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A regularity criterion for the 3D NSE in a local version of the space of functions of bounded mean oscillations

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Abstract

A spatio-temporal localization of the *BMO*-version of the Beale–Kato–Majda criterion for the regularity of solutions to the 3D Navier–Stokes equations obtained by Kozono and Taniuchi, i.e., the time-integrability of the *BMO*-norm of the vorticity, is presented.

Résumé

Une localisation spatio-temporelle de la version *BMO* du critère de Beale–Kato–Majda pour la régularité des solutions des équations de Navier–Stokes obtenue par Kozono et Taniuchi, c.-à-d., l'intégrabilité en temps de la norme *BMO* de la vorticité, est présentée.

1. Introduction

The Beale–Kato–Majda (BKM) regularity criterion, originally derived for solutions to the 3D Euler equations (cf. [1]), holds for solutions to the 3D Navier–Stokes equations (NSE) as well. The criterion can be viewed as a continuation principle for strong solutions stating that as long as the time-integral of the L^∞-norm of the vorticity is bounded, no blow-up can occur.

A refinement of the BKM criterion was obtained in [13] where the condition on time-integrability of the L^∞-norm of the vorticity was replaced by the time-integrability of the *BMO*-norm of the vorticity (*BMO* is the space of bounded mean oscillations). The proof in [13] is based on various bilinear estimates in *BMO* obtained by the authors which in turn rely on continuity of a class of convolution-type pseudodifferential operators with the symbol vanishing on one of the components from $L^2 \times BMO$ to L^2 (cf. [6]).

A further generalization was presented in [14,15] where the regularity condition is expressed in terms of the time-integrability of the homogeneous Besov norm $\dot{B}^0_{\infty,\infty}$ of the vorticity.

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In all the aforementioned results, the spatial domain was the whole space $\mathbb{R}^3$. In this paper, utilizing the localization of the vorticity–velocity formulation of the 3D NSE developed in [12,10] (see also [11]), we present a spatio-temporal localization of the *BMO*-criterion on the vorticity. Instead of trying to localize the original proof in the global case given in [13], we will exploit the non-homogeneous div–curl structure of the leading order vortex-stretching term, a variant of the local non-homogeneous div–curl lemma (cf. [5]), and the duality between a local version of the Hardy space $\mathcal{H}^1$ and a local version of the space of bounded mean oscillations *BMO*.

In a very recent work [3], utilizing a localization of the velocity–pressure form of the 3D NSE, the authors obtained a localization of another *BMO* regularity criterion; namely, the time-integrability of the square of the *BMO*-norm of the velocity (cf. [13]). Their proof was based on bilinear estimates in *BMO* obtained in [13] and an estimate on the *BMO*-norm of a product of a *BMO*-function with a smooth function of compact support.

As in the previous works [12,10,11], for simplicity of the exposition, the calculations are presented on smooth solutions. More precisely, we consider a weak solution on a space–time domain $\Omega \times (0,T)$ and suppose that u is smooth in an open parabolic cylinder $Q_{2R}(x_0,t_0) = B(x_0,2R) \times (t_0-(2R)^2,t_0)$ contained in $\Omega \times (0,T)$. The goal is to show that, under a suitable local condition on $Q_{2R}(x_0,t_0)$ (in this case, the time-integrability of a local version of the *BMO*-norm of the vorticity), the localized enstrophy remains uniformly bounded up to $t = t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} \left|\omega(x,t)\right|^2 dx < \infty.$$

Alternatively, we can consider, e.g., a class of suitable weak solutions constructed in [2] as a limit of a family of delayed mollifications (see also [7]), and perform the calculations on the smooth approximations.

The only *a priori* bounds on weak solutions needed in [12,10,11] are the Leray bounds, u in $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. Here, we will also make use of the vorticity counterpart of the $L^\infty(0,T;L^2)$-bound on the velocity, i.e., the $L^\infty(0,T;L^1)$-bound on the vorticity. This bound holds for suitable weak solutions for which the initial vorticity is a finite Radon measure (cf. [7]).

In Section 2 we recall some facts about global and local versions of the Hardy space $\mathcal{H}^1$ and *BMO*, as well as a local, non-homogeneous div–curl lemma from [5]. Section 3 contains the statement and the proof of our localization result.

2. Preliminaries

Definition 1. Let h in $C_0^\infty = C_0^\infty(\mathbb{R}^n)$ be a function supported in the unit ball $B(0,1)$ such that $\int h = 1$. The maximal function of a distribution f is defined by

$$M_h(f)(x) = \sup_{t>0}\left|f * h_t(x)\right| \quad \text{for all } x \in \mathbb{R}^n, \text{ where } h_t(x) = t^{-n}h\left(t^{-1}x\right).$$

f belongs to $\mathcal{H}^1$ if the maximal function $M_h(f)$ belongs to L^1 and the $\mathcal{H}^1$-norm of f is given by $\|f\|_{\mathcal{H}^1} = \|M_h(f)\|_{L^1}$.

The local Hardy space h^1 was introduced by Goldberg (cf. [9]).

Definition 2. The local maximal function of a distribution f is defined by

$$m_h(f)(x) = \sup_{0<t<1}\left|f * h_t(x)\right| \quad \text{for all } x \in \mathbb{R}^n.$$

f belongs to h^1 if the local maximal function $m_h(f)$ belongs to L^1 and the h^1-norm of f is given by $\|f\|_{h^1} = \|m_h(f)\|_{L^1}$.

On the other hand, $\mathcal{H}^1_{loc}$ is defined to be the space of all locally integrable functions such that the local maximal function $m_h(f)$ is in L^1_{loc}.

Note that the norms are independent of the choice of h up to equivalence; hence the spaces $\mathcal{H}^1$, $\mathcal{H}^1_{loc}$ and h^1 are well defined.

The dual space of $\mathcal{H}^1$ is the space of functions of bounded mean oscillation, abbreviated to *BMO* (cf. [8]).

Definition 3. A locally integrable function f on $\mathbb{R}^n$ is of bounded mean oscillation if

$$\|f\|_{BMO} = \sup_{x,r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| dy < \infty$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$.

Similarly, the dual space of h^1 is the space *bmo*; this is the localized version of *BMO* (see [9]).

Definition 4. A locally integrable function f on $\mathbb{R}^n$ is in *bmo* if

$$\|f\|_{bmo} = \sup_{x,0<r<1} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| dy + \sup_{x,r\geqslant 1} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y)\right| dy < \infty.$$

For any bounded Lipschitz domain Ω, the same duality holds for the spaces $bmo_r(\Omega)$ and $h^1_z(\Omega)$ [4].
Here g is in $bmo_r(\Omega)$ if there exists a function f in *bmo* such that $g = f$ on Ω and

$$\|g\|_{bmo_r(\Omega)} = \inf\bigl\{\|f\|_{bmo}\colon\ f \text{ in } bmo,\ f = g \text{ on } \Omega\bigr\}.$$

The space h^1_z consists of all functions on Ω whose extensions to the constant function 0 on $\mathbb{R}^3\setminus\overline{\Omega}$ are in h_1. Alternatively, this space can be thought of as the space of all distributions in h_1 that are supported in $\overline{\Omega}$.
The following is a variant of a local, non-homogeneous div–curl lemma presented in [5, III.2].

Lemma 1 *(Coifman, Lions, Meyer, Semmes). Suppose that u, v are in $L^2(B(0,R))$ with* $\operatorname{div} u$ *in $W^{-1,s}(B(0,R))$ for some $s > 2$ and* $\operatorname{curl} v = 0$. *Then,*

$$\left\|m_h(u\cdot v)\right\|_{L^1(B(0,R))} \leqslant c(R)\bigl(\|u\|_{L^2(B(0,R))} + \|\operatorname{div} u\|_{W^{-1,s}(B(0,R))}\bigr)\|v\|_{L^2(B(0,R))}$$

where c is an increasing function of R.

Remark 1. A simple consequence of the lemma is the following bound on the div–curl products in $h^1_z(B(0,r))$.
Let $r \leqslant 1$, and suppose that u, v are in $L^2(B(0,r))$ with $\operatorname{div} u$ in $W^{-1,s}(B(0,r))$ for some $s > 2$ and $\operatorname{curl} v = 0$. Then,

$$\|u\cdot v\|_{h^1_z(B(0,r))} \leqslant c\bigl(\|u\|_{L^2(B(0,r))} + \|\operatorname{div} u\|_{W^{-1,s}(B(0,r))}\bigr)\|v\|_{L^2(B(0,r))}.$$

3. A local version of the *BMO* regularity criteria on the vorticity

Theorem 1. *Let u be a weak solution on a space–time domain $\Omega \times (0,T)$ such that $\sup_{t\in(0,T)} \|\omega(t)\|_{L^1(\Omega)}$ is finite (e.g. a suitable weak solution with initial vorticity a finite Radon measure), (x_0,t_0) in $\Omega \times (0,T)$ and $0 < R < 1$ such that the parabolic cylinder $Q_{2R}(x_0,t_0) = B(x_0,2R) \times (t_0 - (2R)^2, t_0)$ is contained in $\Omega \times (0,T)$.*
Suppose that u is smooth in $Q_{2R}(x_0,t_0)$ and that

$$\|\omega\|_{bmo_r(B(x_0,2R))} \text{ is in } L^1\bigl(\bigl(t_0 - (2R)^2, t_0\bigr)\bigr).$$

Then the localized enstrophy remains uniformly bounded up to $t = t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} |\omega|^2(x,t)\, dx < \infty.$$

Proof. Let (x_0,t_0) be a point in $\Omega \times (0,T)$ and $0 < R < 1$ such that $Q_{2R}(x_0,t_0) = B(x_0,2R) \times (t_0 - (2R)^2, t_0)$ is contained in $\Omega \times (0,T)$. Given $0 < r \leqslant R$, let $\psi(x,t) = \phi(x)\eta(t)$ be a smooth cut-off function with the following properties,

$$\operatorname{supp}\phi \subset B(x_0,2r), \qquad \phi = 1 \quad \text{on } B(x_0,r), \qquad \frac{|\nabla\phi|}{\phi^\rho} \leqslant \frac{c}{r} \quad \text{for some } \rho \in (0,1),\ 0 \leqslant \phi \leqslant 1,$$

and

$$\operatorname{supp}\eta \subset \big(t_0-(2r)^2,t_0\big], \qquad \eta = 1 \quad \text{on } \big[t_0-r^2,t_0\big], \qquad \big|\eta'\big| \leqslant \frac{c}{r^2}, \quad 0 \leqslant \eta \leqslant 1.$$

Taking the curl of the velocity–pressure formulation we obtain the vorticity–velocity form of the 3D NSE,

$$\omega_t - \Delta\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u. \tag{1}$$

It has been shown in [12] that multiplying the vorticity equations by $\psi^2\omega$ and integrating over $Q_{2r}^s = B(x_0,2r) \times (t_0-(2r)^2,s)$, for a fixed s in $(t_0-(2r)^2,t_0)$, yields

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0,2r)} \phi^2(x)|\omega|^2(x,s)\,dx + \int_{Q_{2r}^s} \big|\nabla(\psi\omega)\big|^2\,dx\,dt \\
&\quad\leqslant \int_{Q_{2r}} \big(|\eta||\partial_t\eta| + |\nabla\psi|^2\big)|\omega|^2\,dx\,dt + \bigg|\int_{Q_{2r}^s} (u\cdot\nabla)\omega\cdot\psi^2\omega\,dx\,dt\bigg| + \bigg|\int_{Q_{2r}^s} (\omega\cdot\nabla)u\cdot\psi^2\omega\,dx\,dt\bigg| \\
&\quad\leqslant c(r)\int_{Q_{2r}} |\omega|^2\,dx\,dt + \frac{1}{2}\int_{Q_{2r}} \big|\nabla(\psi\omega)\big|^2\,dx\,dt + \bigg|\int_{Q_{2r}^s} (\omega\cdot\nabla)u\cdot\psi^2\omega\,dx\,dt\bigg|,
\end{aligned} \tag{2}$$

where the bound on the localized transport term $(u\cdot\nabla)\omega\cdot\psi^2\omega$ holds for any $\frac{1}{2} \leqslant \rho < 1$. This condition was used to estimate the lower order terms whereas the leading order term vanishes after the integration by parts due to the incompressibility of the fluid. In order to estimate the localized vortex stretching term we will use the explicit localization formula obtained in [10],

$$\begin{aligned}
\phi^2(x)(\omega\cdot\nabla)u\cdot\omega(x) &= -c\,\text{P.V.}\int_{B(x_0,2r)} \epsilon_{jkl}\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\phi\omega_l\,dy\,\phi(x)\omega_i(x)\omega_j(x) \\
&\quad - c\int_{B(x_0,2r)} \epsilon_{jkl}\frac{\partial}{\partial x_i}\frac{1}{|x-y|}\frac{\partial}{\partial y_k}\phi\omega_l\,dy \\
&\quad + c\int_{B(x_0,2r)} \frac{\partial}{\partial x_i}\frac{1}{|x-y|}(2\nabla\phi\cdot\nabla u_j + \Delta\phi u_j)\,dy \\
&\quad - \frac{\partial}{\partial x_i}\phi(x)u_j(x)\phi(x)\omega_i(x)\omega_j(x) \\
&= -c\,\text{P.V.}\int_{B(x_0,2r)} \epsilon_{jkl}\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\phi\omega_l\,dy\,\phi(x)\omega_i(x)\omega_j(x) + \text{LOT} \\
&= \text{VST}_{loc} + \text{LOT}
\end{aligned}$$

where LOT is comprised of the terms that are either lower order with respect to VST_{loc} for at least one order of the differentiation and/or less singular than VST_{loc} for at least one power of $|x-y|$.

Hence, for the vortex stretching term in (2) we get

$$\bigg|\int_{Q_{2r}^s} (\omega\cdot\nabla)u\cdot\psi^2\omega\,dx\,dt\bigg| \leqslant \bigg|\int_{Q_{2r}^s} \eta^2\,\text{VST}_{loc}\,dx\,dt\bigg| + \bigg|\int_{Q_{2r}^s} \eta^2\,\text{LOT}\,dx\,dt\bigg|.$$

It has been shown in [10] that for ρ close enough to 1 it is possible to bound the integral of the lower order terms with a bounded term that depends only on $\|\nabla u\|_{L^2(Q_{2r})}$ and r, and a term that can be absorbed by the left-hand side of (2) for sufficiently small r. It remains to estimate the leading vortex-stretching term.

The claim is that the leading order vortex-stretching term has a div–curl structure amenable to the application of Remark 1. Indeed, we can write

$$\text{P.V.}\int\limits_{B(x_0,2r)} \epsilon_{jkl}\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\phi\omega_l\,dy\,\phi(x)\omega_i(x) = \nabla_x\Bigg(\int\limits_{B(x_0,2r)}\nabla_y\frac{1}{|x-y|}\times\phi\omega\,dy\Bigg)_j\cdot\phi\omega$$
$$= \nabla_x E_j\cdot B. \tag{3}$$

$\mathrm{Curl}(\nabla_x E_j)=0$; on the other hand, for $\operatorname{div} B = \operatorname{div}(\phi\omega) = \nabla\phi\cdot\omega$, the Sobolev embedding and the properties of the cut-off ϕ yield

$$\begin{aligned}\big\|\nabla\phi\cdot\omega(t)\big\|_{W^{-1,3}(B(x_0,2r))} &\leqslant c\big\|\nabla\phi\cdot\omega(t)\big\|_{L^{\frac{3}{2}}(B(x_0,2r))}\\ &\leqslant \frac{c(\rho)}{r}\big\|\big(\phi|\omega(t)|\big)^{\rho}|\omega(t)|^{1-\rho}\big\|_{L^{3/2}(B(x_0,2r))}.\end{aligned}$$

For any $\frac{2}{3}<\rho<1$, several applications of the Hölder inequality imply

$$\begin{aligned}\big\|\nabla\phi\cdot\omega(t)\big\|_{W^{-1,3}(B(x_0,2r))} &\leqslant \frac{c(\rho)}{r}\big\|\phi\omega(t)\big\|^{\rho}_{L^{\frac{3\rho}{3\rho-1}}(B(x_0,2r))}\big\|\omega(t)\big\|^{1-\rho}_{L^1(B(x_0,2r))}\\ &\leqslant \frac{c(\rho)}{r}r^{\frac{3\rho-2}{2}}\big\|\phi\omega(t)\big\|^{\rho}_{L^2(B(x_0,2r))}\big\|\omega(t)\big\|^{1-\rho}_{L^1(B(x_0,2r))}\\ &= c(\rho)r^{\frac{3\rho-4}{2}}\big\|\phi\omega(t)\big\|^{\rho}_{L^2(B(x_0,2r))}\big\|\omega(t)\big\|^{1-\rho}_{L^1(B(x_0,2r))}\\ &\leqslant \big\|\phi\omega(t)\big\|_{L^2(B(x_0,2r))} + c(\rho)r^{\frac{3\rho-4}{2(1-\rho)}}\big\|\omega(t)\big\|_{L^1(B(x_0,2r))}.\end{aligned} \tag{4}$$

Using the fact that $(h^1_z(B(x_0,2r)))^* = bmo_r(B(x_0,2r))$, Remark 1, the fact that each component of $\nabla_x E_j(t)$ is the image of a component of $\phi\omega(t)$ under the Calderon–Zygmund operator with the kernel $\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}$ and (4), we arrive at the following string of inequalities,

$$\begin{aligned}\Bigg|\int\limits_{Q^s_{2r}}\eta^2\,\mathrm{VST}_{loc}\,dx\,dt\Bigg| &\leqslant \Bigg|\int\limits_{Q^s_{2r}}\big(\eta^2\nabla_x E_j\cdot B\big)(t)\cdot\omega(t)\,dx\,dt\Bigg|\\ &\leqslant \int\limits_{(t_0-2r^2,t_0)}\big\|\big(\eta^2\nabla_x E_j\cdot B\big)(t)\big\|_{h^1_z(B(x_0,2r))}\big\|\omega(t)\big\|_{bmo_r(B(x_0,2r))}\\ &\leqslant C\int\limits_{(t_0-2r^2,t_0)}\big\|(\psi\omega)(t)\big\|_{L^2(B(x_0,2r))}\big(\big\|(\psi\omega)(t)\big\|_{L^2(B(x_0,2r))}\\ &\quad+\big\|(\nabla\psi\cdot\omega)(t)\big\|_{W^{-1,3}(B(x_0,2r))}\big)\big\|\omega(t)\big\|_{bmo_r(B(x_0,2r))}\,dt\\ &\leqslant C\int\limits_{(t_0-2r^2,t_0)}\big(\big\|(\psi\omega)(t)\big\|^2_{L^2(B(x_0,2r))}+c(\rho)r^{\frac{3\rho-4}{1-\rho}}\big\|\omega(t)\big\|^2_{L^1(B(x_0,2r))}\big)\\ &\quad\times\big\|\omega(t)\big\|_{bmo_r(B(x_0,2r))}\,dt\\ &\leqslant C\sup_{t\in(t_0-(2r)^2,t_0)}\big\|\phi\omega(t)\big\|^2_{L^2(B(x_0,2r))}\int\limits_{(t_0-2r^2,t_0)}\big\|\omega(t)\big\|_{bmo_r(B(x_0,2r))}\,dt\\ &\quad+c(\rho,r)\sup_{t\in(t_0-(2r)^2,t_0)}\big\|\omega(t)\big\|^2_{L^1(B(x_0,2r))}\int\limits_{(t_0-2r^2,t_0)}\big\|\omega(t)\big\|_{bmo_r(B(x_0,2r))}\,dt.\end{aligned}$$

Due to our assumptions, the second term is finite. Since $\|\omega\|_{bmo_r(B(x_0,2R))}$ is in $L^1((t_0-(2R)^2,t_0))$, the Lebesgue dominated convergence implies that we can choose r small enough so that the first term gets absorbed by the left-hand side of (2). If R is small enough, we are done. If not, we simply cover $B(x_0,2R)$ with finitely many balls of radius r. □

Remark 2. The non-homogeneous div–curl approach to localization of the *BMO* regularity criteria presented here also leads to an alternative proof of the localization of the velocity *BMO* criterion given in [3]. More precisely, in the localized evolution of the enstrophy (2), write

$$\int_{Q_{2r}^s} (\omega \cdot \nabla) u \cdot \psi^2 \omega \, dx \, dt = -\int_{Q_{2r}^s} \nabla\big((\psi\omega)_j\big) \cdot (\psi\omega) u_j \, dx \, dt + \text{LOT};$$

this form of the vortex-stretching term has a non-homogeneous div–curl structure that is after utilizing the h_z^1–bmo_r duality amenable to the application of Remark 1.

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