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Finite time blow-up for a one-dimensional quasilinear parabolic–parabolic chemotaxis system

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Abstract

Finite time blow-up is shown to occur for solutions to a one-dimensional quasilinear parabolic–parabolic chemotaxis system as soon as the mean value of the initial condition exceeds some threshold value. The proof combines a novel identity of virial type with the boundedness from below of the Liapunov functional associated to the system, the latter being peculiar to the one-dimensional setting.

1. Introduction

We study the possible occurrence of blow-up in finite time for solutions to a one-dimensional parabolic system modeling chemotaxis [17]. More precisely, we consider the Keller–Segel chemotaxis model with nonlinear diffusion which describes the space and time evolution of a population of cells moving under the combined effects of diffusion (random motion) and a directed motion in the direction of high gradients of a chemical substance (chemoattractant) secreted by themselves. If $u \geqslant 0$ and v denote the density of cells and the (rescaled) concentration of chemoattractant, respectively, the Keller–Segel model with nonlinear diffusion reads

$$\partial_t u = \operatorname{div}\big(a(u)\nabla u - u\nabla v\big) \qquad \text{in } (0,\infty)\times\Omega, \tag{1}$$

$$\varepsilon\partial_t v = D\Delta v - \gamma v + u - M \qquad \text{in } (0,\infty)\times\Omega, \tag{2}$$

$$a(u)\partial_\nu u = \partial_\nu v = 0 \qquad \text{on } (0,\infty)\times\partial\Omega, \tag{3}$$

$$(u,v)(0) = (u_0,v_0) \qquad \text{in } \Omega. \tag{4}$$

In general, Ω is an open bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, a is a smooth non-negative function, and the parameters ε, D, γ, and M are non-negative real numbers with $D > 0$ and $M > 0$. In addition, the initial data u_0 and v_0 satisfy

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$$u_0 \geqslant 0, \quad \int_\Omega u_0(x)\,dx = |\Omega| M, \quad \text{and} \quad \int_\Omega v_0(x)\,dx = 0. \tag{5}$$

The constraints (5) ensure in particular that a solution (u, v) to (1)–(4) satisfies (at least formally) the same properties for positive times, that is,

$$u(t) \geqslant 0, \quad \int_\Omega u(t,x)\,dx = |\Omega| M, \quad \text{and} \quad \int_\Omega v(t,x)\,dx = 0. \tag{6}$$

The main feature of (1) is that it involves a competition between the diffusive term $\operatorname{div}(a(u)\nabla u)$ (spreading the population of cells) and the chemotactic drift term $-\operatorname{div}(u\nabla v)$ (concentrating the population of cells) that may lead to the blow-up in finite time of the solution to (1)–(4). The possible occurrence of such a singular phenomenon is actually an important mathematical issue in the study of (1)–(4) which is also relevant from a biological point of view: indeed, it corresponds to the experimentally observed concentration of cells in a narrow region of the space which is a preamble to a change of state of the cells. From a mathematical point of view, the blow-up issue has been the subject of several studies in the last twenty years, see the survey [13] and the references therein.

Still, it is far from being fully understood, in particular when $\varepsilon > 0$ (the so-called parabolic–parabolic Keller–Segel model). In that case, the only finite time blow-up result available seems to be that of Herrero and Velázquez who showed in [9,10] that, when Ω is a ball in $\mathbb{R}^2$, $D = 1$, and $a \equiv 1$, there are $M > 8\pi$ and radially symmetric solutions (u, v) to (1)–(4) which blow up in finite time. These solutions are constructed as small perturbations of time rescaled stationary solutions to (1)–(4) and a similar result is also true when $\varepsilon = 0$ [8]. The result in [10] actually goes far beyond the mere occurrence of blow-up in finite time as the shape of the blow-up profile is also identified. Recall that the condition $M > 8\pi$ is necessary for the finite blow-up to take place: indeed, it is shown in [21] that, if Ω is a ball in $\mathbb{R}^2$, $D = 1$, and $a \equiv 1$, radially symmetric solutions to (1)–(4) are global as soon as $M < 8\pi$. We refer to [7,21] for additional global existence results when Ω is a bounded domain in $\mathbb{R}^2$, $\varepsilon > 0$, and $a \equiv 1$. In [12,14,22] the existence of unbounded solutions is shown for $\varepsilon > 0$ and $a \equiv 1$, but it is not known whether the blow-up takes place in finite or infinite time. The same approach is employed in [15] to obtain unbounded solutions to quasilinear Keller–Segel systems, still without knowing whether the blow-up time is finite or infinite. The finite time blow-up result proved in this paper (Theorem 1) is thus the first one of this kind for quasilinear parabolic–parabolic Keller–Segel systems.

In contrast, for the parabolic–elliptic Keller–Segel system corresponding to $\varepsilon = 0$, several finite time blow-up results are available. There is thus a discrepancy between the two cases $\varepsilon > 0$ and $\varepsilon = 0$ which may be explained as follows. On the one hand, as observed in [16] when $\varepsilon = 0$, Ω is a ball of $\mathbb{R}^2$, $a \equiv 1$, and u_0 is radially symmetric, it is possible to reduce (1)–(4) to a single parabolic equation for the cumulative distribution function

$$U(t,r) := \int_{B(0,r)} u(t,x)\,dx.$$

Finite time blow-up is then shown with the comparison principle by constructing appropriate subsolutions. This approach was extended to nonlinear diffusions (non-constant a) and arbitrary space dimension $N \geqslant 1$ in [6]. On the other hand, it has been noticed in [2,18] that, still for $a \equiv 1$, the moment M_k of u defined by

$$M_k(t) := \int_\Omega |x|^k u(t,x)\,dx, \quad k \in (0,\infty),$$

satisfies a differential inequality which cannot hold true for all times for a suitably chosen value of $k > 0$, for it would imply that u reaches negative values in finite time in contradiction with (6). In contrast to the previous approach, this is an obstructive method which provides no information on the blow-up profile and is somehow reminiscent of the celebrated virial identity available for the nonlinear Schrödinger equation (see, e.g., [4, Section 6.5] and the references therein). Nevertheless, it applies to more general sets Ω [19,20,22]. We recently develop further this technique to establish finite time blow-up of radially symmetric solutions to (1)–(4) with $\varepsilon = 0$ in a ball of $\mathbb{R}^N$, $N \geqslant 2$, when the diffusion is nonlinear [5], the main idea being to replace the moments by nonlinear functions of the cumulative distribution function U. For a related model in $\mathbb{R}^N$ with nonlinear diffusion $a(u) = mu^{m-1}$, $m > 1$, finite time blow-up results were recently established in [3,24] by looking at the evolution of the second moment M_2.

Coming back to the parabolic–parabolic Keller–Segel system (1)–(4) ($\varepsilon > 0$), it seems unlikely that the first approach described above (reduction to a single equation) could work and the purpose of this paper is to show that finite time blow-up results can be established by the second approach in the one-dimensional case ($N = 1$). More precisely, we consider the initial-boundary value problem

$$\partial_t u = \partial_x\big(a(u)\partial_x u - u\partial_x v\big) \qquad \text{in } (0,\infty)\times(0,1), \tag{7}$$

$$\varepsilon\partial_t v = D\partial_x^2 v - \gamma v + u - M \qquad \text{in } (0,\infty)\times(0,1), \tag{8}$$

$$a(u)\partial_x u = \partial_x v = 0 \qquad \text{on } (0,\infty)\times\{0,1\}, \tag{9}$$

$$(u,v)(0) = (u_0,v_0) \qquad \text{in } (0,1), \tag{10}$$

and assume that

$$\varepsilon > 0, \quad D > 0, \quad \gamma \geqslant 0, \quad M > 0, \tag{11}$$

and the initial data $(u_0, v_0) \in W^{1,2}(0,1;\mathbb{R}^2)$ satisfy

$$u_0 \geqslant 0, \quad \int_0^1 u_0(x)\,dx = M, \quad \text{and} \quad \int_0^1 v_0(x)\,dx = 0. \tag{12}$$

We further assume that $a \in \mathcal{C}^2(\mathbb{R})$ and that there are $p \in (1,2]$, and $c_1 > 0$ such that

$$0 < a(r) \leqslant c_1(1+r)^{-p} \quad \text{for } r \geqslant 0. \tag{13}$$

Our main result then reads as follows.

Theorem 1. *Assume that the parameters ε, D, γ, M, the initial data (u_0, v_0), and the function a fulfil the conditions* (11), (12), *and* (13), *respectively. Then there is a unique classical maximal solution*

$$(u,v) \in \mathcal{C}\big([0,T_m)\times[0,1];\mathbb{R}^2\big) \cap \mathcal{C}^{1,2}\big((0,T_m)\times[0,1];\mathbb{R}^2\big)$$

to (7)–(10) *with the maximal existence time $T_m \in (0,\infty]$. It also satisfies*

$$u(t,x) \geqslant 0, \quad \int_0^1 u(t,x)\,dx = M, \quad \text{and} \quad \int_0^1 v(t,x)\,dx = 0 \tag{14}$$

for $(t,x) \in [0,T_m)\times[0,1]$. Introducing

$$\begin{aligned} F(z_1,z_2) &:= c_1(1+M) + \frac{M^2}{2D} + z_1 + Mz_2 + \frac{D+\gamma}{2}z_2^2, \\ \mathcal{P}_q(z_1,z_2,z_3) &:= \left(1 + \frac{\gamma}{D} + \frac{\gamma}{M}z_2 + \frac{M^{q-2}}{4qD}z_3\right)F(z_1,z_2) \\ &\quad + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}}F(z_1,z_2)^{(q-2)/q} - \frac{M^q}{q(q+1)} \end{aligned} \tag{15}$$

and

$$m_q(0) := \frac{1}{q}\int_0^1\left(\int_0^x u_0(y)\,dy\right)^q dx,$$

for $(z_1,z_2,z_3) \in [0,\infty)^3$ and $q \geqslant 2$, we have $T_m < \infty$ as soon as $\mathcal{P}_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0$ for some finite $q \in (2, 2/(2-p)]$. In particular, if u_0 is such that

$$\mathcal{P}_q\big(m_q(0),0,0\big) < 0 \quad \text{for some finite } q \in \big(2, 2/(2-p)\big], \tag{16}$$

there is $\vartheta > 0$ such that $\varepsilon M \in (0,\vartheta)$ and $\|v_0\|_{H^1} < \vartheta$ imply that $\mathcal{P}_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0$ and thus $T_m < \infty$.

There are functions u_0 satisfying (12) and (16) if M is sufficiently large. Indeed, observe that

$$\mathcal{P}_q(0,0,0) = \left(1 + \frac{\gamma}{D}\right)\left(c_1(1+M) + \frac{M^2}{2D}\right) + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}}\left(c_1(1+M) + \frac{M^2}{2D}\right)^{(q-2)/q} - \frac{M^q}{q(q+1)}$$

is negative for sufficiently large M as $q > 2$. Given such an $M > 0$ and choosing the function $u_0(x) = 2M\max\{x+\delta-1, 0\}/\delta^2$, $x \in (0,1)$, we have $m_q(0) = (2M)^q\delta/(2q+1)$ and $\mathcal{P}_q(m_q(0), 0, 0) < 0$ for $\delta > 0$ small enough. In fact, if u_0 fulfils (16), then the same computation as the one leading to Theorem 1 shows that the corresponding solution to the parabolic–elliptic Keller–Segel system ($\varepsilon = 0$) blows up in a finite time and the last assertion of Theorem 1 states that this property remains true for the parabolic–parabolic Keller–Segel system ($\varepsilon > 0$) provided ε and v_0 are small, that is, in a kind of neighbourhood of the parabolic–elliptic case.

Remark 2. The growth condition required on a in (13) is seemingly optimal: indeed, it is proved in [6] that $T_m = \infty$ if $a(r) \geqslant c_0(1+r)^{-p}$ for some $p < 1$ and $\varepsilon = 0$, and the proof is likely to extend to the case $\varepsilon > 0$. Global existence of solutions to (7)–(10) is actually shown in [23] for $\varepsilon > 0$ under the stronger assumption that $a(r) \geqslant c_0(1 + r^p)$ for some $c_0 > 0$ and $p > 0$.

The proof of Theorem 1 relies on two properties of the Keller–Segel system (7)–(10): first, there is a Liapunov functional [7,11] which is bounded from below in the one-dimensional case [6] and which provides information on the time derivative of v. This will be the content of Section 2 where we also recall the local well-posedness of (7)–(10). We next derive an identity of virial type for the L^q-norm of the indefinite integral of u in Section 3 which involves in particular the time derivative of v. The information obtained on this quantity in the previous section then allow us to derive a differential inequality for the L^q-norm of the indefinite integral of u for a suitable value of q which cannot be satisfied for all times if the parameters ε, D, γ, M, and the initial data (u_0, v_0) are suitably chosen.

2. Well-posedness and Liapunov functional

In this section, we recall the local well-posedness of (7)–(10) in $W^{1,2}(0,1;\mathbb{R}^2)$ [1,11] and the availability of a Liapunov functional for this system [7,11]. To this end, we assume that

$$0 < a \in \mathcal{C}^2(\mathbb{R}) \tag{17}$$

and define $b \in \mathcal{C}^2((0,\infty))$ by

$$b(1) = b'(1) := 0 \quad \text{and} \quad b''(r) := \frac{a(r)}{r} \quad \text{for } r > 0. \tag{18}$$

Proposition 3. *Assume that the parameters ε, D, γ, M, and the function a fulfil* (11) *and* (17)*, respectively. Given the initial data $(u_0, v_0) \in W^{1,2}(0,1;\mathbb{R}^2)$ satisfying* (12)*, there is a unique classical maximal solution*

$$(u,v) \in \mathcal{C}\big([0,T_m) \times [0,1];\mathbb{R}^2\big) \cap \mathcal{C}^{1,2}\big((0,T_m) \times [0,1];\mathbb{R}^2\big)$$

to (7)–(10) *with the maximal existence time $T_m \in (0,\infty]$ and (u,v) satisfies* (14) *for $t \in [0,T_m)$. In addition, if $T_m < \infty$, we have*

$$\lim_{t\to T_m}\left(\|u(t)\|_\infty + \|v(t)\|_\infty\right) = \infty. \tag{19}$$

Owing to the assumptions on a and the initial data, the existence and uniqueness of a maximal solution to (7)–(10) readily follow from [1, Theorems 14.4 & 14.6], see [11, Theorem 1]. As for the last statement (19), it is a consequence of the upper triangular structure of the system (in the sense that the second equation (8) does not involve the second-order derivative of u) and [1, Theorem 15.5].

Next, an important property of (7)–(10) first noticed in [7] for $a \equiv 1$ and further developed in [11, Theorem 2] in a more general setting is the availability of a Liapunov functional.

Lemma 4. *Assume that the parameters ε, D, γ, M, and the function a fulfil* (11) *and* (17)*, respectively. Given the initial data* $(u_0, v_0) \in W^{1,2}(0,1;\mathbb{R}^2)$ *satisfying* (12) *and such that* $b(u_0) \in L^1(0,1)$*, the corresponding classical solution* (u, v) *to* (7)–(10) *satisfies*

$$L\big(u(t), v(t)\big) + \varepsilon \int_0^t \big\|\partial_t v(s)\big\|_2^2 \, ds \leqslant L(u_0, v_0) \quad \text{for } t \in [0, T_m), \tag{20}$$

where

$$L(u,v) := \int_0^1 \left(b(u) - uv + \frac{D}{2}|\partial_x v|^2 + \frac{\gamma}{2}|v|^2 \right) dx. \tag{21}$$

Proof. We sketch the proof for the sake of completeness. It follows from (7)–(9) that

$$\begin{aligned}
\frac{d}{dt} L(u,v) &= \int_0^1 \big(b'(u) - v\big)\partial_t u \, dx + \int_0^1 \big(D\partial_x v \partial_x \partial_t v + (\gamma v - u)\partial_t v\big) \, dx \\
&= -\int_0^1 \big(b''(u)\partial_x u - \partial_x v\big)\big(a(u)\partial_x u - u\partial_x v\big) \, dx \\
&\quad + \int_0^1 \partial_t v \big(-D\partial_x^2 v + \gamma v - u\big) \, dx \\
&= -\int_0^1 u \big|\partial_x \big(b'(u) - v\big)\big|^2 \, dx - \int_0^1 (M + \varepsilon \partial_t v)\partial_t v \, dx \\
&\leqslant -\varepsilon \|\partial_t v\|_2^2,
\end{aligned} \tag{22}$$

the last inequality being a consequence of (14). Integrating the previous inequality with respect to time gives (20). □

We next take advantage of the one-dimensional setting to show that L is bounded from below without prescribing growth conditions on a. This fact has already been observed in [6] and is peculiar to the one-dimensional case. Indeed, as shown in [7,12], the occurrence of blow-up is closely related to the unboundedness of the Liapunov functional.

Lemma 5. *Assume that the parameters ε, D, γ, M, and the function a fulfil* (11) *and* (17)*, respectively. Given the initial data* $(u_0, v_0) \in W^{1,2}(0,1;\mathbb{R}^2)$ *satisfying* (12) *and such that* $b(u_0) \in L^1(0,1)$*, the corresponding classical solution* (u, v) *to* (7)–(10) *satisfies*

$$L\big(u(t), v(t)\big) \geqslant -\frac{M^2}{2D} \quad \text{for } t \in [0, T_m). \tag{23}$$

Proof. Owing to (14), the Poincaré inequality ensures that $\|v(t)\|_\infty \leqslant \|\partial_x v(t)\|_2$ for $t \in [0, T_m)$ so that

$$\int_0^1 u(t)v(t) \, dx \leqslant \big\|v(t)\big\|_\infty \big\|u(t)\big\|_1 \leqslant \big\|\partial_x v(t)\big\|_2 \big\|u(t)\big\|_1.$$

We use again (14) as well as the non-negativity of b to conclude that

$$L\big(u(t), v(t)\big) \geqslant \frac{D}{2}\big\|\partial_x v(t)\big\|_2^2 - M\big\|\partial_x v(t)\big\|_2 = \frac{D}{2}\left(\big\|\partial_x v(t)\big\|_2 - \frac{M}{D}\right)^2 - \frac{M^2}{2D}$$

for $t \in [0, T_m)$, from which (23) readily follows. □

3. Finite time blow-up

As already mentioned, the main novelty in this paper is a new identity of virial type which is the cornerstone of the proof that blow-up takes place in finite time under suitable assumptions. Specifically, we assume that the parameters ε, D, γ, M, and the function a fulfil the conditions (11) and (13), respectively. Recalling the definition (18) of b, we deduce from (13) that

$$b(r) \leqslant c_1(r\ln r - r + 1)\mathbf{1}_{[0,1]}(r) + \frac{c_1(r-1)}{p}\mathbf{1}_{[1,\infty)}(r) \leqslant c_1(1+r), \quad r \geqslant 0. \tag{24}$$

We also define

$$A(r) := -\int_r^\infty a(s)\,ds, \quad r \geqslant 0, \tag{25}$$

and infer from (13) that A is well-defined and satisfies

$$0 \leqslant -A(r)r \leqslant \frac{c_1}{p-1} r^{2-p}, \quad r \geqslant 0. \tag{26}$$

Consider next the initial data $(u_0, v_0) \in W^{1,2}(0,1;\mathbb{R}^2)$ satisfying (12). If (u, v) denotes the corresponding classical solution to (7)–(10) given by Proposition 3, we define the cumulative distribution functions U and V by

$$U(t,x) := \int_0^x u(t,y)\,dy \quad \text{and} \quad V(t,x) := \int_0^x v(t,y)\,dy \tag{27}$$

for $(t,x) \in [0,T_m) \times [0,1]$. It readily follows from (7)–(9) and (14) that (U, V) solves

$$\partial_t U = \partial_x A(u) - u\partial_x v \qquad \text{in } (0,T_m)\times(0,1), \tag{28}$$

$$\varepsilon\partial_t V = D\partial_x v - \gamma V + U - Mx \qquad \text{in } (0,T_m)\times(0,1), \tag{29}$$

the function A being defined in (25), and

$$U(t,0) = M - U(t,1) = 0 \quad \text{and} \quad V(t,0) = V(t,1) = 0, \quad t \in [0,T_m). \tag{30}$$

Lemma 6. *Introducing $m_q(t) := \|U(t)\|_q^q/q$ for $q \geqslant 2$, we have*

$$\begin{aligned}\frac{dm_q}{dt} = {} & \frac{M}{D}m_q - \frac{M^{q+1}}{q(q+1)D} + M^{q-1}A\big(u(t,1)\big) - (q-1)\int_0^1 U^{q-2}uA(u)\,dx \\ & + \frac{\varepsilon}{qD}\int_0^1 U^q\partial_t v\,dx - \frac{\gamma}{D}\int_0^1 U^{q-1}uV\,dx\end{aligned} \tag{31}$$

for $t \in [0,T_m)$.

Proof. We infer from (28), (29), and (30) that

$$\begin{aligned}\frac{dm_q}{dt} = {} & \big[U^{q-1}A(u)\big]_{x=0}^{x=1} - (q-1)\int_0^1 U^{q-2}uA(u)\,dx \\ & - \frac{1}{D}\int_0^1 uU^{q-1}(\varepsilon\partial_t V + \gamma V - U + Mx)\,dx\end{aligned}$$

$$= M^{q-1}A\big(u(t,1)\big) - (q-1)\int_0^1 U^{q-2}uA(u)\,dx - \frac{\varepsilon}{qD}\big[U^q\partial_t V\big]_{x=0}^{x=1}$$

$$+ \frac{\varepsilon}{qD}\int_0^1 U^q\partial_t v\,dx - \frac{\gamma}{D}\int_0^1 U^{q-1}uV\,dx + \frac{1}{(q+1)D}\big[U^{q+1}\big]_{x=0}^{x=1} - \frac{M}{qD}\big[U^q x\big]_{x=0}^{x=1} + \frac{M}{D}m_q$$

$$= M^{q-1}A\big(u(t,1)\big) - (q-1)\int_0^1 U^{q-2}uA(u)\,dx + \frac{\varepsilon}{qD}\int_0^1 U^q\partial_t v\,dx$$

$$- \frac{\gamma}{D}\int_0^1 U^{q-1}uV\,dx - \frac{M^{q+1}}{q(q+1)D} + \frac{M}{D}m_q,$$

which is the expected identity. □

At this point, we notice that the solution to the ordinary differential equation $D\dot{X} = MX - (M^{q+1}/(q(q+1)))$ (obtained by neglecting several terms in (31)) is given by

$$X(t) = \frac{M^q}{q(q+1)} + e^{Mt/D}\left(X(0) - \frac{M^q}{q(q+1)}\right),$$

and thus vanishes at a finite time if $X(0) < M^q/(q(q+1))$. If a similar argument could be used for (31), we would obtain a positive time t_0 such that $m_q(t_0) = 0$ which clearly contradicts the properties of $U(t_0)$: indeed, by (27) and (30), $x \mapsto U(t_0,x)$ is continuous with $U(t_0,1) = M$. Consequently, the solution (u,v) to (7)–(10) no longer exists at this time t_0 and blow-up shall have occurred at an earlier time, thus establishing Theorem 1. For this approach to work, we shall of course control the other terms on the right-hand side of (31) which will in turn give rise to the blow-up criterion stated in Theorem 1. The latter is actually a simple consequence of the following result:

Theorem 7. *Assume that the parameters ε, D, γ, M, and the initial data (u_0, v_0) are such that*

$$E\left(m_q(0) + L(u_0,v_0) + \frac{M^2}{2D}\right) < 0 \tag{32}$$

for some finite $q \in (2, 2/(2-p)]$, where

$$E(z) := \left(1 + \frac{\gamma}{D} + \frac{\gamma}{M}\|v_0\|_{H^1} + \frac{\varepsilon M^{q-1}}{4qD}\right)z + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}}z^{(q-2)/q} - \frac{M^q}{q(q+1)}$$

for $z \geqslant 0$. Then $T_m < \infty$.

Proof. The starting point of the proof being the identity (31), we first derive upper bounds for the terms on the right-hand side of (31) involving A, ε, and γ. Thanks to (26) and the non-negativity of U, it follows from the Hölder inequality that

$$M^{q-1}A\big(u(t,1)\big) - (q-1)\int_0^1 U^{q-2}uA(u)\,dx \leqslant \frac{c_1(q-1)}{p-1}\int_0^1 U^{q-2}u^{2-p}\,dx$$

$$\leqslant \frac{c_1(q-1)q^{(q-2)/q}}{(p-1)}m_q^{(q-2)/q}\left(\int_0^1 u^{((2-p)q)/2}\,dx\right)^{2/q}.$$

Since $q \in (2, 2/(2-p)]$, we may use the Jensen inequality and (14) to conclude that

$$M^{q-1}A\big(u(t,1)\big) - (q-1)\int_0^1 U^{q-2}uA(u)\,dx \leqslant \frac{c_1(q-1)q^{(q-2)/q}}{(p-1)}M^{2-p}m_q^{(q-2)/q}. \tag{33}$$

Next, to estimate the term involving γ, we adapt an argument from [18] and first claim that

$$V(t,x) \geqslant V_m(t,x) := \frac{M}{6D}\left(x^3 - x\right) + h(t,x), \quad (t,x) \in [0,T_m) \times [0,1], \tag{34}$$

where h denotes the unique solution to

$$\varepsilon \partial_t h - D\partial_x^2 h + \gamma h = 0, \quad (t,x) \in (0,\infty) \times (0,1), \tag{35}$$

$$h(t,0) = h(t,1) = 0, \quad t \in (0,\infty), \tag{36}$$

$$h(0,x) = \min\left\{V(0,x) + \frac{M}{6D}\left(x - x^3\right), 0\right\} \leqslant 0, \quad x \in (0,1). \tag{37}$$

Indeed, $V_m \leqslant V$ on $[0,T_m) \times \{0,1\}$ and $\{0\} \times [0,1]$, and it follows from the non-negativity of U and the negativity of h that

$$\begin{aligned}\varepsilon \partial_t V_m - D\partial_x^2 V_m + \gamma V_m &= \varepsilon \partial_t h - Mx - D\partial_x^2 h + \frac{M\gamma}{6D}\left(x^3 - x\right) + \gamma h \\ &\leqslant -Mx \leqslant U - Mx = \varepsilon \partial_t V - D\partial_x^2 V + \gamma V.\end{aligned}$$

The comparison principle then implies (34). We next infer from (34) and the non-negativity of u and U that

$$\begin{aligned}-\frac{\gamma}{D}\int_0^1 U^{q-1} u V\,dx &\leqslant -\frac{\gamma}{D}\int_0^1 U^{q-1} u V_m\,dx \\ &= -\frac{\gamma}{qD}\left[U^q V_m\right]_{x=0}^{x=1} + \frac{\gamma}{qD}\int_0^1 U^q \partial_x V_m\,dx \\ &\leqslant \frac{\gamma}{D}\left(\frac{M}{2D} + \|\partial_x h\|_\infty\right) m_q.\end{aligned}$$

We next note that $\partial_x h$ also solves (35) with homogeneous Neumann boundary conditions, the latter property being a consequence of (35) and (36). Since

$$\left|\partial_x h(0,x)\right| \leqslant \left|v_0(x) + \frac{M}{6D}\left(1 - 3x^2\right)\right| \leqslant \|v_0\|_\infty + \frac{M}{3D},$$

the comparison principle and the non-negativity of γ warrant that $\|\partial_x h(t)\|_\infty \leqslant \|v_0\|_\infty + (M/3D)$ for $t \geqslant 0$. Consequently, recalling the Sobolev embedding $\|v_0\|_\infty \leqslant \|v_0\|_{H^1}$, we end up with

$$-\frac{\gamma}{D}\int_0^1 U^{q-1} u V\,dx \leqslant \frac{\gamma M}{D^2}\left(1 + \frac{D}{M}\|v_0\|_{H^1}\right) m_q. \tag{38}$$

We finally infer from (14), (27), (30), and the Hölder inequality that

$$\frac{\varepsilon}{qD}\int_0^1 U^q \partial_t v\,dx \leqslant \frac{\varepsilon M^{q/2}}{qD}\int_0^1 U^{q/2}|\partial_t v|\,dx \leqslant \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2}\|\partial_t v\|_2. \tag{39}$$

It now follows from (31), (33), (38), and (39) that

$$\begin{aligned}\frac{dm_q}{dt} &\leqslant \frac{M}{D}\left[\left(1 + \frac{\gamma}{D} + \frac{\gamma}{M}\|v_0\|_{H^1}\right) m_q + \frac{c_1(q-1)q^{(q-2)/q} D}{(p-1)M^{p-1}} m_q^{(q-2)/q} - \frac{M^q}{q(q+1)}\right] + \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2}\|\partial_t v\|_2 \\ &\leqslant \frac{M}{D}E(m_q) - \frac{\varepsilon M^q}{4qD^2} m_q + \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2}\|\partial_t v\|_2.\end{aligned}$$

Owing to (12) and (24), we have $b(u_0) \in L^1(0,1)$ and it follows from (22), (23), and the above inequality that

$$\begin{aligned}\frac{d}{dt}\left(m_q + L(u,v) + \frac{M^2}{2D}\right) &\leqslant \frac{M}{D}E(m_q) - \frac{\varepsilon M^q}{4qD^2}m_q + \frac{\varepsilon M^{q/2}}{q^{1/2}D}m_q^{1/2}\|\partial_t v\|_2 - \varepsilon\|\partial_t v\|_2^2 \\ &= \frac{M}{D}E(m_q) - \varepsilon\left(\|\partial_t v\|_2 - \frac{M^{q/2}}{2q^{1/2}D}m_q^{1/2}\right)^2 \\ &\leqslant \frac{M}{D}E(m_q).\end{aligned}$$

Using now the monotonicity of E and (23), we end up with

$$\frac{d}{dt}\left(m_q + L(u,v) + \frac{M^2}{2D}\right) \leqslant \frac{M}{D}E\left(m_q + L(u,v) + \frac{M^2}{2D}\right).$$

Assume now for contradiction that $T_m = \infty$. The previous inequality and (32) then warrant that there is a time $t_0 > 0$ such that $m_q(t_0) + L(u(t_0), v(t_0)) + (M^2/2D) = 0$ and hence $m_q(t_0) = 0$ by (23). This in turn implies that $U(t_0, x) = 0$ for all $x \in [0,1]$ and contradicts (30). Consequently, $T_m < \infty$. □

The remaining step towards Theorem 1 is to use the properties of a to simplify the condition (32) derived in Theorem 7.

Proof of Theorem 1. It follows from (12), (24), and the Sobolev embedding $\|v_0\|_\infty \leqslant \|v_0\|_{H^1}$ that

$$\begin{aligned}L(u_0,v_0) + \frac{M^2}{2D} &\leqslant \int_0^1 \left(c_1(1+u_0) + \frac{D}{2}|\partial_x v_0|^2 + \frac{\gamma}{2}|v_0|^2 + u_0\|v_0\|_\infty\right)dx + \frac{M^2}{2D} \\ &\leqslant c_1(1+M) + \frac{M^2}{2D} + \frac{D+\gamma}{2}\|v_0\|_{H^1}^2 + M\|v_0\|_{H^1} \\ &= F\big(m_q(0), \|v_0\|_{H^1}\big) - m_q(0),\end{aligned}$$

the function F being defined in Theorem 1. Therefore,

$$E\left(m_q(0) + L(u_0,v_0) + \frac{M^2}{2D}\right) \leqslant (E\circ F)\big(m_q(0), \|v_0\|_{H^1}\big) = \mathcal{P}_q\big(m_q(0), \|v_0\|_{H^1}, \varepsilon M\big),$$

and the condition $\mathcal{P}_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0$ clearly implies (32) and hence $T_m < \infty$. □

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