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A remark on gauge transformations and the moving frame method

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Abstract

In this note we give a shorter proof of recent regularity results on elliptic partial differential equations with antisymmetric structure presented in Rivière (2007) [23], Rivière and Struwe (2008) [24]. We differ from the mentioned articles in using the direct method of Hélein's moving frame, i.e. minimizing a certain variational energy-functional, in order to construct a suitable gauge transformation. Though this is neither new nor surprising, it enables us to describe a proof of regularity using elementary arguments of calculus of variations and algebraic identities.

Moreover, we remark that in order to prove Hildebrandt's conjecture on regularity of critical points of 2D-conformally invariant variational problems one can avoid the application of the Nash–Moser imbedding theorem.

Résumé

La note contient une démonstation plus courte des résultats récents sur la régularité des solutions d'équations aux dérivées partielles ayant une structure antisymétrique comme dans Rivière (2007) [23], Rivière and Struwe (2008) [24]. La différence par rapport méthodes dans ces articles est qu'on utilise directement les « repères mobiles » developpés par Hélein, c'est – à – dire la minimisation d'une énergie variationnelle, dans la but de construire une transformation de Jauge. Même si ce n'est ni nouveau ni étonnant, ceci nous permet de mener une démonstration de régularité par des arguments élémentaires du calcul variationnel et des identités algébriques.

De plus, nous remarquons que la conjecture d'Hildebrandt, concernant la regularité des points critiques des problèmes variationnels invariants sous des transformations conformes, ne nécessite pas l'application du théorème dimmersion de Nash et Moser.

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1. Introduction

In the influential article [23] Rivière discovered that Euler equations of conformally invariant variational functionals acting on maps $U \in W^{1,2}(\mathcal{M},\mathcal{N})$ from two-dimensional manifolds $\mathcal{M}$ into n-dimensional manifolds $\mathcal{N}$ can locally be written in the form

$$\Delta u^i = \Omega_{ik} \cdot \nabla u^k \quad \text{in } B_1(0),\ 1 \leqslant i \leqslant n, \tag{1.1}$$

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where $\Omega_{ij} = -\Omega_{ji} \in L^2(B_1(0), \mathbb{R}^2)$ and $u \in W^{1,2}(B_1(0), \mathcal{N})$ is a local representation of U. Here and in the following we adopt Einstein's summation convention, summing over repeated indices. For an overview of the geometric problems and the development towards the regularity result finally achieved, the interested reader is referred to the detailed introduction in [23].

The right-hand side of (1.1) is only in L^1, and hence there is no standard theory in order to conclude better regularity as e.g. continuity of u. Using an algebraic feature, namely the antisymmetry of Ω, one can construct a gauge transformation $P \in W^{1,2}(B_1(0), SO(n))$ which pointwise almost everywhere is an orthogonal matrix in $\mathbb{R}^{n\times n}$ such that

$$\operatorname{div}\bigl(P^T_{ik}\nabla P_{kj} - P^T_{ik}\Omega_{kl}P_{lj}\bigr) = 0 \quad \text{in } B_1(0),\ 1 \leqslant i, j \leqslant n. \tag{1.2}$$

Statements on matrices like the last one will often be abbreviated by omitting matrix indices. That is, instead of (1.2) we will write

$$\operatorname{div}\bigl(P^T\nabla P - P^T\Omega P\bigr) = 0 \quad \text{in } B_1(0). \tag{1.3}$$

Then, by solving an extra system of PDEs Rivière finds an invertible matrix $A \in W^{1,2} \cap L^\infty(B_1(0), GL(n))$ such that

$$\operatorname{div}(\nabla A + A\Omega) = 0 \quad \text{in } B_1(0). \tag{1.4}$$

Using this, (1.1) transforms into

$$\operatorname{div}(A\nabla u) = (\nabla A + A\Omega)\cdot\nabla u \quad \text{in } B_1(0). \tag{1.5}$$

By [21,5] the right-hand side lies in the Hardy-space $\mathcal{H}$. This is a strict subspace of L^1 featuring a good behavior when being convoluted with Calderon–Zygmund kernels, and this implies in particular continuity of u. (A great source on this is e.g. [26], for an overview with a focus on PDE one might also want to look into [25].) The way of constructing A seems to be purely two-dimensional, as it crucially relies on L^∞-bounds of Wente's inequality (for the statement see [23, Lemma A.1], for proofs see [30], [27, Chapter II], [1, Lemma A.1] or [13, Chapter 3]). Adapting Rivière's idea in its spirit to higher dimensions, in [24] it is shown how to prove regularity by a Dirichlet growth approach without having to construct A but working with P instead.

In order to construct P, in [23] a beautiful yet involved technique by Uhlenbeck [28] is applied, which relies on a continuity argument and the implicit function theorem.

One purpose of this note is to establish in Section 2, Theorem 2.1, the existence of the Coulomb-gauge P as in (1.3) by means of a rather elementary variational approach as in Hélein's moving frame method developed in the 90's ([12], see also [13, Chapter 4] and the appendix of [3]): We simply minimize the following energy integral

$$E(Q) := \int\limits_{B_1(0)} \bigl|Q^T\nabla Q - Q^T\Omega Q\bigr|^2, \quad Q \in W^{1,2}\bigl(B_1(0), SO(n)\bigr), \tag{1.6}$$

whose critical points $P \in W^{1,2}(B_1(0), SO(n))$ satisfy (1.3). Here, we denote $W^{1,2}(B_1(0), SO(n))$ to be all those functions $Q \in W^{1,2}(B_1(0), \mathbb{R}^{n\times n})$ such that $Q(x)$ is an orthogonal matrix with $\det Q(x) > 0$ almost everywhere in $B_1(0)$. Neither is there any theory of Hardy and BMO spaces necessary at this stage, nor do we use an approximation of Ω or some kind of smallness conditions on Ω, all of which is needed in the proof of Theorem 2.1 as done in [23, Lemma A.3].

From the existence of P minimizing (1.6) and thus satisfying (1.3) one gets regularity of solutions to (1.1) by applying a Dirichlet growth estimate to

$$\operatorname{div}\bigl(P^T\nabla u\bigr) = -\bigl(P^T\nabla P - P^T\Omega P\bigr)P^T\nabla u. \tag{1.7}$$

This approach using P instead of A and (1.4) is due to [24] and was used there to extend the results of [23] to higher dimensions.

All in all, constructing P by minimizing (1.6) as in [12], and then using the Dirichlet growth theorem as in [24] one gets a simplified proof of [23, Theorem I.1].

Interestingly, this simplification can be applied as well to the case of dimensions greater than two: In order to prove [24, Theorem 1.1] one does not need to prove that P belongs to some Morrey-space. The L^2-estimates on the gradient of P resulting from minimizing (1.6) are sufficient – this will be sketched in Section 3.

Moreover, in this note we describe an observation regarding Hildebrandt's conjecture that critical points $u \in W^{1,2}(B_1(0), \mathbb{R}^n)$, $B_1(0) \subset \mathbb{R}^2$, of conformally invariant variational problems are in fact continuous. Problems of this type are equivalent to a Dirichlet-energy minimization for $u \in W^{1,2}(B_1(0), (\mathbb{R}^n, g))$ perturbed by an antisymmetric term where g is a certain Riemannian metric. This was shown in [9]. Using in a first step the Nash–Moser isometric imbedding theorem in order to avoid in the associated Euler–Lagrange equation the appearance of non-antisymmetric terms with Christoffel-symbols stemming from the metric g, the conjecture was completely solved in [23]. As the proof of the Nash–Moser theorem is very involved, we remark on how to avoid this deep result by a simple decomposition of the metric components $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ into products of linearly independent vectors $(e_i)_{i=1}^n$. By this, one can construct a system similar to (1.1). More details can be found in Section 4.

Let us conclude this introduction by comparing the two seemingly very different methods of Rivière's proof and Hélein's moving frame method. For this purpose we consider as a prominent example the case of weakly harmonic maps $v \in W^{1,2}(B_1(0), \mathcal{N})$ from the unit ball $B_1(0) \subset \mathbb{R}^2$ into an n-dimensional compact manifold $\mathcal{N}$ which is isometrically embedded in $\mathbb{R}^N$. By [13, Lemma 1.4.10] a map like this satisfies

$$\Delta v \perp T_v\mathcal{N} \quad \text{in } B_1(0). \tag{1.8}$$

Orthogonality means orthogonality in the sense of the Euclidean metric in $\mathbb{R}^N$. As explained in [13, Section 4.1] we can assume furthermore that there is some moving frame on $(\mathcal{N}, T\mathcal{N})$: That is, there are smooth tangent vectors $e_i : \mathcal{N} \to T\mathcal{N}$, $1 \leqslant i \leqslant n$, such that at any point $y \in \mathcal{N}$ the $e_i(y)$ form an orthonormal basis of the tangential space $T_y\mathcal{N}$. It is then not too difficult to see, that by (1.8)

$$\operatorname{div}\big(\langle e_i(v), \nabla v\rangle\big) = \big\langle e_i(v), \nabla e_k(v)\big\rangle\big\langle e_k(v), \nabla v\big\rangle, \quad 1 \leqslant i \leqslant n.$$

The scalar product $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^N$, that is $\langle e_i(v), \nabla v\rangle := \sum_{a=1}^N e_i^a(v)\nabla v^a$. Setting $\Omega_{ij} := \langle e_i, \nabla e_j\rangle$ one observes the similarity with (1.1) – instead of ∇u^i in (1.1), here we have $\langle e_i(v), \nabla v\rangle$. But from the point of view of growth estimates regarding ∇v this is not a big difference: Pointwise a.e. one can compare the size of $(\langle e_i(v), \nabla v\rangle)_{i=1}^n$ to the size of ∇v.

Thus in some way what was done up to this point is to bring (1.8) into a form similar to (1.1). One of the key observations in [23] is that this can be done for more general situations than (1.8) – even in cases where one would not be able to speak of "moving frames" as e.g. in the case of H-surfaces.

The next step is to transform the moving frame $(e_i \circ v)_{i=1}^n$ into one that is more suitable for our equation, namely we seek $f_i = P_{ik}^T e_k \circ v$, where $P \in W^{1,2}(B_1(0), SO(n))$ is almost everywhere an orthogonal matrix in $\mathbb{R}^{n\times n}$, such that

$$0 = \operatorname{div}\big(\langle f_i, \nabla f_j\rangle\big) = \operatorname{div}\big(P_{ik}^T \nabla P_{kj} + P_{ik}^T\big\langle e_k(v), \nabla e_l(v)\big\rangle P_{lj}\big). \tag{1.9}$$

Again, one should compare the latter expression to (1.3) with Ω_{ij} replaced by $-\langle e_i(v), \nabla e_j(v)\rangle$. By antisymmetry of $\langle e_i(v), \nabla e_j(v)\rangle$, $1 \leqslant i, j \leqslant n$, Eq. (1.9) is the Euler–Lagrange equation of (1.6) for a critical point P which motivates the variational approach as observed in [12]. In [23], on the other hand, a solution to (1.9) was constructed by the continuity argument developed by Uhlenbeck in [28].

The connection between the techniques of minimizing the energy as in (1.6) and the construction of a Coulomb-gauge by methods of Uhlenbeck is not new. In fact, in [29] in order to construct a moving frame for m-harmonic maps Uhlenbeck's approach [28] is used: Although minimization of (1.6) is possible in any dimension, and the Euler–Lagrange equations stay the same, the regularity of a minimizing transformation P seems to be a priori only of class $W^{1,2}$, even if $\Omega \in L^p$, $p > 2$. In this case, results obtained by the methods in [28] are better: If $\Omega \in L^m$, m the dimension of the underlying space, then one can construct $P \in W^{1,m}$ satisfying (1.3). Some two-dimensional regularity results of P minimizing (1.6) for smooth Ω can be found in [6].

Let us stress that in the original proof of regularity in [23] which from the gauge transformation P constructs the somewhat more elegant transformation A satisfying (1.4), the main focus lies on the construction of good conservation laws for equations like (1.1). In particular, while for (1.7) testfunctions have to be of class $W^{1,2} \cap L^\infty$ the right-hand side of (1.5) belongs to the Hardy-space and thus the equation can be tested with BMO-functions such as u itself. That way one can e.g. avoid a Dirichlet growth estimate below the natural exponent. Moreover, convergence issues become easier – once the preliminary work of constructing P and then A is done.

As for our *notation*, for a matrix or tensor A we will denote $|A|$ to be the Hilbert–Schmidt-norm of this quantity. Mappings like the solution u of (1.1) will usually map the unit ball $B_1(0) \subset \mathbb{R}^m$ into the n-dimensional target manifold

$\mathcal{N} \subset \mathbb{R}^N$ or simply into $\mathbb{R}^n$. Most of the time, instead of the ball $B_1(0)$ one could use other kinds of sets to obtain the same results. By $\nabla = [\partial_1, \partial_2, \ldots, \partial_m]^T$ we denote the gradient. If $m = 2$ the formally orthogonal gradient will be denoted by $\nabla^\perp = [-\partial_2, \partial_1]^T$. The special orthogonal group in $\mathbb{R}^{n\times n}$ is denoted by $SO(n)$; $so(n)$ are all those matrices $(A_{ij})_{ij} \in \mathbb{R}^{n\times n}$ such that $A_{ij} = -A_{ji}$. Many times, our constants depend on the dimensions involved. Further dependencies are usually clarified by a subscript. That is, a constant C_p may depend on the dimensions as well as on p. Without further notice constants denoted by C may change from line to line.

2. Direct construction of Coulomb-gauge

In this section we prove, by elementary methods, the following theorem:

Theorem 2.1. *(See [12], [3, Lemmas A.4, A.5], [13, Chapter 4], [28, Lemma 2.7], [23, Lemma A.3].) Let $D \subset \mathbb{R}^m$ be a smoothly bounded domain, $\Omega_{ij} \in L^2(D, \mathbb{R}^m)$, $\Omega_{ij} = -\Omega_{ji}$. Then there exists $P \in W^{1,2}(D, SO(n))$ such that*

$$\operatorname{div}\bigl(P^T \nabla P - P^T \Omega P\bigr) = 0 \quad \text{in } D,$$

and

$$\|\nabla P\|_{L^2(D)} + \bigl\|P^T \nabla P - P^T \Omega P\bigr\|_{L^2(D)} \leqslant 3\|\Omega\|_{L^2(D)}$$

holds.

The proof of Theorem 2.1 which we like to present here, follows from the next two lemmata which use only standard calculus of variation and a bit of linear algebra.

Lemma 2.2. *(Cf. [3, Lemma A.4].) Let $D \subset \mathbb{R}^m$ be a bounded domain. For any $\Omega_{ij} \in L^2(D, \mathbb{R}^m)$, $1 \leqslant i, j \leqslant n$, there exists $P \in W^{1,2}(D, SO(n))$ minimizing the variational functional*

$$E(Q) = \int_D \bigl|Q^T \nabla Q - Q^T \Omega Q\bigr|^2, \quad Q \in W^{1,2}\bigl(D, SO(n)\bigr).$$

Furthermore, $\|\nabla P\|_{L^2(D)} \leqslant 2\|\Omega\|_{L^2(D)}$.

Remark 2.3. Of course, this lemma holds as well, if one takes ‘Dirichlet’-boundary data, that is, if one assumes $Q - I \in W_0^{1,2}(D, \mathbb{R}^{n\times n})$, where I is the n-dimensional identity matrix.

Lemma 2.4. *(Cf. [3, Lemma A.5].) Critical points $P \in W^{1,2}(D, SO(n))$ of*

$$E(Q) = \int_D \bigl|Q^T \nabla Q - Q^T \Omega Q\bigr|^2, \quad Q \in W^{1,2}\bigl(D, SO(n)\bigr),$$

satisfy

$$\operatorname{div}\bigl(P^T_{ik} \nabla P_{kj} - P^T_{ik} \Omega_{kl} P_{lj}\bigr) = 0, \quad 1 \leqslant i, j \leqslant n,$$

provided that $\Omega_{ij} \in L^2(D, \mathbb{R}^m)$ and $\Omega_{ij} = -\Omega_{ji}$ for any $1 \leqslant i, j \leqslant n$.

Proof of Lemma 2.2. The function $Q \equiv I := (\delta_{ij})_{ij}$ is clearly admissible. Thus, there exists a minimizing sequence $Q_k \in W^{1,2}(D, SO(n))$ such that

$$E(Q_k) \leqslant E(I) = \|\Omega\|^2_{L^2}, \quad k \in \mathbb{N}.$$

By a.e. orthogonality of $Q_k(x) \in SO(n)$ we know that $Q_k(x)$ is bounded and

$$|\nabla Q_k| = \bigl|Q_k^T \nabla Q_k\bigr| \leqslant \bigl|Q_k^T \nabla Q_k - Q_k^T \Omega Q_k\bigr| + |\Omega| \quad \text{a.e. in } D;$$

thus

$$\|\nabla Q_k\|^2_{L^2(D)} \leqslant 2\big(E(Q_k) + \|\Omega\|^2_{L^2(D)}\big) \leqslant 4\|\Omega\|^2_{L^2(D)}.$$

Up to choosing a subsequence, we can assume that Q_k converges weakly in $W^{1,2}$ to $P \in W^{1,2}(D, \mathbb{R}^{m\times m})$. At the same time it shall converge strongly in L^2, and pointwise almost everywhere. The latter implies $P^T P = \lim_{k\to\infty} Q_k^T Q_k = I$, and $\det(P) = 1$, that is $P \in SO(n)$ almost everywhere.

Denoting $\Omega^P := P^T \nabla P - P^T \Omega P$ we obtain

$$Q_k^T \nabla Q_k - Q_k^T \Omega Q_k = \big(P^T Q_k\big)^T \nabla\big(P^T Q_k\big) + \big(P^T Q_k\big)^T \Omega^P \big(P^T Q_k\big),$$

and consequently

$$\big|Q_k^T \nabla Q_k - Q_k^T \Omega Q_k\big|^2 = \big|\nabla\big(P^T Q_k\big) + \Omega^P P^T Q_k\big|^2 = \big|\nabla\big(P^T Q_k\big)\big|^2 + 2\big\langle\nabla\big(P^T Q_k\big), \Omega^P P^T Q_k\big\rangle + \big|\Omega^P\big|^2,$$

where in this case $\langle\cdot,\cdot\rangle$ is just the Hilbert–Schmidt scalar product for matrices. This implies

$$\begin{aligned} E(Q_k) &= \int_D \big|\nabla\big(P^T Q_k\big)\big|^2 + 2\big\langle\nabla\big(P^T Q_k\big), \Omega^P P^T Q_k\big\rangle + E(P) \\ &\geqslant \int_D \big|\nabla\big(P^T Q_k\big)\big|^2 + 2\int_D \big\langle\nabla\big(P^T Q_k\big), \Omega^P P^T Q_k\big\rangle + \inf_Q E(Q). \end{aligned}$$

The middle part of the right-hand side converges to zero as $k \to \infty$. To see this, one can check that $\Omega^P P^T Q_k$ converges to Ω^P almost everywhere. Lebesgue's dominated convergence theorem implies strong convergence in L^2. On the other hand, $\nabla(P^T Q_k)$ converges to zero weakly in L^2.

Hence, using $E(Q_k) \xrightarrow{k\to\infty} \inf_Q E(Q)$, we have strong $W^{1,2}$-convergence of $P^T Q_k$ to I: Thus, Q_k converges strongly to P, which readily implies minimality of P. $\square$

Proof of Lemma 2.4. Let P be a critical point of $E(Q)$. A valid perturbation P_ε is the following

$$P_\varepsilon := P e^{\varepsilon\varphi\alpha} = P + \varepsilon\varphi P\alpha + o(\varepsilon) \in W^{1,2}\big(D, SO(n)\big)$$

for any $\varphi \in C^\infty(\overline{D})$, $\alpha \in so(n)$ and $\varepsilon \to 0$. This uses the simple algebraic fact that the exponential function applied to a skew-symmetric matrix is an orthogonal matrix; or from the point of view of geometry, that the space of skew-symmetric matrices is the tangential space to the manifold $SO(n) \subset \mathbb{R}^{n\times n}$ at the identity matrix. Then,

$$\begin{aligned} P_\varepsilon^T &= P^T - \varepsilon\varphi\alpha P^T + o(\varepsilon), \\ \nabla P_\varepsilon &= \nabla P + \varepsilon\varphi\nabla P\alpha + \varepsilon\nabla\varphi P\alpha + o(\varepsilon). \end{aligned}$$

Thus, denoting again $\Omega^P := P^T \nabla P - P^T \Omega P \in so(n) \otimes \mathbb{R}^m$, we obtain

$$\Omega^{P_\varepsilon} = \Omega^P + \varepsilon\varphi\big(\Omega^P\alpha - \alpha\Omega^P\big) + \varepsilon\nabla\varphi\alpha + o(\varepsilon).$$

Antisymmetry of Ω^P yields

$$\sum_{i,j} \big(\Omega^P\big)_{ij} \cdot \big(\Omega^P\alpha - \alpha\Omega^P\big)_{ij} = 0 \quad \text{pointwise almost everywhere.}$$

It follows that,

$$\big|\Omega^{P_\varepsilon}\big|^2 = \big|\Omega^P\big|^2 + 2\varepsilon\big(\Omega^P\big)_{ij}\alpha_{ij}\nabla\varphi + o(\varepsilon),$$

which readily implies

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(P_\varepsilon) = \int_D \big(\Omega^P\big)_{ij}\alpha_{ij} \cdot \nabla\varphi.$$

This is true for any $\varphi \in C^\infty(\overline{D})$ and $\alpha \in so(n)$. Setting for arbitrary $1 \leqslant s, t \leqslant n$ our $\alpha_{ij} := \delta_i^s \delta_j^t - \delta_j^s \delta_i^t$, we arrive at

$$\operatorname{div}\big(\Omega^P\big)_{st} = 0 \quad \text{in } D,\ 1 \leqslant s, t \leqslant n. \qquad \square$$

Remark 2.5. The disadvantage of this method is the fact that we do not know of a short and direct way to get better estimates on P than the ones obtained here. That is, it does not seem to be clear that $\Omega \in L^p$ yields $P \in W^{1,p}$.

This is the advantage of the more involved method due to Uhlenbeck [28, Lemma 2.7]; for the version needed here one best consults [23, Lemma A.3]. In the appendix of [20] this technique is also explained in some detail. This method works in similar ways for different integrability exponents in higher dimensions. In [19,24] there is a Morrey-space version of it.

On the other hand, the technique presented here can be easily adapted to e.g. the case of different measures instead of the Lebesgue measure.

Interestingly, the knowledge that $\|\nabla P\|_{L^2} \leqslant C\|\Omega\|_{L^2}$ is sufficient also for partial regularity in dimensions $m > 2$. We will observe this in Section 3 by a tiny modification of the proof in [24].

3. Application of Dirichlet growth theorem

In this section we will sketch how to apply the Dirichlet growth theorem (cf. [18, Theorem 3.5.2]) in order to derive regularity for solutions of (1.1), given the existence of P as in the proof of Theorem 2.1. A detailed proof can be found in [24]. As a slight modification, we will remark on how to avoid Morrey-space estimates on the gradient of the gauge transformation P. Those Morrey-space estimates can be obtained via the Uhlenbeck-approach, but it is not obvious how to get them by a method as in Theorem 2.1. We will show that the L^2-estimates of Theorem 2.1 are sufficient.

We will use one non-elementary technique, namely the duality between Hardy-space and BMO. But in fact we need only a special case. For $p \in (1, \infty)$ set

$$\mathcal{J}_p(x, \rho; f) := \frac{1}{\rho^{m-p}} \int_{B_\rho(x)} |f|^p,$$

$$\mathcal{M}_p(y, \varrho; f) := \sup_{B_\rho(x) \subset B_\varrho(y)} \mathcal{J}_p(x, \rho; f).$$

Lemma 3.1 *(Hardy-BMO-inequality). For any $p > 1$, there is a uniform constant $C_{m,p}$ such that the following holds:*

For any ball $B \equiv B_\varrho(y) \subset \mathbb{R}^m$, $2B = B_{2\varrho}(y)$ the ball with same center and twice the radius, $a \in W^{1,2}(2B)$, $\Gamma \in L^2(B, \mathbb{R}^m)$, $\operatorname{div} \Gamma = 0$ in B, $c \in W_0^{1,2} \cap L^\infty(B)$

$$\left| \int_B (\nabla a \cdot \Gamma) c \right| \leqslant C_{m,p} \|\Gamma\|_{L^2(B)} \|\nabla c\|_{L^2(B)} \big(\mathcal{M}_p(y, 2\varrho, \nabla a)\big)^{\frac{1}{p}},$$

whenever the right-hand side is finite.

For a proof we refer to [5, Theorem II.1], [7, Chapter II.2], [26, Chapter IV, Section 1.2]. One might also look into [2,4].

Theorem 3.2. *(See [24, Theorem 1.1].) There is $\varepsilon \equiv \varepsilon(m) \in (0, 1)$ such that the following holds:*

Let $D \subset \mathbb{R}^m$ be open and let $u \in W^{1,2}(D, \mathbb{R}^n)$ be a solution of

$$\Delta u^i = \Omega_{ik} \cdot \nabla u^k \quad \text{in } D,\ 1 \leqslant i \leqslant n,$$

such that

$$\sup_{B_r(x) \subset D} \frac{1}{r^{m-2}} \int_{B_r(x)} |\Omega|^2 \leqslant \varepsilon^2 \tag{3.1}$$

and

$$\sup_{B_r(x)\subset D} \frac{1}{r^{m-2}} \int_{B_r(x)} |\nabla u|^2 < \infty. \tag{3.2}$$

If $\Omega_{ij} = -\Omega_{ji} \in L^2(D,\mathbb{R}^m)$ *then* $u \in C^{0,\alpha}(D,\mathbb{R}^n)$ *for some* $\alpha \in (0,1)$.

Sketch of the proof. Most parts of the following are a copy of the proof in [24, Theorem 1.1].

Let $z \in D$, $0 < r < R < \frac{1}{2}\operatorname{dist}(z,\partial D)$. Apply Theorem 2.1 on $B_R(z)$: There exists $P \in W^{1,2}(B_R(z), SO(n))$ such that

$$\operatorname{div}\big(\Omega^P\big) \equiv \operatorname{div}\big(P^T\nabla P - P^T\Omega P\big) = 0 \quad \text{weakly in } B_R(z), \tag{3.3}$$

with the estimate

$$\|\nabla P\|_{L^2(B_R(z))} + \big\|\Omega^P\big\|_{L^2(B_R(z))} \leqslant 3\|\Omega\|_{L^2(B_R(z))}. \tag{3.4}$$

We have weakly

$$\operatorname{div}\big(P^T\nabla u\big) = \Omega^P \cdot P^T\nabla u \quad \text{in } B_R(z). \tag{3.5}$$

Use Hodge decomposition to find $f \in W_0^{1,2}(B_R(z),\mathbb{R}^n)$, $g \in W_0^{1,2}(B_R(z), \bigwedge^2\mathbb{R}^n)$, $h \in C^\infty(B_R(z), \mathbb{R}^n \otimes \mathbb{R}^m)$ such that

$$P^T\nabla u = \nabla f + \operatorname{Curl} g + h \quad \text{a.e. in } B_R(z), \tag{3.6}$$

$$\begin{cases} \Delta f = \operatorname{div}\big(P^T\nabla u\big) \overset{(3.5)}{=} \Omega^P \cdot P^T\nabla u & \text{in } B_R(z),\\ f = 0 & \text{on } \partial B_R(z), \end{cases}$$
$$\begin{cases} \Delta g = \operatorname{curl}\big(P^T\nabla u\big) & \text{in } B_R(z),\\ g = 0 & \text{on } \partial B_R(z), \end{cases}$$
$$\begin{cases} \operatorname{div} h = 0 & \text{in } B_R(z),\\ \operatorname{curl} h = 0 & \text{in } B_R(z). \end{cases} \tag{3.7}$$

For more on Hodge-decompositions we refer to [16, Corollary 10.5.1]. Fix $1 < p < \frac{m}{m-1}$. One estimates

$$\int_{B_r(z)} |\nabla u|^p = \int_{B_r(z)} \big|P^T\nabla u\big|^p \overset{(3.6)}{\leqslant} C_p\bigg(\int_{B_r(z)} |h|^p + \int_{B_R(z)} |\nabla f|^p + \int_{B_R(z)} |\nabla g|^p\bigg).$$

By harmonicity we have (cf. [8, Theorem 2.1, p. 78])

$$\int_{B_r(z)} |h|^p \leqslant C_p\left(\frac{r}{R}\right)^m \int_{B_R(z)} |h|^p.$$

Consequently, again by (3.6),

$$\int_{B_r(z)} |\nabla u|^p \leqslant C_p\bigg(\left(\frac{r}{R}\right)^m \int_{B_R(z)} |\nabla u|^p + \int_{B_R(z)} |\nabla f|^p + |\nabla g|^p\bigg). \tag{3.8}$$

In order to estimate $\int_{B_R(z)} |\nabla f|^p$ note that since $f = 0$ on $\partial B_R(z)$, by duality

$$\|\nabla f\|_{L^p(B_R(z))} \leqslant C_p \sup_{\substack{\varphi\in C_0^\infty(B_R(z))\\ \|\varphi\|_{W^{1,q}}\leqslant 1}} \int_{B_R(z)} \nabla f \cdot \nabla\varphi. \tag{3.9}$$

Here, $q = \frac{p}{p-1}$ denotes the Hölder-conjugate exponent of p. If $\|\varphi\|_{W^{1,q}(B_R(z))} \leqslant 1$ one calculates

$$\|\varphi\|_{L^\infty(B_R(z))} \leqslant C_p R^{1+\frac{m}{p}-m}, \qquad \|\nabla\varphi\|_{L^2(B_R(z))} \leqslant C_p R^{\frac{m}{p}-\frac{m}{2}}. \tag{3.10}$$

Note that the L^∞-bound holds only as $q > m$ by choice of p. In particular, the constant C_p blows up as p approaches $\frac{m}{m-1}$ from below.

Recall our notation

$$\mathcal{J}_p(x,\rho) \equiv \mathcal{J}_p\big(x,\rho;|\nabla u|\big) = \frac{1}{\rho^{m-p}} \int\limits_{B_\rho(x)} |\nabla u|^p,$$

$$\mathcal{M}_p(y,\varrho) \equiv \mathcal{M}_p\big(y,\varrho,|\nabla u|\big) := \sup_{B_\rho(x)\subset B_\varrho(y)} \mathcal{J}_p(x,\rho).$$

By (3.7),

$$\int\limits_{B_R(z)} \nabla f \cdot \nabla\varphi = \int\limits_{B_R(z)} \Omega^P \cdot P^T \nabla u \varphi.$$

As of (3.3) Lemma 3.1 can be applied to this quantity by choosing $c = P^T_{kl}\varphi$, $a = u^l$, $\Gamma = (\Omega^P)_{ik}$ for any $1 \leqslant i,k,l \leqslant n$. Then (3.9) is further estimated by

$$\begin{aligned}
\|\nabla f\|_{L^p(B_R(z))} &\leqslant C_p \big\|\Omega^P\big\|_{L^2(B_R(z))}\big(\|\nabla P\|_{L^2(B_R(z))}\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2}\big)\big(\mathcal{M}_p(z,2R)\big)^{\frac{1}{p}} \\
&\overset{(3.4)}{\leqslant} C_p \|\Omega\|_{L^2(B_R(z))}\big(\|\Omega\|_{L^2(B_R(z))}\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2}\big)\big(\mathcal{M}_p(z,2R)\big)^{\frac{1}{p}} \\
&\overset{\substack{(3.1)\\(3.10)}}{\leqslant} C_p \varepsilon R^{\frac{m}{p}-1}\big(\mathcal{M}_p(z,2R)\big)^{\frac{1}{p}}.
\end{aligned}$$

Note again that the constant C_p blows up as p approaches $\frac{m}{m-1}$ from below. The last step is the only qualitative albeit tiny difference to the proof in [24]: Instead of using an a priori estimate on $\sup_r \frac{1}{r^{m-2}} \int_{B_r} |\nabla P|^2$ and $\sup_r \frac{1}{r^{m-2}} \int_{B_r} |\Omega^P|^2$, we use the domain-independent estimate (3.4) of the L^2-norm of ∇P and Ω^P, respectively. By a similar argument

$$\|\nabla g\|_{L^p(B_R(z))} \leqslant C_p \varepsilon R^{\frac{m}{p}-1}\big(\mathcal{M}_p(z,2R)\big)^{\frac{1}{p}}.$$

Plugging these estimates into (3.8) we arrive at

$$\int\limits_{B_r(z)} |\nabla u|^p \leqslant C_p \left(\frac{r}{R}\right)^m \int\limits_{B_R(z)} |\nabla u|^p + C_p \varepsilon^p R^{m-p} \mathcal{M}_p(z,2R).$$

The right-hand side of this estimate is finite by (3.2). We divide by r^{m-p} to get

$$\frac{1}{r^{m-p}} \int\limits_{B_r(z)} |\nabla u|^p \leqslant C_p \left(\frac{r}{R}\right)^p \frac{1}{R^{m-p}} \int\limits_{B_R(z)} |\nabla u|^p + C_p \varepsilon^p \left(\frac{R}{r}\right)^{m-p} \mathcal{M}_p(z,2R).$$

Hence,

$$\mathcal{J}_p(z,r) \leqslant C_p \left(\left(\frac{r}{R}\right)^p + \varepsilon^p \left(\frac{R}{r}\right)^{m-p}\right) \mathcal{M}_p(z,2R).$$

Choose $\gamma \in (0,\frac{1}{2})$ such that $C_p\gamma^p \leqslant \frac{1}{4}$ and set $\varepsilon := \gamma^{\frac{m}{p}}$. Then for $r := \gamma R$ we have shown

$$\mathcal{J}_p(z,\gamma R) \leqslant \frac{1}{2}\mathcal{M}_p(z,2R).$$

This is valid for any $R > 0$, $z \in D$ such that $B_{2R}(z) \subset D$. For arbitrary $\rho \in (0,1)$, $y \in D$, $B_{2\rho}(y) \subset D$ this implies

$$\mathcal{J}_p(z,\gamma R) \leqslant \frac{1}{2}\mathcal{M}_p(y,\rho) \quad \text{whenever } B_{2R}(z) \subset B_\rho(y),$$

that is

$$\mathcal{M}_p\left(y, \frac{\gamma}{2}\rho\right) \leqslant \frac{1}{2}\mathcal{M}_p(y, \rho).$$

This gives Hölder-continuity as claimed. □

Remark 3.3. With the presented techniques one can prove slight generalizations of this. For example, in order to prove regularity for systems of the type

$$\partial_\alpha\big(g_{\alpha\beta}\partial_\beta u^i\big) = g_{\alpha\beta}\Omega^\beta_{ik}\nabla u^k,$$

one would minimize

$$E(P) = \int\limits_D \big(P^T_{ik}\partial_\alpha P_{kj} - P^T_{ik}\Omega^\alpha_{kl}P_{lj}\big)g_{\alpha\beta}\big(P^T_{ik}\partial_\beta P_{kj} - P^T_{ik}\Omega^\beta_{kl}P_{lj}\big).$$

Remark 3.4. Slightly modifying this approach, one also can check the following: Let $\xi^i := A_{ik}\nabla u^k$, $A \in W^{1,2} \cap L^\infty(D, \mathbb{R}^{n\times n})$, and $u \in W^{1,2}(D, \mathbb{R}^m)$ satisfy (3.2). Assume that ξ is a solution of a system like

$$\operatorname{div}\big(\xi^i\big) = \Omega_{ik}\cdot\xi^k \quad \text{in } D,\ 1 \leqslant i \leqslant n.$$

This implies better regularity of u, if (3.1) holds for Ω and A and under the additional condition that there is a uniform constant $\Lambda > 0$ such that

$$\frac{1}{\Lambda}|\xi| \leqslant |\nabla u| \leqslant \Lambda|\xi| \quad \text{a.e. in } D.$$

The last condition is used to switch in growth estimates like (3.8) between $|\xi|$ and $|\nabla u|$.

4. Hildebrandt's conjecture

In this section we sketch a proof of Hildebrandt's conjecture [14,15] stating that critical points of conformally invariant variational functionals on maps $v \in W^{1,2}(D, \mathbb{R}^n)$ where $D \subset \mathbb{R}^2$ are continuous: We construct from Grüter's [9] characterization directly a Rivière-type system – avoiding the Nash–Moser embedding theorem as in e.g. [3] and [23, Theorem I.2].

As explained for example in [13, Section 1.2], the Nash–Moser theorem is used to avoid the appearance of terms involving Christoffel-symbols in the Euler–Lagrange equations of harmonic maps or – more generally – conformally invariant variational functionals: Let $D \subset \mathbb{R}^2$ be an open set. For $v \in W^{1,2}(D, \mathbb{R}^n)$ we define the functional

$$\mathcal{F}(v) \equiv \mathcal{F}_D(v) = \int\limits_D F\big(v(x), \nabla v(x)\big)\,dx,$$

where $F : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ is of class C^1 with respect to the first entry and of class C^2 with respect to the second entry. The functional $\mathcal{F}$ is called conformally invariant if

$$\mathcal{F}_D(v) = \mathcal{F}_{D'}(v \circ \phi)$$

for every smooth $v : D \to \mathbb{R}^n$ and every smooth conformal diffeomorphism $\phi : D' \to D$. Suppose $\mathcal{F}$ is conformally invariant and that for some $\Lambda > 0$

$$\frac{1}{\Lambda}|p|^2 \leqslant F(v, p) \leqslant \Lambda|p|^2 \quad \text{for all } v \in \mathbb{R}^n,\ p \in \mathbb{R}^{2n}.$$

Then, by [9, Theorem 1], there exists a positive, symmetric matrix (g_{ij}) and a skew symmetric matrix (b_{ij}) such that

$$F(v, p) = g_{ij}(v)p^i \cdot p^k + b_{ij}(v)\det\big(p^i, p^j\big),$$

and hence

$$\mathcal{F}(v) = \int\limits_D g_{ij}(v)\nabla v^i \cdot \nabla v^k + b_{ij}(v)\nabla v^i \cdot \nabla^\perp v^j.$$

Recall that $\nabla^\perp = (-\partial_y, \partial_x)^\perp$. Let us interpret $(g_{ij})_{i,j=1}^n$ as a metric of the target space $\mathbb{R}^n$. As in [9, (2.7)] Euler–Lagrange-equation could then be written as

$$2\Delta u^i + \Gamma^i_{kl}(u)\nabla u^k \cdot \nabla u^l = g^{ij}\{\partial_l b_{jk} + \partial_j b_{kl} + \partial_k b_{li}\}(u)\nabla u^k \cdot \nabla^\perp u^l, \tag{4.1}$$

where

$$\Gamma^i_{kl} = g^{ij}\{\partial_l g_{jk} - \partial_j g_{kl} + \partial_k g_{lj}\}$$

are the Christoffel symbols corresponding to the metric (g_{ij}). Here, we have denoted the inverse of (g_{ij}) by (g^{ij}). Let

$$\Omega_{jk} := \{\partial_l b_{jk} + \partial_j b_{kl} + \partial_k b_{lj}\}(u)\nabla^\perp u^l$$

which is antisymmetric. Eq. (4.1) then reads as

$$2\Delta u^i + \Gamma^i_{kl}(u)\nabla u^k \cdot \nabla u^l = g^{ij}(u)\Omega_{jk} \cdot \nabla u^k. \tag{4.2}$$

At first glance, (4.2) does not seem to fit into the setting of (1.1) because in general (g_{ij}) is not the standard Euclidean metric on $\mathbb{R}^n$.

The Nash–Moser theorem (cf. [22,17,10,11]) solves this problem: It states that there is a manifold $\mathcal{N} \subset \mathbb{R}^N$, $N \geqslant n$, and a C^1-diffeomorphism T mapping $(\mathbb{R}^n, g_{ij})$ isometrically into $(\mathcal{N}, c_{ij})$ where c_{ij} is the induced $\mathbb{R}^N$-metric on $\mathcal{N}$. That is, $T : (\mathbb{R}^n, g_{ij}) \to \mathcal{N}$ and

$$\left\langle dT_x\left(\frac{\partial}{\partial x^i}\right), dT_x\left(\frac{\partial}{\partial x^j}\right)\right\rangle_{\mathbb{R}^N} = g_{ij}(x), \quad x \in \mathbb{R}^n,\ 1 \leqslant i, j \leqslant n. \tag{4.3}$$

Here, $(\frac{\partial}{\partial x^i})_{i=1}^n$ denotes the standard Euclidean basis in $\mathbb{R}^n$. Using this isometric diffeomorphism T, we introduce an adapted functional $\widetilde{\mathcal{F}}$ defined on mappings $\tilde{v} \in W^{1,2}(D, \mathcal{N})$ of which $T(u)$ is a critical point. Looking at the Euler–Lagrange equations of this new $\widetilde{\mathcal{F}}$, the fact that the metric on $\mathcal{N}$ is induced by the surrounding space $\mathbb{R}^N$ will imply trivial Christoffel-symbols. On the other hand, the additional side-condition $\tilde{v}(x) \in \mathcal{N}$ a.e. will bring up a term involving the second fundamental form of the embedding $\mathcal{N} \subset \mathbb{R}^N$. This new term can be rewritten into the form of the right-hand side of (1.1) as was observed in [23].

In fact, setting

$$\tilde{b}_{ab} := \left(dT^a\left(\frac{\partial}{\partial x^k}\right)g^{ki} b_{ij} g^{jl} dT^b\left(\frac{\partial}{\partial x^l}\right)\right) \circ T^{-1}$$

we obtain

$$\mathcal{F}(v) = \int\limits_D \left|\nabla T(v)\right|^2_{\mathbb{R}^N} + \sum_{a,b=1}^N \int\limits_D \tilde{b}_{ab}(Tv)\nabla T^a(v) \cdot \nabla^\perp T^b(v).$$

Consequently, u is a critical point of $\mathcal{F}$ if and only if $T(u)$ is a critical point of

$$\widetilde{\mathcal{F}}(\tilde{v}) = \int\limits_D |\nabla \tilde{v}|^2 + \sum_{a,b=1}^N \tilde{b}_{ab}(\tilde{v})\nabla \tilde{v}^a \cdot \nabla^\perp \tilde{v}^b, \quad \tilde{v} \in W^{1,2}(D, \mathcal{N}).$$

One checks that $\tilde{b}$ is antisymmetric. Hence, assuming that the second fundamental form of the embedding $\mathcal{N} \subset \mathbb{R}^N$ is bounded, one can proceed as in [23, Theorem I.2] to see that the Euler–Lagrange equation of $\widetilde{\mathcal{F}}$ is a system of type (1.1). Thus, regularity of $T(u)$, u is implied.

The proof of the Nash–Moser embedding is quite involved. However, it can be avoided easily by the following approach: A critical point $u \in W^{1,2}(D, \mathbb{R}^n)$ of $\mathcal{F}$ weakly satisfies (4.1) or equivalently for $1 \leqslant j \leqslant n$

$$-\operatorname{div}\bigl(2g_{jk}(u)\nabla u^k\bigr) + (\partial_j g_{kl})(u)\nabla u^k \cdot \nabla u^l = \operatorname{div}\bigl(2b_{jk}(u)\nabla^\perp u^k\bigr) - (\partial_j b_{kl})(u)\nabla u^k \cdot \nabla^\perp u^l. \tag{4.4}$$

By algebraic calculations one constructs vector functions $e_i : \mathbb{R}^n \to \mathbb{R}^n$, $1 \leqslant i, j \leqslant n$, such that pointwise

$$\langle e_i, e_j \rangle_{\mathbb{R}^n} = g_{ij}. \tag{4.5}$$

In order to construct T as in (4.3) one would be tempted to integrate, that is, to set

$$dT\left(\frac{\partial}{\partial x^i}\right) := e_i,$$

and therefore one would need e_i satisfying (4.5) and

$$\partial_j e_i - \partial_i e_j = 0, \quad 1 \leqslant i, j \leqslant n. \tag{4.6}$$

One observes now that the latter quantity is a skew symmetric one. That is, the error one would make in (4.4) assuming (4.6) to hold is not a bad one – it fits into the setting of Rivière's system (1.1). In fact, the following lemma holds, which by the techniques of [24], see also Section 3, Remark 3.4, implies regularity.

Lemma 4.1. *Let $u \in W^{1,2}(D, \mathbb{R}^n)$ be a weak solution of*

$$-\operatorname{div}\big(2g_{ik}(u)\nabla u^k\big) + (\partial_i g_{kl})(u)\nabla u^k \cdot \nabla u^l = \Omega_{ik} \cdot \nabla u^k + \nabla^\perp b_{ik} \nabla u^k. \tag{4.7}$$

Assume that $g, g^{-1} \in W^{1,\infty}(\mathbb{R}^n, GL(n))$ are symmetric and positive definite, $b_{jk} \in W^{1,2}(D)$, $\Omega_{ij} = -\Omega_{ji} \in L^2(D, \mathbb{R}^2)$.

Then there are $A \in W^{1,2} \cap L^\infty(D, GL(m))$, $\widetilde{\Omega}_{ij} = -\widetilde{\Omega}_{ji} \in L^2(D, \mathbb{R}^2)$ such that

$$\operatorname{div}\big(A_{ik}\nabla u^k\big) = \widetilde{\Omega}_{ik} \cdot A_{kl}\nabla u^l + \nabla^\perp b_{ik} \cdot \nabla u^k.$$

Sketch of the proof. By easy algebraic transformations using symmetry and positive definiteness of g one can choose $e_i \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\big\langle e_i(x), e_j(x)\big\rangle_n = g_{ij}(x), \quad x \in \mathbb{R}^n,\ 1 \leqslant i, j \leqslant n. \tag{4.8}$$

The A_{ia} from the claim will be $e_i^a \circ u$. Let us abbreviate as follows

$$\xi^a := A_{ak}\nabla u^k = e_k^a(u)\nabla u^k, \tag{4.9}$$

which is equivalent to

$$\nabla u^j = g^{jk}(u)e_k^a(u)\xi^a. \tag{4.10}$$

Let φ be any admissible testfunction. The first term on the left-hand side of (4.7)

$$I := 2g_{ik}(u)\nabla u^k \cdot \nabla\varphi^i \overset{(4.10)}{=} 2\xi^a \cdot \big(e_i^a(u)\nabla\varphi^i\big).$$

On the other hand, the second term of (4.7)

$$\begin{aligned} II &:= \partial_i g_{kl}(u)\nabla u^k \cdot \nabla u^l \varphi^i \\ &\overset{(4.8)}{=} 2\big(\partial_i e_k^a\big)(u)e_l^a(u)\nabla u^k \cdot \nabla u^l \varphi^i \\ &= 2\big(\partial_k e_i^a\big)(u)e_l^a(u)\nabla u^k \cdot \nabla u^l \varphi^i + 2\big(\partial_i e_k^a - \partial_k e_i^a\big)(u)e_l^a(u)\nabla u^k \cdot \nabla u^l \varphi^i \\ &=: II_1 + II_2. \end{aligned}$$

One computes

$$II_1 \overset{(4.9)}{=} 2\nabla\big(e_i^a(u)\big)\varphi^i \cdot \xi^a,$$

and thus

$$I + II_1 = 2\xi^a \cdot \nabla\big(e_i^a(u)\varphi^i\big).$$

For arbitrary $\tilde{\varphi} \in C_0^\infty(D, \mathbb{R}^n)$ one sets

$$\varphi^i := g^{ij}(u)\langle e_j(u), \tilde{\varphi}\rangle_n \tag{4.11}$$

which is an admissible testfunction. One checks that

$$\langle \tilde{\varphi} - e_j(u)\varphi^j, e_s(u)\rangle_n \overset{(4.8)}{=} 0, \quad 1 \leqslant s \leqslant n.$$

Pointwise in $\mathbb{R}^n$ the vectors $e_i \in \mathbb{R}^n$, $1 \leqslant i \leqslant n$, are linearly independent, which implies $\tilde{\varphi} = e_j(u)\varphi^j$ almost everywhere. Then

$$I + II_1 = 2\xi^a \cdot \nabla\tilde{\varphi}^a.$$

Rewriting the quantity II_2 in terms of ξ^a and $\tilde{\varphi}$ yields

$$II_2 = 2\big(\partial_i e_k^a - \partial_k e_i^a\big)(u)\xi^a \cdot g^{ks}(u)e_s^b(u)\xi^b g^{it}(u)e_t^c(u)\tilde{\varphi}^c =: 2\omega_{bc}\xi^b\tilde{\varphi}^c,$$

where $\omega_{bc} = (\partial_i e_k^a - \partial_k e_i^a)(u)\xi^a \cdot g^{ks}(u)e_s^b(u)g^{it}(u)e_t^c(u)$ is antisymmetric and of class L^2.

For the right-hand side of (4.7) one observes just by plugging in (4.11) and (4.10)

$$\Omega_{ik} \cdot \nabla u^k \varphi^i = \Omega_{ik} g^{kl}(u)e_l^a(u)g^{is}(u)e_s^c(u)\tilde{\varphi}^c \cdot \xi^a$$

and $\widetilde{\Omega}_{ac} := \Omega_{ik} g^{kl}(u)e_l^a(u)g^{is}(u)e_s^c(u)$ is antisymmetric and belongs to L^2.

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