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Singular solution to Special Lagrangian Equations

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Abstract

We prove the existence of non-smooth solutions to three-dimensional Special Lagrangian Equations in the non-convex case.

Résumé

Nous démontrons l'existence de solutions singulières d'équations speciales lagrangiennes en dimension trois, dans le cas non convexe.

MSC: 35J60; 53C38

1. Introduction

In this paper we study a fully nonlinear second-order elliptic equations of the form (where $h \in \mathbf{R}$)

$$\mathbf{F}_h\big(D^2u\big) = \det\big(D^2u\big) - Tr\big(D^2u\big) + h\sigma_2\big(D^2u\big) - h = 0 \tag{1}$$

defined in a smooth-bordered domain of $\Omega \subset \mathbf{R}^3$, $\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$ being the second symmetric function of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of D^2u. Here D^2u denotes the Hessian of the function u. This equation is equivalent to the Special Lagrangian potential equation [10]:

$$SLE_\theta: \quad Im\big\{e^{-i\theta}\det\big(I + iD^2u\big)\big\} = 0$$

for $h := -\tan(\theta)$ which can be re-written as

$$\mathbf{F}_\theta = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 - \theta = 0.$$

The set

$$\big\{A \in Sym^2\big(\mathbf{R}^3\big)\colon\ \mathbf{F}_h(A) = 0\big\} \subset Sym^2\big(\mathbf{R}^3\big)$$

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has three connected components, C_i, $i = 1, 2, 3$, which correspond to the values $\theta_1 = -\arctan(h) - \pi$, $\theta_2 = -\arctan(h)$, $\theta_3 = -\arctan(h) + \pi$.

We study the Dirichlet problem

$$\begin{cases} \mathbf{F}_\theta\big(D^2u\big) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

For $\theta_1 = -\arctan(h) - \pi$ and $\theta_3 = -\arctan(h) + \pi$ the operator $\mathbf{F}_\theta$ is concave or convex, and the Dirichlet problem in these cases was treated in [5]; smooth solutions are established there for smooth boundary data on appropriately convex domains.

The middle branch $C_2, \theta_2 = -\arctan(h)$ is never convex (neither concave), and the classical solvability of the Dirichlet problem remained open.

In the case of uniformly elliptic equations a theory of weak (viscosity) solutions for the Dirichlet problem gives the uniqueness of such solutions, see [8], moreover these solutions lie in $C^{1,\varepsilon}$ [4,16,17]. However, the recent results [13–15] show that at least in 12 and more dimensions the viscosity solution of the Dirichlet problem for a uniformly elliptic equation can be singular, even in the case when the operator depends only on eigenvalues of the Hessian.

One can define viscosity solutions for strictly elliptic equations (such as SLE_θ); in this case the uniqueness and the existence for C^0 viscosity solutions are known to experts in the field. For example, strict ellipticity alone is enough for the argument leading to comparison on pp. 45 and 46 of Caffarelli and Cabre [4]; cf. [6, p. 594]. However, we prefer to use a new very interesting approach to degenerate elliptic equations suggested recently by Harvey and Lawson [11]. They introduced a new notion of a weak solution for the Dirichlet problem for such equations and proved the existence, the continuity and the uniqueness of these solutions.

The main purpose of the present note is to show that the classical solvability for Special Lagrangian Equations *does not* hold.

More precisely, we show the existence for any $\theta \in]-\pi/2, \pi/2[$ of a small ball $B \subset \mathbf{R}^3$ and of an analytic function ϕ on ∂B for which the unique Harvey–Lawson solution u_θ of the Dirichlet problem satisfies:

(i) $u_\theta \in C^{1,1/3}$;
(ii) $u_\theta \notin C^{1,\delta}$ for $\forall\delta > 1/3$.

Our construction use the Legendre Transform for solutions of $\mathbf{F}_{\frac{1}{h}}(D^2u) = 0$ which gives solutions of $\mathbf{F}_h(D^2u) = 0$; in particular, for $h = 0$ it transforms solutions of $\sigma_2(D^2u) = 1$ into solutions of $\det(D^2u) = Tr(D^2u)$. This construction could be of interest by itself. The Legendre Transform was already used in [3, p. 312] (as was kindly pointed to us by the anonymous referee) to convert solutions of $\det(D^2u) = Tr(D^2u)$ into solutions of $\sigma_2(D^2u) = 1$, but only in the convex case.

Finally, we think that the following conjecture is quite plausible.

Conjecture. *Any Harvey–Lawson solution of SLE_θ on a ball B lies in $C^1(B)$ (if φ is sufficiently smooth).*

In the case $\theta = 0$ these solutions lie in $C^{0,1}(B)$ by Corollary 1.2 in [18] (note that in this case the convexity of the equation is not really necessary there).

2. Harvey–Lawson Dirichlet Duality Theory

In this section we recall the "Dirichlet duality" theory by Harvey and Lawson [11], which establishes (under an appropriate explicit geometric assumption on the domain Ω) the existence and uniqueness of continuous solutions of the Dirichlet problem for fully nonlinear, degenerate elliptic equations

$$\mathbf{F}\big(D^2u\big) = 0. \tag{2}$$

Following the method by Krylov [12] this theory takes a geometric approach to the equation which eliminates the operator $\mathbf{F}$ and replaces it with a closed subset F of the space $Sym^2(\mathbf{R}^n)$ of real symmetric $n \times n$ matrices, with the property that ∂F is contained in $\{\mathbf{F}=0\}$. We need only the case when Ω is a ball when the geometric assumption is automatically true and thus we do not discuss it below.

The general set-up of the theory is the following. Let F be a given closed subset of the space of real symmetric matrices $Sym^2(\mathbf{R}^n)$. The theory formulates and solves the Dirichlet problem for the equation

$$Hess_x(u) \in \partial F \quad \text{for all } x \in \Omega$$

using the functions of "type F", i.e., which satisfy

$$Hess_x(u) \in F \quad \text{for all } x.$$

A priori these conditions make sense only for C^2-functions u. The theory extends the notion to functions which are only upper semi-continuous.

A closed subset $F \subset Sym^2(\mathbf{R}^n)$ is called a *Dirichlet set* if it satisfies the condition

$$F + \mathcal{P} \subset F$$

where

$$\mathcal{P} = \{A \in Sym^2(\mathbf{R}^n)\colon\ A \geqslant 0\}$$

is the subset of non-negative matrices. This condition corresponds to degenerate ellipticity in modern fully nonlinear theory; it implies that the maximum of two functions of type F is again of type F which is the key requirement for solving the Dirichlet problem. Note that translates, unions (when closed) and intersections of Dirichlet sets are Dirichlet sets. The *Dirichlet dual* set $\tilde{F}$ is defined as

$$\tilde{F} := -\big(Sym^2(\mathbf{R}^n)\backslash Int(F)\big).$$

By Lemma 4.3 in [11] this is equivalent to the condition

$$\tilde{F} := \{A \in Sym^2(\mathbf{R}^n)\colon\ \forall B \in F,\ A + B \in \tilde{\mathcal{P}}\},$$

$\tilde{\mathcal{P}}$ being the set of all quadratic forms except those that are negative definite.

An upper semi-continuous (USC) function u is called a *subaffine function* if it verifies locally the condition:

For each affine function a, if $u \leqslant a$ on the boundary of a ball B, then $u \leqslant a$ on B.

Note that a C^2-function is subaffine if and only if $Hess(u)$ has at least one non-negative eigenvalue at each point. A USC function u is *of type F* if $u + v$ is subaffine for all C^2-functions v of type $\tilde{F}$. In other words, u is of type F if for any "test function" $v \in C^2$ of dual type $\tilde{F}$, the sum $u + v$ satisfies the maximum principle. A function u on a domain is said to be *F-Dirichlet* if u is of type F and $-u$ is of type $\tilde{F}$. Such a function u is automatically continuous, and at any point x where u is C^2, it satisfies the condition

$$Hess_x(u) \in \partial F \quad \text{for all } x \in \Omega.$$

The main result of the theory (in our restricted setting) is Theorem 6.2 in [11].

Theorem *(The Dirichlet problem). Let $B \subset \mathbf{R}^n$ be a ball, and let F be a Dirichlet set. Then for each $\varphi \in C(\partial B)$, there exists a unique $u \in C(B)$ which is an F-Dirichlet function on B and equals φ on ∂B.*

Besides, one has [11, Remark 4.9]:

Proposition *(Viscosity solutions). In the conditions of the theorem u is a viscosity solution of* (2).

Krylov's idea [12, Theorem 3.2] permits to reconstruct from F a *canonical form* of the operator $\mathbf{F}$ such that:

1) $\partial F = \{\mathbf{F} = 0\}$;
2) $F = \{\mathbf{F} \geqslant 0\}$.

It is sufficient to define

$\mathbf{F}(A) := dist(A, \partial F)$ for $A \in F$;
$\mathbf{F}(A) := -dist(A, \partial F)$ for $A \notin F$.

The operator $\mathbf{F}$ (in its canonical form) is *strictly elliptic* if for any $A \in F$ there exists $\delta(A) > 0$ s.t. $\mathbf{F}(A + P) \geqslant \delta(A) \cdot \|P\|$ for all $P \in \mathcal{P}$, and *uniformly elliptic* if $\mathbf{F}(A + P) \geqslant \delta \cdot \|P\|$ for all $P \in \mathcal{P}$, $A \in F$ and an absolute constant $\delta > 0$ (note that for $\mathbf{F}$ in its canonical form $\mathbf{F}(A + P) - \mathbf{F}(A) \leqslant \|P\|$ by definition). Moreover, the function $\mathbf{F}$ is concave iff F is concave, and is convex iff $\tilde{F}$ is convex.

Below we will use the Harvey–Lawson theory only in the case of Hessian equations, i.e. when $\mathbf{F}(A)$ depends only on the eigenvalues $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A)$ of A. Then the sets $\{\mathbf{F} = 0\}$, F, $\tilde{F}$ are stable under the action of the orthogonal group $O_n(\mathbf{R})$ by conjugation. Consider the map

$$Sym^2(\mathbf{R}^n) \longrightarrow D_n \subset \mathbf{R}^n,$$
$$A \longmapsto \big(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\big)$$

where

$$D_n := \big\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{R}^n\colon\ \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n\big\}.$$

The images $\{\mathbf{F}_\lambda = 0\}$, F_λ, $\tilde{F}_\lambda$ of $\{\mathbf{F} = 0\}$, F, $\tilde{F}$ determine completely their preimages. The sets

$$\{\mathbf{F}_\Lambda = 0\} := \bigcup_{\sigma \in S_n} \{\mathbf{F}_{\sigma(\lambda)} = 0\} \subset \mathbf{R}^n,$$
$$F_\Lambda := \bigcup_{\sigma \in S_n} F_{\sigma(\lambda)} \subset \mathbf{R}^n,$$
$$\tilde{F}_\Lambda := \bigcup_{\sigma \in S_n} \tilde{F}_{\sigma(\lambda)} \subset \mathbf{R}^n,$$

where $\sigma(\lambda) := (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)})$ are S_n-invariant subsets in $\mathbf{R}^n$ which determine $\{\mathbf{F} = 0\}$, F and $\tilde{F}$ as well. Moreover, it is well known (see, e.g., [2,5]) that the set F is convex iff F_Λ is convex.

3. Some properties of Special Lagrangian Equations

In this section we give some properties of the Special Lagrangian Equation

$$\mathbf{F}_{-c}\big(D^2u\big) = \det\big(D^2u\big) - Tr\big(D^2u\big) - c\sigma_2\big(D^2u\big) + c = 0.$$

Note first that the set $\{\mathbf{F}_{-c,\Lambda} = 0\}$ is a real cubic surface $\mathbf{S}_{c,\Lambda}$ with three components ("branches") which can be presented as a graph:

$$\lambda_3 = \frac{c(1 - \lambda_1\lambda_2) + \lambda_1 + \lambda_2}{\lambda_1\lambda_2 - 1 + c(\lambda_1 + \lambda_2)}. \tag{3}$$

One easily proves the following by brute force computations:

Lemma 3.1.

1) *The components of* $\mathbf{S}_{c,\Lambda}$ *are given by*

$$C_1 = \big\{(\lambda_1, \lambda_2, \lambda_3)\colon\ \lambda_1\lambda_2 - 1 + c(\lambda_1 + \lambda_2) > 0,\ \lambda_1 > -c,\ \lambda_2 > -c\big\},$$
$$C_2 = \big\{(\lambda_1, \lambda_2, \lambda_3)\colon\ \lambda_1\lambda_2 - 1 + c(\lambda_1 + \lambda_2) < 0\big\},$$
$$C_3 = \big\{(\lambda_1, \lambda_2, \lambda_3)\colon\ \lambda_1\lambda_2 - 1 + c(\lambda_1 + \lambda_2) > 0,\ \lambda_1 < -c,\ \lambda_2 < -c\big\};$$

equivalently,

$$C_1 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = \pi + \arctan c\},$$
$$C_2 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = \arctan c\},$$
$$C_3 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = -\pi + \arctan c\}.$$

2) *For any $c \in \mathbf{R}$, C_1 is convex, C_3 is concave, C_2 is neither.*

Proof. 1) is straightforward, 2) follows from the Hessian of λ_3 in (3):

$$\frac{\partial^2\lambda_3}{\partial\lambda_1^2} = \frac{2(\lambda_2 + c)(\lambda_2^2 + 1)(1 + c^2)}{(\lambda_1\lambda_2 - 1 + c\lambda_1 + c\lambda_2)^3},$$
$$\frac{\partial^2\lambda_3}{\partial\lambda_2^2} = \frac{2(\lambda_1 + c)(\lambda_1^2 + 1)(1 + c^2)}{(\lambda_1\lambda_2 - 1 + c\lambda_1 + c\lambda_2)^3},$$
$$\det(D^2\lambda_3) = \frac{4(c^2 + 1)^2(c\lambda_1 + c\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2 + \lambda_2^2 + \lambda_1^2)}{(\lambda_1\lambda_2 - 1 + c\lambda_1 + c\lambda_2)^5},$$

which implies e.g. that the point with $\lambda_1 = \lambda_2 = -c - 1/10$ for $c \geqslant 0$, $\lambda_1 = \lambda_2 = -c + 1/10$ for $c \leqslant 0$ is a saddle point on C_2. □

The corresponding Dirichlet sets F_c^i, $i = 1, 2, 3$, are given (via $F_{c,\Lambda}^i$) by

$$F_{c,\Lambda}^1 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant \pi + \arctan c\},$$
$$F_{c,\Lambda}^2 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant \arctan c\},$$
$$F_{c,\Lambda}^3 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant -\pi + \arctan c\},$$

and their duals by [11, Prop. 10.4.]:

$$\tilde{F}_{c,\Lambda}^1 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant -\pi - \arctan c\},$$
$$\tilde{F}_{c,\Lambda}^2 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant -\arctan c\},$$
$$\tilde{F}_{c,\Lambda}^3 = \{(\lambda_1, \lambda_2, \lambda_3):\ \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 \geqslant \pi - \arctan c\}.$$

A simple calculation gives

Lemma 3.2. *F_c^i is strictly, but not uniformly, elliptic.*

Indeed, the derivatives $\frac{1}{\lambda_i^2+1} > 0$ tend to 0 at infinity.

Remark 3.1. If we (artificially) impose the uniform ellipticity condition, we get a smooth solution. Indeed, if $\mathbf{F}_{-c}$ verifies this condition on u, the derivatives

$$\frac{1}{\lambda_1^2 + 1}, \frac{1}{\lambda_2^2 + 1}, \frac{1}{\lambda_3^2 + 1} = \frac{(\lambda_1\lambda_2 - 1 + c\lambda_1 + c\lambda_2)^2}{(1 + \lambda_2^2)(\lambda_1^2 + 1)(c^2 + 1)} \in \left[\frac{1}{M}, M\right]$$

for some ellipticity constant M, which implies that $u \in C^{1,1}$ and thus is smooth by [19].

We give now the principal technical result of this section which permits to construct in the next section a singular solution of SLE.

Proposition 3.1. *There exists a ball $B = B(0, \varepsilon)$ centered at the origin s.t.*

1) *The equation*

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \sigma_2(D^2u) = 1$$

has an analytic solution u_0 in B verifying

(i) $$u_0 = -\frac{y^4}{3} + 5y^2z^2 - x^4 + 7x^2z^2 - \frac{z^4}{3} + 2y^2z - 2zx^2 + \frac{y^2}{2} + \frac{x^2}{2} + O(r^5),$$

(ii) $$\lambda_1 = 1 + O(r), \qquad \lambda_2 = 1 + O(r), \qquad \lambda_3 = -\frac{x^2}{2} - \frac{3y^2}{2} - z^2 + O(r^3).$$

2) *The equation*

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 + c(\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3) = \sigma_2(D^2u) + c(\det(D^2u) - Tr(D^2u)) = 1$$

for $c \neq 0, -1$ *has an analytic solution* u_c *in* B *verifying*

(i) $$\begin{aligned} u_c = {} & \frac{-z^4}{(c+1)(c^2+2c+2)(c^2+c+1)(c^2+1)} + \frac{2z^2y^2(4c^5+4c^4+8c^3+5c^2+4c+4)}{(c+1)(c^2+c+1)} \\ & + \frac{2x^2z^2(4c^2+4c+3)}{(c+1)(c^2+c+1)} + \frac{y^4(c^2+1)(3c^4+2c^3+2c^2-4c-4)}{(c+1)(c^2+c+1)} - \frac{x^4(3c^2+2c+2)}{(c+1)(c^2+c+1)} \\ & - 2(c^2+1)zy^2 + 2zx^2 + \frac{(c^2+c+1)y^2}{2} + \frac{(c+1)x^2}{2} + O(r^5), \end{aligned}$$

(ii) $$\lambda_1 = c^2 + c + 1 + O(r), \qquad \lambda_2 = c + 1 + O(r),$$
$$\lambda_3 = -\frac{x^2(c^2+1)(c^2+2c+2) + 3y^2c^2(c^2+1)(c^2+2c+2) + 3z^2}{2(c+1)(c^2+c+1)(c^2+1)(c^2+2c+2)} + O(r^3).$$

3) *The equation* $(c = -1)$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 + c(\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3) = \sigma_2(D^2u) - \det(D^2u) + Tr(D^2u) = 1$$

has an analytic solution u_{-1} *in* B *verifying*

(i) $$u_{-1} = 48y^2x^2 - 12y^2z^2 - \frac{119x^4}{2} + 93x^2z^2 + \frac{z^4}{2} + 2y^2z - 9x^2z - \frac{y^2}{6} + x^2 + O(r^5),$$

(ii) $$\lambda_1 = 2 + O(r), \qquad \lambda_2 = 6y^2 + 6x^2 + \frac{3z^2}{2} + O(r^3), \qquad \lambda_3 = -\frac{1}{3} + O(r),$$

for $r = |(x, y, z)|$.

Proof. Let us note that

$$v_0 := -\frac{y^4}{3} + 5y^2z^2 - x^4 + 7x^2z^2 - \frac{z^4}{3} + 2y^2z - 2zx^2 + \frac{y^2}{2} + \frac{x^2}{2},$$
$$v_{-1} := 48y^2x^2 - 12y^2z^2 - \frac{119x^4}{2} + 93x^2z^2 + \frac{z^4}{2} + 2y^2z - 9x^2z - \frac{y^2}{6} + x^2,$$

and

$$\begin{aligned} v_c = {} & \frac{-z^4}{(c+1)(c^2+2c+2)(c^2+c+1)(c^2+1)} + \frac{2z^2y^2(4c^5+4c^4+8c^3+5c^2+4c+4)}{(c+1)(c^2+c+1)} \\ & + \frac{2x^2z^2(4c^2+4c+3)}{(c+1)(c^2+c+1)} + \frac{y^4(c^2+1)(3c^4+2c^3+2c^2-4c-4)}{(c+1)(c^2+c+1)} - \frac{x^4(3c^2+2c+2)}{(c+1)(c^2+c+1)} \\ & - 2(c^2+1)zy^2 + 2zx^2 + \frac{(c^2+c+1)y^2}{2} + \frac{(c+1)x^2}{2} \end{aligned}$$

verify their respective equations up to the second-order, i.e.

$$\sigma_2(D^2v_0) - 1 = O(r^3),$$
$$\sigma_2(D^2v_{-1}) - \det(D^2v_{-1}) + Tr(D^2v_{-1}) - 1 = O(r^3),$$
$$\sigma_2(D^2v_c) + c(\det(D^2u_c) - Tr(D^2u_c)) - 1 = O(r^3),$$

which can be proven by a brute force (e.g., MAPLE) calculation (e.g.,

$$\sigma_2\big(D^2v_0\big)-1=4\big(-10y^4-32y^2x^2-50y^2z^2-42x^4-130x^2z^2+11z^4-36y^2z+4x^2z+4z^3\big)\big).$$

To prove 1) one considers the following Cauchy problem for the equation $F_0=\sigma_2(D^2u)-1=0$:

$$u|_{z=0}=v_0|_{z=0}=-\frac{y^4}{3}-x^4+\frac{y^2}{2}+\frac{x^2}{2},$$
$$\left(\frac{\partial u}{\partial z}\right)_{z=0}=\left(\frac{\partial v_0}{\partial z}\right)_{z=0}=2y^2-2x^2.$$

Since the equation is elliptic, we get by the Cauchy–Kowalevskaya theorem a unique local analytic solution u_0 which should coincide with v_0 within to fourth-order.

The same argument is valid for

$$F_c=\sigma_2\big(D^2u\big)+c\big(\det\big(D^2u\big)-Tr\big(D^2u\big)\big)-1=0$$

and the Cauchy problem

$$u|_{z=0}=v_c|_{z=0}$$
$$=\frac{y^4(c^2+1)(3c^4+2c^3+2c^2-4c-4)}{3(c+1)(c^2+c+1)}-\frac{x^4(3c^2+2c+2)}{3(c+1)(c^2+c+1)}+\frac{y^2((c^2+c+1)}{2}+\frac{x^2(c+1)}{2},$$
$$\left(\frac{\partial u}{\partial z}\right)_{z=0}=\left(\frac{\partial v_c}{\partial z}\right)_{z=0}=-2\big(c^2+1\big)y^2+2x^2.$$

The claim on the eigenvalues follows directly from the formulas

$$\det\big(D^2v_0\big)=-\frac{x^2}{2}-\frac{3y^2}{2}-z^2+O\big(r^3\big),$$
$$\det\big(D^2v_c\big)=-\frac{x^2}{2}-\frac{3y^2c^2}{2}-\frac{3z^2}{(c^2+1)(c^2+2c+2)}+O\big(r^3\big)$$

which are straightforward (e.g.

$$\begin{aligned}\det\big(D^2v_0\big)={}&120y^4x^2-140y^4z^2+168y^2x^4+1440y^2x^2z^2-994y^2z^4-420x^4z^2-1350x^2z^4-140z^6\\&+40y^4z+192y^2x^2z-136y^2z^3-168x^4z-120x^2z^3-16z^5-10y^4+20y^2x^2+28y^2z^2\\&-42x^4+28x^2z^2-8z^4-24y^2z+40x^2z-\frac{3y^2}{2}-\frac{x^2}{2}-z^2).\end{aligned}$$

The argument works for 3) as well. □

4. Legendre Transform

Let us recall some essential properties see, e.g. §1.6 in [7], of the Legendre Transform (for simplicity of notation we consider here only the case of 3 dimensions used below). Let f be a C^2-function defined in a domain $D\subset\mathbf{R}^3$ s.t. its gradient map $\nabla f:D\longrightarrow\mathbf{R}^3$ maps bijectively D onto a domain G. Let $g=(\nabla f)^{-1}=(P,Q,R):G\longrightarrow D$ be the map inverse to the gradient. Then the Legendre Transform $\tilde f:G\longrightarrow\mathbf{R}$ is given by

$$\tilde f(u,v,w):=uP(u,v,w)+vQ(u,v,w)+wR(u,v,w)-f\big(g(u,v,w)\big).\tag{4}$$

Suppose also that $\det(D^2f)\neq 0$ except for a point $a\in D$ with $b=(\nabla f)(a)$. Then $(D^2\tilde f)=(D^2f)^{-1}$ on $G-\{b\}$.

We want then to apply the Legendre Transform to the solutions u_c on a small ball centered at zero. We need thus to verify that ∇u_c is injective. One finds

$$\nabla u_c=\big[U(x,y,z)x+(c+1)x,\,V(x,y,z)y+\big(c^2+c+1\big)y,\,-4z^3m_c+x^2W_1(z)+y^2W_2(z)\big],$$

where $U(x,y,z),V(x,y,z)\in\mathbf{R}\{\{x,y,z\}\}$, $W_1(z),W_2(z)\in\mathbf{R}\{\{z\}\}$, $U(0,0,0)=V(0,0,0)=0$, $m_c:=1/((c+1)(c^2+2c+2)(c^2+c+1)(c^2+1))>0$ for $c\neq 0,-1$, $m_{-1}:=1/2$, $m_0:=1/3$.

To prove the injectivity of the gradient map we use Theorem 1.1 of [9] which says in our situation that the degree of ∇u_c equals $\dim_{\mathbf{R}} Q(\nabla u_c) - 2\dim_{\mathbf{R}} I$ where I is an ideal of $Q(\nabla u_c)$ which is maximal with respect to the property $I^2 = 0$, the ring $Q(\nabla u_c)$ being defined as

$$Q(\nabla u_c) := \mathbf{R}\big\{\{x, y, z\}\big\}/(\partial u_c/\partial x, \partial u_c/\partial y, \partial u_c/\partial z).$$

Therefore, to prove the injectivity it is sufficient to prove

Lemma 4.1. *The ring $Q(\nabla u_c)$ is isomorphic to $\mathbf{R}[h]/(h^3)$.*

Proof. For simplicity of notation we consider only the case $c = 0$, the general case being completely similar. Then

$$\nabla u_0 = \left[U(x, y, z)x + x, V(x, y, z)y + y, -\frac{4z^3}{3} + x^2 W_1(z) + y^2 W_2(z)\right]$$

and

$$Q(\nabla u_c) = \mathbf{R}\big\{\{x, y, z\}\big\}/\left(U(x, y, z)x + x, V(x, y, z)y + y, -\frac{4z^3}{3} + x^2 W_1(z) + y^2 W_2(z)\right).$$

If one sets $p := x + U(x, y, z)x$, $q := y + V(x, y, z)y$ one sees that $Q(\nabla u_0)$ is isomorphic to $\mathbf{R}\{\{p, q, z\}\}/(p, q, -\frac{4z^3}{3} + p^2 W_1'(p, q, z) + q^2 W_2'(p, q, z))$ and thus to $\mathbf{R}\{\{z\}\}/(-\frac{4z^3}{3})$ which implies the result. □

We can now prove our main result.

Theorem 4.1. *Let $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, and let*

$$\mathbf{F}_\theta(u) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 - \theta = 0.$$

Then for some ball B_ε centered at the origin there exists an analytic function f_θ on ∂B_ε s.t. the unique (Harvey–Lawson) solution u_θ of the Dirichlet problem

$$\begin{cases} \mathbf{F}_\theta(u) = 0 & \text{in } B_\varepsilon, \\ u = f_\theta & \text{on } \partial B_\varepsilon \end{cases}$$

verifies:

(i) $u_\theta \in C^{1,1/3}$;
(ii) $u_\theta \notin C^{1,\delta}$ *for* $\forall\delta > 1/3$.

Proof. We can apply the Legendre Transform to u_c with $c = \cot(\theta)$ for $\theta \neq 0$, and to u_0 for $\theta = 0$ thanks to the injectivity of ∇u_c. Set $u_\theta := \tilde{u}_c$ in this situation. Since u_c with $c \neq 0$ verifies the equation

$$\sigma_2\big(D^2 u\big) + c\big(\det\big(D^2 u\big) - Tr\big(D^2 u\big)\big) - 1 = 0,$$

its Legendre Transform $\tilde{u}_c$ verifies

$$c\big(\sigma_2\big(D^2 u\big) - 1\big) + \det\big(D^2 u\big) - Tr\big(D^2 u\big) = 0,$$

the signature of $(\lambda_1(\tilde{u}_c), \lambda_2(\tilde{u}_c), \lambda_3(\tilde{u}_c))$ being $(+, +, -)$ for $c \geqslant -1$ and $(-, -, +)$ for $c < -1$ which implies that $\tilde{u}_c$ lies on the middle branch of this equation. The same is true for $\tilde{u}_0$ and the equations

$$\sigma_2\big(D^2 u\big) - 1 = 0, \qquad \det\big(D^2 u\big) - Tr\big(D^2 u\big) = 0.$$

The function $\tilde{u}_c$ is analytic outside zero and belongs to $C^{1,1/3}(B_\varepsilon)$ which proves (i). Indeed, it is sufficient to prove the boundness of the $C^{1,1/3}$-norm of $\tilde{u}_c$ on a small ball. Due to the elementary relation $3|f|^2|\nabla f| = |\nabla(f^3)|$ which holds for C^1-functions, it will be sufficient to prove the boundness of the product $|D\tilde{u}_c|^2|D^2\tilde{u}_c|$. To prove this last assertion we note that

$$\frac{\partial \tilde{u}_c}{\partial u} = \frac{\partial}{\partial u}\bigl(uP + vQ + wR - u_c(P,Q,R)\bigr) = P,$$
$$\frac{\partial \tilde{u}_c}{\partial v} = \frac{\partial}{\partial v}\bigl(uP + vQ + wR - u_c(P,Q,R)\bigr) = Q,$$
$$\frac{\partial \tilde{u}_c}{\partial w} = \frac{\partial}{\partial w}\bigl(uP + vQ + wR - u_c(P,Q,R)\bigr) = R,$$

since $\nabla u_c = (u, v, w)$. Thus, $|D\tilde{u}_c|^2 = P^2 + Q^2 + R^2$. On the other hand, since the Hessian $D^2(\tilde{u}_c) = D^2(u_c)^{-1}$, the matrix $\det(D^2(u_c))D^2(\tilde{u}_c)$ has bounded entries and thus $\|\tilde{u}_c\|_{C^2} \leqslant \frac{C}{|\det(D^2(u_c))|}$ for an absolute constant C (e.g. for $C = 10$ in the case $c = 0$). Since by Proposition 3.1 $|\det(D^2(u_c))| \geqslant C'(c)(P^2 + Q^2 + R^2)$ for an absolute constant $C'(c)$ (e.g., $C'(0) = 1/2 - \varepsilon$) we get the boundness of the product. We need then to prove that $\tilde{u}_c$ is a Harvey–Lawson solution of the corresponding Dirichlet problem.

This is implied by the following form of the Alexandrov maximum principle [1]:

Proposition 4.1. *Let F be a Dirichlet domain, and let $u = v + w$ where v is of $\tilde{F}$-type. If*

$$u \in C^2\bigl(B - \{0\}\bigr) \cap C^1(B)$$

and $D^2 u$ is non-negatively defined on $B - \{0\}$ then

$$\sup_B u \leqslant \sup_{\partial B} u.$$

(ii) Let $u = 0$, $v = 0$, $w \neq 0$. Then

$$\lambda_3(u_\theta) = -2m_c^{-1} w^{-2/3}/3 + o\bigl(w^{-2/3}\bigr)$$

which contradicts the condition $u_\theta \in C^{1,\delta}$ for $\delta > 1/3$. □

Remark 4.1. Let us consider the Special Lagrangian submanifold $L_{u,c} \subset \mathbf{C}^3$ corresponding to our singular solution u_c i.e. the graph of the map

$$i\nabla u_c : B \longrightarrow i\mathbf{R}^3.$$

It is easy to show that it is smooth, and the singularity of u_c implies only that the projection $L_{u,c} \longrightarrow B$ is singular (map between smooth manifolds).

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