(0), X_v(T)) + 1 \tag{3.27}$$

the following inequality holds:

$$|X_v(t)| \leqslant M_1, \quad t \in [0,T]. \tag{3.28}$$

Since the function $U_{\tau/4}^f$ s continuous, it is uniformly continuous on compact subsets of $R^2 \times R^2$ and there exists $\delta > 0$ such that

$$|U_{\tau/4}^f(x,y) - U_{\tau/4}^f(\bar{x},\bar{y})| \leqslant \epsilon/4 \tag{3.29}$$

for each $x, y, \bar{x}, \bar{y} \in R^2$ satisfying

$$|x|, |y|, |\bar{x}|, \bar{y}| \leqslant M_1, \qquad |x-\bar{x}|, |y-\bar{y}| \leqslant \delta. \tag{3.30}$$

Assume that $x, y, \bar{x}, \bar{y} \in R^2$ satisfy (3.25) and that $T \geqslant \tau$. In order to prove the proposition it is sufficient to show that

$$U_T^f(\bar{x},\bar{y}) \leqslant U_T^f(x,y) + \epsilon.$$

There exists $v \in W^{2,1}([0,T])$ such that

$$X_v(0) = x, \qquad X_v(T) = y, \qquad I^f(0,T,v) = U_T^f(x,y). \tag{3.31}$$

By (3.31), (3.25) and the choice of M_1, (3.28) is valid. There exists $u \in W^{2,1}([0,T])$ such that

$$\begin{aligned}
&X_u(0) = \bar{x}, \qquad X_u(\tau/4) = X_v(\tau/4), \qquad I^f(0,\tau/4,u) = U_{\tau/4}^f(\bar{x}, X_v(\tau/4)),\\
&u(t) = v(t), \quad t \in [\tau/4, T-\tau/4],\\
&X_u(T-\tau/4) = X_v(T-\tau/4), \qquad X_u(T) = \bar{y},\\
&I^f(T-\tau/4, T, u) = U_{\tau/4}^f(X_v(T-\tau/4), \bar{y}).
\end{aligned} \tag{3.32}$$

It follows from (3.25) and (3.28) and the choice of δ (see (3.29) and (3.30)) that

$$\begin{aligned}
&|U_{\tau/4}^f(\bar{x}, X_v(\tau/4)) - U_{\tau/4}^f(x, X_v(\tau/4))| \leqslant \epsilon/4,\\
&|U_{\tau/4}^f(X_v(T-\tau/4), \bar{y}) - U_{\tau/4}^f(X_v(T-\tau/4), y)| \leqslant \epsilon/4.
\end{aligned}$$

It follows from the inequalities above, (3.32) and (3.31) that

$$\begin{aligned}
U_T^f(\bar{x},\bar{y}) &\leqslant I^f(0,T,u) = I^f(0,\tau/4,u) + I^f(\tau/4, T-\tau/4, u) + I^f(T-\tau/4, T, u)\\
&= U_{\tau/4}^f(\bar{x}, X_v(\tau/4)) + I^f(\tau/4, T-\tau/4, u) + U_{\tau/4}^f(X_v(T-\tau/4), \bar{y})\\
&\leqslant U_{\tau/4}^f(x, X_v(\tau/4)) + \epsilon/4 + I^f(\tau/4, T-\tau/4, u) + U_{\tau/4}^f(X_v(T-\tau/4), y) + \epsilon/4\\
&= I^f(0,T,v) + \epsilon/2 = U_T^f(x,y) + \epsilon/2.
\end{aligned}$$

Proposition 3.4 is proved. □

Proposition 3.5. *Suppose that*

$$\mu(f) < \inf\{f(t,0,0):\ t \in R^1\}.$$

Let $\epsilon > 0$. *Then there exist* $q > 0$ *and* $\delta > 0$ *such that the following assertion holds*:

Let $T \geqslant q$, $w \in \mathcal{A}(f)$,

$$x, y \in R^2, \quad d(x, \Omega(w)), d(y, \Omega(w)) \leqslant \delta. \tag{3.33}$$

Then there exists $v \in W^{2,1}([0,T])$ *which satisfies*

$$X_v(0) = x, \qquad X_v(\tau) = y, \qquad \Gamma^f(0,\tau,v) \leqslant \epsilon. \tag{3.34}$$

Proof. By Proposition 2.8 for each $w \in \mathcal{A}(f)$ there exist $\delta(w), q(w) > 0$ such that the following property holds:

(P1) If $T \geqslant q(w)$ and if $x, y \in R^2$ satisfy $d(x, \Omega(w)), d(y, \Omega(w)) \leqslant \delta(w)$, then there exists $v \in W^{2,1}([0,T])$ which satisfies (3.34).

By Propositions 2.4 and 3.1,

$$\bar{T} := \sup\{\tau(w):\ w \in \mathcal{A}(f)\} < \infty, \tag{3.35}$$

$$\inf\{\tau(w):\ w \in \mathcal{A}(f)\} > 0. \tag{3.36}$$

Define

$$E = \bigcup\{\Omega(w) \times \Omega(w):\ w \in \mathcal{A}(f)\}. \tag{3.37}$$

We will show that E is compact. In view of (3.2) it is sufficient to show that E is closed.

Let

$$\{(x_i, y_i)\}_{i=1}^{\infty} \subset E, \quad \lim_{i\to\infty}(x_i, y_i) = (x, y). \tag{3.38}$$

We show that $(x, y) \in E$. For each natural number i there exist $w_i \in \mathcal{A}(f)$, $s_i, t_i \in [0, \infty)$ such that

$$x_i = \big(w_i(t_i), w_i'(t_i)\big), \qquad y_i = \big(w_i(s_i), w_i'(s_i)\big). \tag{3.39}$$

In view of (3.35) we may assume that

$$t_i, s_i \in [0, \bar{T}], \quad i = 1, 2, \ldots. \tag{3.40}$$

By (3.2) and the continuity of $U^f_{\bar{T}}$, the sequence $\{I^f(0, \bar{T}, w_i)\}_{i=1}^{\infty}$ is bounded. Combined with (3.2) and (1.1) this implies that the sequence $\{\int_0^{\bar{T}} |w_i''(t)|^{\gamma}\, dt\}_{i=1}^{\infty}$ is bounded. Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that there exist

$$t_* = \lim_{i\to\infty} t_i, \qquad s_* = \lim_{i\to\infty} s_i, \qquad \tau_* = \lim_{i\to\infty} \tau(w_i) \tag{3.41}$$

and there exists $u \in W^{2,\gamma}([0, \bar{T}])$ such that

$$\begin{aligned} &w_i \to u \quad \text{as } i \to \infty \text{ weakly in } W^{2,\gamma}\big([0,\bar{T}]\big),\\ &\big(w_i, w_i'\big) \to (u, u') \quad \text{as } i \to \infty \text{ uniformly on } [0, \bar{T}]. \end{aligned} \tag{3.42}$$

By (3.42), (3.2), the continuity of π^f, and the lower semicontinuity of integral functionals [4],

$$\Gamma^f(0, \bar{T}, u) \leqslant \liminf_{i\to\infty} \Gamma^f(0, \bar{T}, w_i) = 0$$

and $\Gamma^f(0, \bar{T}, u) = 0$.

It follows from (3.38), (3.39), (3.40), (3.42) and (3.41) that

$$x = \lim_{i\to\infty} x_i = \lim_{i\to\infty}\big(w_i(t_i), w_i'(t_i)\big) = \lim_{i\to\infty}\big(u(t_i), u'(t_i)\big) = \big(u(t_*), u'(t_*)\big), \tag{3.43}$$

$$y = \lim_{i\to\infty} y_i = \lim_{i\to\infty}\big(w_i(s_i), w_i'(s_i)\big) = \lim_{i\to\infty}\big(u(s_i), u'(s_i)\big) = \big(u(s_*), u'(s_*)\big). \tag{3.44}$$

By (3.42), the inclusion $w_i \in \mathcal{A}(f)$, $i = 1, 2, \ldots$, (3.35) and (3.41),

$$X_u(0) = \lim_{i\to\infty} X_{w_i}(0) = \lim_{i\to\infty} X_{w_i}\big(\tau(w_i)\big) = \lim_{i\to\infty} X_u\big(\tau(w_i)\big) = X_u(\tau_*).$$

In view of (3.41), (3.40) and (3.36),

$$0 < \tau_* \leqslant \bar{T}.$$

We have shown that

$$X_u(0) = X_u(\tau_*), \quad 0 \leqslant \Gamma^f(0, \tau_*, u) \leqslant \Gamma^f(0, \bar{T}, u) = 0.$$

This implies that u can be extended on the infinite interval $[0, \infty)$ as a periodic (f)-good function with the period τ_*. Thus we have that $u \in \mathcal{A}(f)$ and in view of (3.43), (3.44) and (3.37)

$$(x, y) \in \Omega(u) \times \Omega(u) \subset E.$$

Therefore E is compact. For each $w \in \mathcal{A}(f)$ define an open set $\mathcal{U}(w) \subset R^4$ by

$$\mathcal{U}(w) = \{(x, y) \in R^4\colon\ d(x, \Omega(w)) < \delta(w)/4,\ d(y, \Omega(w)) < \delta(w)/4\}. \tag{3.45}$$

Then $\mathcal{U}(w)$, $w \in \mathcal{A}(f)$ is an open covering of the compact E and there exists a finite set $\{w_1, \ldots, w_n\} \in \mathcal{A}(f)$ such that

$$E \subset \bigcup_{i=1}^{n} \mathcal{U}(w_i). \tag{3.46}$$

Set

$$q = \max\{q(w_i)\colon\ i = 1, \ldots, n\}, \qquad \delta = \min\{\delta(w_i)/4\colon\ i = 1, \ldots, n\}. \tag{3.47}$$

Let $T \geqslant q$, $w \in \mathcal{A}(f)$ and let $x, y \in R^2$ satisfy (3.33). There exist

$$\tilde{x}, \tilde{y} \in \Omega(w) \tag{3.48}$$

such that

$$|x - \tilde{x}|, |y - \tilde{y}| \leqslant \delta. \tag{3.49}$$

In view of (3.37), (3.46) and (3.48), $(\tilde{x}, \tilde{y}) \in E$ and there is $j \in \{1, \ldots, n\}$ such that

$$(\tilde{x}, \tilde{y}) \in \mathcal{U}(w_j). \tag{3.50}$$

Relations (3.50) and (3.45) imply that there exist

$$\bar{x}, \bar{y} \in \Omega(w_j) \tag{3.51}$$

such that

$$|\tilde{x} - \bar{x}|, |\tilde{y} - \bar{y}| < \delta(w_j)/4. \tag{3.52}$$

By (3.49), (3.52) and (3.47)

$$|x - \bar{x}|, |y - \bar{y}| < \delta + \delta(w_j)/4 \leqslant \delta(w_j)/2.$$

It follows from this inequalities, (3.51), property (P1) with $w = w_j$, (3.47) and the inequality $T \geqslant q$ that there exists $v \in W^{2,1}([0, T])$ satisfying (3.34). Proposition 3.5 is proved. □

4. Proof of Theorem 1.1

By Proposition 3.4 in order to prove the theorem it is sufficient to show that for each $x, y \in R^2$ there exists

$$\lim_{T \to \infty} [U_T^f(x, y) - T\mu(f)].$$

Let $x, y \in R^2$ and fix $\epsilon > 0$. We will show that there exist $\bar{T} > 0$ and $q > 0$ such that

$$U_S^f(x, y) - S\mu(f) \leqslant U_T^f(x, y) - T\mu(f) + \epsilon \tag{4.1}$$

for each $T \geqslant \bar{T}$ and each $S \geqslant T + q$.

By Proposition 3.5 there exist $q > 0$, $\delta_0 > 0$ such that for the following property holds:

(P2) For each $T \geqslant q$, each $w \in \mathcal{A}(f)$ and each $x, y \in R^2$ satisfying

$$d(x, \Omega(w)), d(y, \Omega(w)) \leqslant \delta_0 \tag{4.2}$$

there exists $v \in W^{2,1}([0, T])$ such that

$$X_v(0) = x, \qquad X_v(T) = y, \qquad \Gamma^f(0, T, v) \leqslant \epsilon. \tag{4.3}$$

In view of Proposition 2.4 there exists a real number

$$l > \sup\{\tau(w):\ w \in \mathcal{A}(f)\}. \tag{4.4}$$

Choose

$$M_0 > |x| + |y| + 2. \tag{4.5}$$

By Proposition 2.5 there exists $M_1 > M_0$ such that for each $T \geqslant 1$ and each $v \in W^{2,1}([0,T])$ satisfying

$$|X_v(0)|, |X_v(T)| \leqslant M_0, \qquad I^f(0,T,v) \leqslant U_T^f\big(X_v(0), X_v(T)\big) + 1 \tag{4.6}$$

the following inequality holds:

$$|X_v(T)| \leqslant M_1, \quad t \in [0,T]. \tag{4.7}$$

By Proposition 3.2 there exist $\delta_1 > 0$, $L_1 > l$ such that for each $T \geqslant L_1$ and each $v \in W^{2,1}([0,T])$ satisfying

$$|X_v(0)|, |X_v(T)| \leqslant M_1, \qquad \Gamma^f(0,T,v) \leqslant \delta_1 \tag{4.8}$$

there exist $\sigma \in [0, T-l]$ and $w \in \mathcal{A}(f)$ such that

$$|X_v(\sigma + t) - X_w(t)| \leqslant \delta_0, \quad t \in [0,l]. \tag{4.9}$$

By Proposition 3.3 there exists a natural number n such that for each $T \geqslant 1$ and each $v \in W^{2,1}([0,T])$ satisfying

$$|X_v(0)|, |X_v(T)| \leqslant M_1, \qquad I^f(0,T,v) \leqslant U_T^f\big(X_v(0), X_v(T)\big) + 1 \tag{4.10}$$

there exists a sequence $\{t_i\}_{i=0}^m \subset [0,T]$ with $m \leqslant n$ such that

$$0 = t_0 < \cdots < t_i < t_{i+1} < \cdots < t_m = T, \tag{4.11}$$

$$\Gamma^f(t_i, t_{i+1}, v) = \delta_1 \quad \text{for all integers } i \text{ satisfying } 0 \leqslant i < m-1,$$
$$\Gamma^f(t_{m-1}, t_m, v) \leqslant \delta_1. \tag{4.12}$$

Choose a number

$$\bar{T} > 1 + nL_1. \tag{4.13}$$

Let

$$T \geqslant \bar{T}, \qquad S \geqslant T + q. \tag{4.14}$$

There exists $v \in W^{2,1}([0,T])$ such that

$$X_v(0) = x, \qquad X_v(T) = y, \qquad I^f(0,T,v) = U_T^f(x,y). \tag{4.15}$$

By (4.5), (4.13), (4.14), the choice of M_1 and (4.15), the inequality (4.7) holds. In view of (4.15), the choice of n (see (4.10)–(4.12)), (4.14), (4.13) and (4.5) there exists a sequence $\{t_i\}_{i=0}^m \subset [0,T]$ with $m \leqslant n$ such that (4.11) and (4.12) hold. It follows from (4.14), (4.13) and (4.11) that

$$\max\{t_{i+1} - t_i:\ i = 0,\dots,m-1\} \geqslant T/m \geqslant \bar{T}/n > L_1.$$

Thus there exists $j \in \{0,\dots,m-1\}$ such that

$$t_{j+1} - t_j > L_1. \tag{4.16}$$

By (4.16), (4.7), (4.12) and the choice of δ_1, L_1 (see (4.8), (4.9)) there exist

$$\sigma \in [t_j, t_{j+1} - l], \quad w \in \mathcal{A}(f) \tag{4.17}$$

such that (4.9) holds.

In particular

$$d\big(X_v(\sigma), \Omega(w)\big) \leqslant \delta_0. \tag{4.18}$$

It follows from (4.14), (4.17), the property (P2) and (4.18) that there exists

$$h \in W^{2,1}\big([\sigma, \sigma + S - T]\big)$$

such that

$$X_h(\sigma) = X_v(\sigma), \qquad X_h(\sigma + S - T) = X_v(\sigma),$$
$$\Gamma^f(\sigma, \sigma + S - T, h) \leqslant \epsilon. \tag{4.19}$$

It is easy to see that there exist $u \in W^{2,1}([0, S])$ such that

$$u(t) = v(t), \quad t \in [0, \sigma], \qquad u(t) = h(t), \quad t \in [\sigma, \sigma + S - T],$$
$$u(\sigma + S - T + t) = v(\sigma + t), \quad t \in [0, T - \sigma]. \tag{4.20}$$

By (4.20) and (4.15),

$$X_u(0) = x, \qquad X_u(S) = y. \tag{4.21}$$

By (4.21), (2.2), (4.15), (4.20) and (4.19),

$$\begin{aligned} U_S^f(x, y) - S\mu(f) &\leqslant I^f(0, S, u) - S\mu(f) \\ &= \pi^f\big(X_u(0)\big) - \pi^f\big(X_u(S)\big) + \Gamma^f(0, S, u) \\ &= \pi^f\big(X_u(0)\big) - \pi^f\big(X_u(S)\big) + \Gamma^f(0, \sigma, u) + \Gamma^f(\sigma, \sigma + S - T, u) + \Gamma^f(\sigma + S - T, S, u) \\ &= \pi^f\big(X_v(0)\big) - \pi^f\big(X_v(T)\big) + \Gamma^f(0, \sigma, v) + \epsilon + \Gamma^f(\sigma, T, v) \\ &= \epsilon + I^f(0, T, v) - T\mu(f) = U_T^f(x, y) - T\mu(f) + \epsilon. \end{aligned}$$

Thus we have shown that (4.1) holds for each $T \geqslant \bar{T}$ and each $S \geqslant T + q$. By Proposition 2.7

$$\sup\big\{U_T^f(x, y) - T\mu(f)\colon T \in [1, \infty)\big\} < \infty.$$

On the other hand by (1.6) for each $T \geqslant 1$

$$U_T^f(x, y) - T\mu(f) \geqslant \pi^f(x) - \pi^f(y).$$

Hence the set $\{U_T^f(x, y)\colon T \in [1, \infty)\}$ is bounded. Put

$$d_* = \lim_{T\to\infty} \inf\big\{U_S^f(x, y) - S\mu(f)\colon S \in [T, \infty)\big\}. \tag{4.22}$$

We show that

$$d_* = \lim_{T\to\infty}\big[U_T^f(x, y) - T\mu(f)\big].$$

Let $\epsilon > 0$. We have shown that there exist $\bar{T} > 0$, $q > 0$ such that (4.1) holds for each $T \geqslant \bar{T}$ and each $S \geqslant T + q$. By (4.22) there exists $T_0 \geqslant \bar{T}$ such that

$$d_* \geqslant \inf\big\{U_S^f(x, y) - S\mu(f)\colon S \in [T_0, \infty)\big\} \geqslant d_* - \epsilon. \tag{4.23}$$

There exists $T_1 \geqslant T_0$ such that

$$\big|U_{T_1}^f(x, y) - T_1\mu(f) - \inf\big\{U_S^f(x, y) - S\mu(f)\colon S \in [T_0, \infty)\big\}\big| \leqslant \epsilon. \tag{4.24}$$

Let $T \geqslant T_1 + q$. Then in view of (4.23)

$$U_T^f(x, y) - T\mu(f) \geqslant \inf\big\{U_S^f(x, y) - S\mu(f)\colon S \in [T_0, \infty)\big\} \geqslant d_* - \epsilon.$$

On the other hand by the relation $T \geqslant T_1 + q \geqslant T_0 + q \geqslant \bar{T} + q$, (4.1) (which holds with $T = T_1$, $S = T$), (4.24) and (4.23)

$$\begin{aligned} U_T^f(x, y) - T\mu(f) &\leqslant U_{T_1}^f(x, y) - T_1\mu(f) + \epsilon \\ &\leqslant \inf\big\{U_S^f(x, y) - S\mu(f)\colon S \in [T_0, \infty)\big\} + 2\epsilon \leqslant d_* + 2\epsilon. \end{aligned}$$

Therefore

$$\left|U_T^f(x,y) - T\mu(f) - d_*\right| \leqslant 2\epsilon \quad \text{for all } T \geqslant T_1 + q.$$

Since ϵ is an arbitrary positive number we conclude that

$$d_* = \lim_{T\to\infty}\left[U_T^f(x,y) - T\mu(f)\right].$$

Theorem 1.1 is proved.

5. Proof of Theorem 1.2

Consider the function $U_\infty^f : R^2 \times R^2 \to R^1$ defined in Theorem 1.1:

$$U_\infty^f(x,y) = \lim_{T\to\infty}\left[U_T^f(x,y) - T\mu(f)\right], \quad x, y \in R^2. \tag{5.1}$$

By Proposition 2.10 there exists $M > 0$ such that for each $T \geqslant 1$ and each $v \in W^{2,1}([0,T])$ satisfying

$$I^f(0,T,v) \leqslant \inf\left\{U_T^f(x,y):\ x,y \in R^2\right\} + 1 \tag{5.2}$$

the following inequality holds:

$$\left|X_v(t)\right| \leqslant M, \quad t \in [0,T]. \tag{5.3}$$

Let $x, y \in R^2$ satisfy $\max\{|x|,|y|\} > T \geqslant 1$. Then by the choice of M,

$$U_T^f(x,y) > \inf\left\{U_T^f(z_1,z_2):\ z_1,z_2 \in R^2\right\} + 1.$$

This implies that for each $T \geqslant 1$

$$\inf\left\{U_T^f(x,y):\ x,y \in R^2 \text{ and } \max\left\{|x|,|y|\right\} > M\right\} \geqslant \inf\left\{U_T^f(x,y):\ x,y \in R^2\right\} + 1. \tag{5.4}$$

Put

$$E_1 = \left\{(x,y) \in R^2 \times R^2:\ \max\left\{|x|,|y|\right\} > M\right\}, \qquad E_2 = \left(R^2 \times R^2\right) \setminus E_1. \tag{5.5}$$

In view of (5.5) and (5.4) for any $T \geqslant 1$

$$\inf\left\{U_T^f(x,y) - T\mu(f):\ (x,y) \in E_1\right\} \geqslant \inf\left\{U_T^f(x,y) - T\mu(f):\ (x,y) \in E_2\right\} + 1. \tag{5.6}$$

By Theorem 1.1

$$U_T^f(x,y) - T\mu(f) \to U_\infty^f(x,y) \quad \text{as } T \to \infty \tag{5.7}$$

uniformly on E_2. This implies that

$$\lim_{T\to\infty} \inf\left\{U_T^f(x,y) - T\mu(f):\ x,y \in E_2\right\} = \inf\left\{U_\infty^f(x,y):\ (x,y) \in E_2\right\}. \tag{5.8}$$

Let $(z,\bar z) \in E_1$. Then by (5.1), (5.6) and (5.8)

$$\begin{aligned} U_\infty^f(z,\bar z) &= \lim_{T\to\infty}\left[U_T^f(z_1,\bar z) - T\mu(f)\right] \\ &\geqslant \lim_{T\to\infty}\left[\inf\left\{U_T^f(x,y) - T\mu(f):\ (x,y) \in E_2\right\} + 1\right] \\ &= \inf\left\{U_\infty^f(x,y):\ (x,y) \in E_2\right\} + 1. \end{aligned} \tag{5.9}$$

Since the function U_∞^f is continuous the set

$$E_\infty := \left\{(x,y) \in E_2:\ U_\infty^f(x,y) = \inf\left\{U_\infty^f(z):\ z \in E_2\right\}\right\} \tag{5.10}$$

is nonempty and compact. In view of (5.9) and (5.10)

$$U_\infty^f(z) \geqslant U_\infty^f(y) + 1 \quad \text{for each } z \in E_1 \text{ and each } y \in E_\infty. \tag{5.11}$$

Let $\epsilon > 0$. Using standard arguments and compactness of E_2 we can show that there exists $\delta \in (0, 8^{-1})$ such that

$$\text{if } z \in R^4 \quad \text{satisfies } U^f_\infty(z) \leqslant \inf\{U^f_\infty(y)\colon\ y \in R^4\} + 4\delta, \quad \text{then } d(z, E_\infty) \leqslant \epsilon. \tag{5.12}$$

By Theorem 1.1 there exists $\bar{T} > 1$ such that

$$\left|U^f_T(x, y) - T\mu(f) - U^f_\infty(x, y)\right| \leqslant \delta \quad \text{for any } T \geqslant \bar{T} \text{ and any } (x, y) \in E_2. \tag{5.13}$$

Assume that

$$T \geqslant \bar{T}, \quad (x, y) \in R^2 \times R^2, \qquad U^f_T(x, y) \leqslant \inf\{U^f_T(z)\colon\ z \in R^4\} + \delta. \tag{5.14}$$

In view of (5.14), (5.5) and (5.6),

$$(x, y) \in E_2. \tag{5.15}$$

By (5.15), (5.14) and (5.13),

$$\left|U^f_T(x, y) - \mu(f)T - U^f_\infty(x, y)\right| \leqslant \delta. \tag{5.16}$$

By (5.14), (5.6), (5.9) and (5.13),

$$\begin{aligned}&\left|\inf\{U^f_T(z) - T\mu(f)\colon\ z \in R^4\} - \inf\{U^f_\infty(z)\colon\ z \in R^4\}\right| \\ &\quad = \left|\inf\{U^f_T(z) - T\mu(f)\colon\ z \in E_2\} - \inf\{U^f_\infty(z)\colon\ z \in E_2\}\right| \leqslant \delta.\end{aligned}$$

Combined with (5.16) and (5.14) this implies that

$$\begin{aligned}U^f_\infty(x, y) &\leqslant U^f_T(x, y) - \mu(f)T + \delta \leqslant \inf\{U^f_T(z) - T\mu(f)\colon\ z \in R^4\} + 2\delta \\ &\leqslant \inf\{U^f_\infty(z)\colon\ z \in R^4\} + 3\delta.\end{aligned}$$

By the relation above and (5.12), $d((x, y), E_\infty) \leqslant \epsilon$. Theorem 1.2 is proved.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] S. Aubry, P.Y. Le Daeron, The discrete Frenkel–Kontorova model and its extensions I, Phys. D 8 (1983) 381–422.

[3] J. Baumeister, A. Leitao, G.N. Silva, On the value function for nonautonomous optimal control problem with infinite horizon, Systems Control Lett. 56 (2007) 188–196.

[4] L.D. Berkovitz, Lower semicontinuity of integral functionals, Trans. Amer. Math. Soc. 192 (1974) 51–57.

[5] J. Blot, P. Cartigny, Optimality in infinite-horizon variational problems under sign conditions, J. Optim. Theory Appl. 106 (2000) 411–419.

[6] J. Blot, P. Michel, The value-function of an infinite-horizon linear quadratic problem, Appl. Math. Lett. 16 (2003) 71–78.

[7] B.D. Coleman, M. Marcus, V.J. Mizel, On the thermodynamics of periodic phases, Arch. Ration. Mech. Anal. 117 (1992) 321–347.

[8] A. Leizarowitz, Tracking nonperiodic trajectories with the overtaking criterion, Appl. Math. Optim. 14 (1986) 155–171.

[9] A. Leizarowitz, V.J. Mizel, One dimensional infinite horizon variational problems arising in continuum mechanics, Arch. Ration. Mech. Anal. 106 (1989) 161–194.

[10] V.L. Makarov, A.M. Rubinov, Mathematical Theory of Economic Dynamics and Equilibria, Springer-Verlag, New York, 1977.

[11] M. Marcus, Universal properties of stable states of a free energy model with small parameters, Calc. Var. Partial Differential Equations 6 (1998) 123–142.

[12] M. Marcus, A.J. Zaslavski, On a class of second order variational problems with constraints, Israel J. Math. 111 (1999) 1–28.

[13] M. Marcus, A.J. Zaslavski, The structure of extremals of a class of second order variational problems, Ann. Inst. H. Poincaré Anal. Non Linéare 16 (1999) 593–629.

[14] M. Marcus, A.J. Zaslavski, The structure and limiting behavior of locally optimal minimizers, Ann. Inst. H. Poincaré Anal. Non Linéare 19 (2002) 343–370.

[15] B. Mordukhovich, Minimax design for a class of distributed parameter systems, Autom. Remote Control 50 (1990) 1333–1340.

[16] B. Mordukhovich, I. Shvartsman, Optimization and feedback control of constrained parabolic systems under uncertain perturbations, in: Optimal Control, Stabilization and Nonsmooth Analysis, in: Lecture Notes Control Inform. Sci., Springer, 2004, pp. 121–132.

[17] J. Moser, Minimal solutions of variational problems on a torus, Ann. Inst. H. Poincaré Anal. Non Linéare 3 (1986) 229–272.

[18] P.H. Rabinowitz, E. Stredulinsky, On some results of Moser and of Bangert, Ann. Inst. H. Poincaré Anal. Non Linéare 21 (2004) 673–688.

[19] P.H. Rabinowitz, E. Stredulinsky, On some results of Moser and of Bangert. II, Adv. Nonlinear Stud. 4 (2004) 377–396.

[20] A.J. Zaslavski, Ground states in Frenkel–Kontorova model, Math. USSR Izv. 29 (1987) 323–354.

[21] A.J. Zaslavski, The existence of periodic minimal energy configurations for one-dimensional infinite horizon variational problems arising in continuum mechanics, J. Math. Anal. Appl. 194 (1995) 459–476.

[22] A.J. Zaslavski, The existence and structure of extremals for a class of second order infinite horizon variational problems, J. Math. Anal. Appl. 194 (1995) 660–696.

[23] A.J. Zaslavski, Structure of extremals for one-dimensional variational problems arising in continuum mechanics, J. Math. Anal. Appl. 198 (1996) 893–921.

[24] A.J. Zaslavski, Existence and structure of optimal solutions of variational problems, in: Proceedings of the Special Session on Optimization and Nonlinear Analysis, Joint AMS-IMU Conference, Jerusalem, May 1995, in: Contemp. Math., vol. 204, 1997, pp. 247–278.

[25] A.J. Zaslavski, Turnpike Properties in the Calculus of Variations and Optimal Control, Springer, New York, 2006.

[26] A.J. Zaslavski, On a class of infinite horizon variational problems, Comm. Appl. Nonlinear Anal. 13 (2006) 51–57.