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Equivalence of viscosity and weak solutions for the $p(x)$-Laplacian

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Abstract

We consider different notions of solutions to the $p(x)$-Laplace equation

$$-\operatorname{div}\left(\left|Du(x)\right|^{p(x)-2}Du(x)\right)=0$$

with $1 < p(x) < \infty$. We show by proving a comparison principle that viscosity supersolutions and $p(x)$-superharmonic functions of nonlinear potential theory coincide. This implies that weak and viscosity solutions are the same class of functions, and that viscosity solutions to Dirichlet problems are unique. As an application, we prove a Radó type removability theorem.

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1. Introduction

During the last fifteen years, variational problems and partial differential equations with various types of non-standard growth conditions have become increasingly popular. This is partly due to their frequent appearance in applications such as the modeling of electrorheological fluids [2,25] and image processing [20], but these problems are very interesting from a purely mathematical point of view as well.

In this paper, we focus on a particular example, the $p(x)$-Laplace equation

$$-\Delta_{p(x)}u(x) := -\operatorname{div}\left(\left|Du(x)\right|^{p(x)-2}Du(x)\right)=0 \tag{1.1}$$

with $1 < p(x) < \infty$. This is a model case of a problem exhibiting the so-called $p(x)$-growth, which was first considered by Zhikov in [27]. Our interest is directed at the very notion of a solution to (1.1). Since this equation is of divergence form, the most natural choice is to use the distributional weak solutions, whose definition is based on integration by parts. However, if the variable exponent $x \mapsto p(x)$ is assumed to be continuously differentiable, then

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also the notion of viscosity solutions, defined by means of pointwise touching test functions, is applicable. Our objective is to prove that weak and viscosity solutions to the $p(x)$-Laplace equation coincide. The proof also implies the uniqueness of viscosity solutions of the Dirichlet problem. For a constant p, similar results were proved by Juutinen, Lindqvist, and Manfredi in [17], see also [5].

The modern theory of viscosity solutions, introduced by Crandall and Lions in the eighties, has turned out to be indispensable. It provides a notion of generalized solutions applicable to fully nonlinear equations, and crucial tools for results related to existence, stability, and uniqueness for first and second order partial differential equations, see for example Crandall, Ishii, and Lions [8], Crandall [7], and Jensen [15]. Viscosity theory has also been used in stochastic control problems, and more recently in stochastic games, see for example [24]. The variable exponent viscosity theory is also useful. Indeed, as an application, we prove a Radó type removability theorem: if a function $u \in C^1(\Omega)$ is a solution to (1.1) outside its set of zeroes $\{x\colon u(x) = 0\}$, then it is a solution in the whole domain Ω. A similar result also holds for the zero set of the gradient. We do not know how to prove this result without using both the concept of a weak solution and that of a viscosity solution.

To prove our main result, we show that viscosity supersolutions of the $p(x)$-Laplace equation are the same class of functions as $p(x)$-superharmonic functions, defined as lower semicontinuous functions obeying the comparison principle with respect to weak solutions. The equivalence for the solutions follows from this fact at once. A simple application of the comparison principle for weak solutions shows that $p(x)$-superharmonic functions are viscosity supersolutions. The reverse implication, however, requires considerably more work. To show that a viscosity supersolution obeys the comparison principle with respect to weak solutions, we first show that weak solutions of (1.1) can be approximated by the weak solutions of

$$-\Delta_{p(x)} u = -\varepsilon \tag{1.2}$$

and then prove a comparison principle between viscosity supersolutions and weak solutions of (1.2). The comparison principle for viscosity sub- and supersolutions can be reduced to this result as well.

Although the outline of our proof is largely the same as that of [17] for the constant p case, there are several significant differences in the details. Perhaps the most important of them is the fact that the $p(x)$-Laplace equation is not translation invariant. At first thought, this property may not seem that consequential, but one should bear in mind that we are dealing with generalized solutions, not with classical solutions. The core of our argument, the proof of the comparison principle for viscosity solutions, is based on the maximum principle for semicontinuous functions. Applying this principle is not entirely trivial even for such simple equations as

$$-\operatorname{div}\big(a(x)Du\big) = 0, \tag{1.3}$$

with a smooth and strictly positive coefficient $a(x)$. In the case of the $p(x)$-Laplacian, the proof is quite delicate. We need to carefully exploit the information coming from the maximum principle for semicontinuous functions, and properties such as the local Lipschitz continuity of the matrix square root as well as the regularity of weak solutions of (1.2) play an essential role in the proof.

In addition, we have to take into account the strong singularity of the equation at the points where the gradient vanishes and $p(x) < 2$. Further, since $1 < p(x) < \infty$, the equations we encounter can be singular in some parts of the domain and degenerate in others, and we have to find a way to fit together estimates obtained in the separate cases. For a constant p, this problem never occurs. Finally, if we carefully compute $\Delta_{p(x)}u$, the result is the expression

$$-\Delta_{p(x)}u(x) = -|Du|^{p(x)-2}\big(\Delta u + \big(p(x)-2\big)\Delta_\infty u\big) - |Du|^{p(x)-2} Dp(x) \cdot Du \log|Du|, \tag{1.4}$$

where

$$\Delta_\infty u := |Du|^{-2} D^2u\, Du \cdot Du$$

is the normalized ∞-Laplacian. Obviously, the first order term involving $\log|Du|$ does not appear if $p(x)$ is constant.

The $p(x)$-Laplacian (1.1) is not only interesting in its own right but also provides a useful test case for generalizing the viscosity techniques to a wider class of equations as indicated by (1.3). One important observation is that our proof uses heavily the well established theory of weak solutions. More precisely, we repeatedly exploit the existence, uniqueness and regularity of the weak solutions to (1.1). In particular, we use the fact that weak solutions can be approximated by the weak solutions of (1.2). To emphasize this point, let us mention another variable exponent version of the p-Laplace equation, given by

$$-\Delta^N_{p(x)} u(x) := -\Delta u(x) - \big(p(x)-2\big)\Delta_\infty u(x) = 0. \tag{1.5}$$

This equation, set forth recently in [3], certainly looks simpler than (1.4), and one can indeed quite easily prove a comparison principle for viscosity subsolutions and strict supersolutions. However, owing to the incompleteness of the theory of weak solutions of (1.5), the full comparison principle and the equivalence of weak and viscosity solutions remain open.

2. The spaces $L^{p(x)}$ and $W^{1,p(x)}$

In this section, we discuss the variable exponent Lebesgue and Sobolev spaces. These spaces provide the functional analysis framework for weak solutions. Most of the results below are from [19].

Let $p : \mathbb{R}^n \to [1,\infty)$, called a variable exponent, be in $C^1(\mathbb{R}^n)$ and let Ω be a bounded open set in $\mathbb{R}^n$. We denote

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x),$$

and assume that

$$1 < p^- \leqslant p^+ < \infty. \tag{2.1}$$

Observe that our arguments are local, so it would suffice to assume that (2.1) holds in compact subsets of Ω. However, for simplicity, we use the stronger assumption.

The *variable exponent Lebesgue space* $L^{p(x)}(\Omega)$ consists of all measurable functions u defined on Ω for which the $p(x)$-modular

$$\varrho_{p(x)}(u) = \int_\Omega \big|u(x)\big|^{p(x)}\,\mathrm{d}x$$

is finite. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

Equipped with this norm $L^{p(x)}(\Omega)$ is a Banach space. If $p(x)$ is constant, $L^{p(x)}(\Omega)$ reduces to the standard Lebesgue space.

In estimates, it is often necessary to switch between the norm and the modular. This is accomplished by the coarse but useful inequalities

$$\begin{aligned}\min\big\{\|u\|^{p^+}_{L^{p(x)}(\Omega)}, \|u\|^{p^-}_{L^{p(x)}(\Omega)}\big\} &\leqslant \varrho_{p(x)}(u)\\ &\leqslant \max\big\{\|u\|^{p^+}_{L^{p(x)}(\Omega)}, \|u\|^{p^-}_{L^{p(x)}(\Omega)}\big\},\end{aligned} \tag{2.2}$$

which follow from the definition of the norm in a straightforward manner. Note that these inequalities imply the equivalence of convergence in norm and in modular. More specifically,

$$\|u-u_i\|_{L^{p(x)}(\Omega)} \to 0 \quad \text{if and only if} \quad \varrho_{p(x)}(u-u_i) \to 0$$

as $i \to \infty$.

A version of Hölder's inequality,

$$\int_\Omega fg\,\mathrm{d}x \leqslant C\|f\|_{L^{p(x)}(\Omega)}\|g\|_{L^{p'(x)}(\Omega)}, \tag{2.3}$$

holds for functions $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$, where the conjugate exponent $p'(x)$ of $p(x)$ is defined pointwise. Further, if $1 < p^- \leqslant p^+ < \infty$, the dual of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, and the space $L^{p(x)}(\Omega)$ is reflexive.

The *variable exponent Sobolev space* $W^{1,p(x)}(\Omega)$ consists of functions $u \in L^{p(x)}(\Omega)$ whose weak gradient Du exists and belongs to $L^{p(x)}(\Omega)$. The space $W^{1,p(x)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|Du\|_{L^{p(x)}(\Omega)}.$$

The local Sobolev space $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ consists of functions u that belong to $W^{1,p(x)}_{\mathrm{loc}}(\Omega')$ for all open sets Ω' compactly contained in Ω.

The density of smooth functions in $W^{1,p(x)}(\Omega)$ turns out to be a surprisingly nontrivial matter. However, our assumption $p(x) \in C^1$, or even the weaker log-Hölder continuity, implies that smooth functions are dense in the Sobolev space $W^{1,p(x)}(\Omega)$, see [9,26]. Due to the Hölder inequality (2.3), density allows us to pass from smooth test functions to Sobolev test functions in the definition of weak solutions by the usual approximation argument.

The Sobolev space with zero boundary values, $W^{1,p(x)}_0(\Omega)$, is defined as the completion of $C^\infty_0(\Omega)$ in the norm of $W^{1,p(x)}(\Omega)$. The following variable exponent Sobolev–Poincaré inequality for functions with zero boundary values was first proved by Edmunds and Rákosník in [10].

Theorem 2.1. *For every $u \in W^{1,p(x)}_0(\Omega)$, the inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \leqslant C \operatorname{diam}(\Omega) \|Du\|_{L^{p(x)}(\Omega)} \tag{2.4}$$

holds with the constant C depending only on the dimension n and p.

3. Notions of solutions

In this section, we discuss the notions of weak solutions, $p(x)$-superharmonic functions and viscosity solutions to the equation

$$-\Delta_{p(x)} u(x) = -\operatorname{div}\left(\left|Du(x)\right|^{p(x)-2} Du(x)\right) = 0. \tag{3.1}$$

Definition 3.1. A function $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ is a weak supersolution to (3.1) in Ω if

$$\int_\Omega \left|Du(x)\right|^{p(x)-2} Du(x) \cdot D\varphi(x)\,\mathrm{d}x \geqslant 0 \tag{3.2}$$

for every nonnegative test function $\varphi \in C^\infty_0(\Omega)$. For subsolutions, the inequality in (3.2) is reversed, and a function u is a weak solution if it is both a super- and a subsolution, which means that we have equality in (3.2) for all $\varphi \in C^\infty_0(\Omega)$.

If $u \in W^{1,p(x)}(\Omega)$, then by the usual approximation argument, we may employ test functions belonging to $W^{1,p(x)}_0(\Omega)$. Note also that u is a subsolution if and only if $-u$ is a supersolution.

Since (3.1) is the Euler–Lagrange equation of the functional

$$v \mapsto \int_\Omega \frac{1}{p(x)} |Dv|^{p(x)}\,\mathrm{d}x,$$

the existence of a weak solution $u \in W^{1,p(x)}(\Omega)$ such that $u - g \in W^{1,p(x)}_0(\Omega)$ for a given $g \in W^{1,p(x)}(\Omega)$ readily follows by the direct method of the calculus of variations. Moreover, by regularity theory [1,4,11], there always exists a locally Hölder continuous representative of a weak solution.

Assume for a moment that $p(x)$ is radial, and that $p^+ < n$. In such a case, we may modify a well-known example, the fundamental solution of the p-Laplacian. Indeed, consider the function

$$v(x) = \int_{|x|}^{1} \left(p(r) r^{n-1}\right) r^{-1/(p(r)-1)}\,\mathrm{d}r$$

in the unit ball $B(0,1)$ of $\mathbb{R}^n$. Then v is a solution in $B(0,1)\setminus\{0\}$, but not a weak supersolution in the whole ball $B(0,1)$. See [12, Section 6] for the details. A computation shows that

$$\int_{B(0,\rho)} |Dv|^{p(x)}\,\mathrm{d}x = \infty$$

for any $\rho < 1$. To include functions like v in our discussion, we use the following class of $p(x)$-superharmonic functions.

Definition 3.2. A function $u : \Omega \to (-\infty, \infty]$ is $p(x)$-*superharmonic*, if

(1) u is lower semicontinuous,
(2) u is finite almost everywhere, and
(3) the comparison principle holds: if h is a weak solution to (3.1) in $D \Subset \Omega$, continuous in $\overline{D}$, and

$$u \geqslant h \quad \text{on } \partial D,$$

then

$$u \geqslant h \quad \text{in } D.$$

A function $u : \Omega \to [-\infty, \infty)$ is $p(x)$-*subharmonic*, if $-u$ is $p(x)$-superharmonic.

In the case of the Laplace equation, that is, $p(x) \equiv 2$, this potential theoretic definition of superharmonic functions goes back to F. Riesz. For constant values $p \neq 2$, p-superharmonic functions and, in particular, their relationship to the weak supersolutions were studied by Lindqvist in [21].

The reader should notice that above we have required u to be finite almost everywhere, whereas in the case $p = \text{constant}$, it is normally only assumed that a p-superharmonic function is not identically $+\infty$. This stronger condition is needed for the characterization of $p(x)$-superharmonic functions as pointwise increasing limits of weak supersolutions to (3.1), see [12, Section 6]. Whether the condition that u is not identically $+\infty$ would suffice as well is not known, and this stems from the fact that the constants in the Harnack estimates in the variable exponent case depend on the solution and cannot be taken to be universal as in the case $p = \text{constant}$, see [12, Example 3.10].

We next review briefly some relevant facts about $p(x)$-superharmonic functions. We shall use the lower semicontinuous regularization

$$u^*(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y) = \lim_{R\to 0} \operatorname*{ess\,inf}_{B(x,R)} u. \tag{3.3}$$

First, every weak supersolution has a lower semicontinuous representative which is $p(x)$-superharmonic.

Theorem 3.3. *Let u be a weak supersolution in Ω. Then $u = u^*$ almost everywhere and u^* is $p(x)$-superharmonic.*

This theorem follows from [12, Theorem 6.1] and [13, Theorem 4.1]. For the reverse direction, we have (see [12, Corollary 6.6])

Theorem 3.4. *A locally bounded $p(x)$-superharmonic function is a weak supersolution.*

With these results in hand, we may conclude that being a weak solution is equivalent to being both $p(x)$-super- and $p(x)$-subharmonic. Indeed, any function with the latter properties is continuous and hence locally bounded. Then Theorem 3.4 implies that the function belongs to the right Sobolev space, and verifying the weak formulation is easy. For the converse, it suffices to note that the comparison principle for the continuous representative of a weak solution follows from Theorem 3.3.

Next we define viscosity solutions of the $p(x)$-Laplace equation. To accomplish this, we need to evaluate the operator $\Delta_{p(x)}$ on C^2 functions. Carrying out the differentiations, we see that

$$\Delta_{p(x)}\varphi(x) = |D\varphi|^{p(x)-2}\big(\Delta\varphi + Dp\cdot D\varphi \log|D\varphi| + \big(p(x)-2\big)\Delta_\infty\varphi(x)\big)$$

for functions $\varphi \in C^2(\Omega)$, where

$$\Delta_\infty \varphi(x) = D^2\varphi(x)\frac{D\varphi(x)}{|D\varphi(x)|} \cdot \frac{D\varphi(x)}{|D\varphi(x)|}$$

is the normalized ∞-Laplacian.

In order to use the standard theory of viscosity solutions, $\Delta_{p(x)}\varphi(x)$ should be continuous in x, $D\varphi$, and $D^2\varphi$. Since $Dp(x)$ is explicitly involved, it is natural to assume that $p(x) \in C^1$. However, this still leaves the problem that at the points were $p(x) < 2$ and $D\varphi(x) = 0$, the expression $\Delta_{p(x)}\varphi(x)$ is not well defined. As in [17], it will turn out that we can ignore the test functions whose gradient vanishes at the contact point.

Definition 3.5. A function $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution to (3.1), if

(1) u is lower semicontinuous.
(2) u is finite almost everywhere.
(3) If $\varphi \in C^2(\Omega)$ is such that $u(x_0) = \varphi(x_0)$, $u(x) > \varphi(x)$ for $x \neq x_0$, and $D\varphi(x_0) \neq 0$, it holds that

$$-\Delta_{p(x)}\varphi(x_0) \geqslant 0.$$

A function $u : \Omega \to [-\infty, \infty)$ is a viscosity subsolution to (3.1) if it is upper semicontinuous, finite a.e., and (3) holds with the inequalities reversed.

Finally, a function is a viscosity solution if it is both a viscosity super- and subsolution.

We often refer to the third condition above by saying that φ touches u at x_0 from below. The definition is symmetric in the same way as before: u is a viscosity subsolution if and only if $-u$ is a viscosity supersolution. Observe that nothing is required from u at the points in which it is not finite.

One might wonder if omitting entirely the test functions whose gradient vanishes at the point of touching could allow for "false" viscosity solutions for the equation. Our results ensure that this is not the case. Indeed, we show that the requirements in Definition 3.5 are stringent enough for a comparison principle to hold between viscosity sub- and supersolutions, and, moreover, that the definition is equivalent with the definition of a weak solution. Finally, we want to emphasize that Definition 3.5 is tailor-made for the equation $-\Delta_{p(x)}v(x) = 0$, and it does *not* work as such for example in the case of a non-homogeneous $p(x)$-Laplace equation $-\Delta_{p(x)}v(x) = f(x)$. The reason is that for a constant function $u \equiv c$ all test functions have zero gradient at the contact point, and thus u automatically becomes a solution to whatever equation if such test functions are ignored in the definition.

4. Equivalence of weak solutions and viscosity solutions

We turn next to the equivalence between weak and viscosity solutions to (3.1). This follows from the fact that viscosity supersolutions and $p(x)$-superharmonic functions are the same class of functions. This is our main result.

Theorem 4.1. *A function v is a viscosity supersolution to* (3.1) *if and only if it is $p(x)$-superharmonic.*

As an immediate corollary we have

Corollary 4.2. *A function u is a weak solution of* (3.1) *if and only if it is a viscosity solution of* (3.1).

Let us now start with the *proof of Theorem* 4.1. Proving the fact that a $p(x)$-superharmonic function is a viscosity supersolution is straightforward, cf. for example [22]: Suppose first that v is $p(x)$-superharmonic. To see that v is a viscosity supersolution, assuming the opposite we find a function $\varphi \in C^2(\Omega)$ such that $v(x_0) = \varphi(x_0)$, $v(x) > \varphi(x)$ for all $x \neq x_0$, $D\varphi(x_0) \neq 0$, and

$$-\Delta_{p(x)}\varphi(x_0) < 0.$$

By continuity, there is a radius r such that $D\varphi(x) \neq 0$ and

$$-\Delta_{p(x)}\varphi(x) < 0$$

for all $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r} \big(v(x) - \varphi(x)\big) > 0,$$

and

$$\tilde{\varphi} = \varphi + m.$$

Then $\tilde{\varphi}$ is a weak subsolution in $B(x_0, r)$, and $\tilde{\varphi} \leqslant v$ on $\partial B(x_0, r)$. Thus $\tilde{\varphi} \leqslant v$ in $B(x_0, r)$ by the comparison principle for weak sub- and supersolutions, Lemma 5.1 below, but

$$\tilde{\varphi}(x_0) = \varphi(x_0) + m > v(x_0),$$

which is a contradiction.

The proof of the reverse implication is much more involved. Let us suppose that v is a viscosity supersolution. In order to prove that v is $p(x)$-superharmonic, we need to show that v obeys the comparison principle with respect to weak solutions of (3.1). To this end, let $D \Subset \Omega$ and let $h \in C(\overline{D})$ be a weak solution of (3.1) such that $v \geqslant h$ on ∂D. Owing to the lower semicontinuity of v, for every $\delta > 0$ there is a smooth domain $D' \Subset D$ such that $h \leqslant v + \delta$ in $D \setminus D'$. Observe that $h \in W^{1,p(x)}(D')$, whereas we do not have $h \in W^{1,p(x)}(D)$ in general.

For $\varepsilon > 0$, let h_ε be the unique weak solution to

$$-\Delta_{p(x)} h_\varepsilon = -\varepsilon, \quad \varepsilon > 0$$

such that $h_\varepsilon - h \in W_0^{1,p(x)}(D')$. Then h_ε is locally Lipschitz in D', see [1,6], $v + \delta \geqslant h_\varepsilon$ on $\partial D'$ because of the smoothness of D', and it follows from Lemma 5.2 below that $h_\varepsilon \to h$ locally uniformly in D' as $\varepsilon \to 0$. Hence, in order to prove that $v \geqslant h$ in D, which is our goal, it suffices to prove that $v + \delta \geqslant h_\varepsilon$ in D' and then let first $\varepsilon \to 0$ and then $\delta \to 0$. As $v + \delta$ is also a viscosity supersolution of (3.1), the proof of Theorem 4.1 thus reduces to

Proposition 4.3. *Let $D' \Subset \Omega$, and suppose that v is a viscosity supersolution to the $p(x)$-Laplace equation in D', and let $\varepsilon > 0$. Assume further that h_ε is a locally Lipschitz continuous weak solution of*

$$-\Delta_{p(x)} h_\varepsilon = -\varepsilon \tag{4.1}$$

in D' such that

$$v \geqslant h_\varepsilon \quad \text{on } \partial D'.$$

Then

$$v \geqslant h_\varepsilon \quad \text{in } D'.$$

A similar statement holds for viscosity subsolutions u, and locally Lipschitz continuous weak solutions $\tilde{h}_\varepsilon$ of

$$-\Delta_{p(x)} \tilde{h}_\varepsilon = \varepsilon. \tag{4.2}$$

In other words, if

$$u \leqslant \tilde{h}_\varepsilon \quad \text{on } \partial D',$$

then

$$u \leqslant \tilde{h}_\varepsilon \quad \text{in } D'.$$

The proof for Proposition 4.3 turns out to be both long and technically complicated. It requires three lemmas for weak solutions that we prove in Section 5 below. The proof itself is given in Section 6.

We close this section by showing that Proposition 4.3 in fact implies the comparison principle for viscosity sub- and supersolutions of the $p(x)$-Laplace equation.

Theorem 4.4. *Let Ω be a bounded domain. Assume that u is a viscosity subsolution, and v a viscosity supersolution such that*

$$\limsup_{x\to z} u(x) \leqslant \liminf_{x\to z} v(z) \tag{4.3}$$

for all $z \in \partial\Omega$, where both sides are not simultaneously $-\infty$ or ∞. Then

$$u \leqslant v \quad \text{in } \Omega.$$

Corollary 4.5. *Let Ω be a bounded domain, and $f : \partial\Omega \to \mathbb{R}$ be a continuous function. If u and v are viscosity solutions of* (3.1) *in Ω such that*

$$\lim_{x\to x_0} u(x) = f(x_0) \quad \text{and} \quad \lim_{x\to x_0} v(x) = f(x_0)$$

for all $x_0 \in \partial\Omega$, then $u = v$.

Proof of Theorem 4.4. Owing to (4.3), for any $\delta > 0$ there is a smooth subdomain $D \Subset \Omega$ such that

$$u < v + \delta$$

in $\Omega \setminus D$. By semicontinuity, there is a smooth function φ such that

$$u < \varphi < v + \delta \quad \text{on } \partial D.$$

Let h be the unique weak solution to (3.1) in D with boundary values φ. Then

$$u < h < v + \delta$$

on ∂D, and h is locally Lipschitz continuous in D by the local $C^{1,\alpha}$ regularity of $p(x)$-harmonic functions, see [1,6].

For $\varepsilon > 0$, let h_ε be the unique weak solution to (4.2) such that $h_\varepsilon - h \in W_0^{1,p(x)}(D)$. Then h_ε is locally Lipschitz in D and it follows from Proposition 4.3 that $u \leqslant h_\varepsilon$ in D. In view of Lemma 5.2, this shows that $u \leqslant h$ in D, and a symmetric argument, using Eq. (4.1), gives $h \leqslant v + \delta$ in D. Thus we have $u \leqslant v + \delta$ in D, and since this inequality was already known to hold in $\Omega \setminus D$, we finally have $u \leqslant v + \delta$ in Ω. The claim now follows by letting $\delta \to 0$. □

5. Three lemmas for weak solutions

In this section, we prove three lemmas that are needed in the proofs of Theorem 4.1 above, and of Proposition 4.3 in the next section. The following well-known vector inequalities will be used several times below:

$$\left(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta\right)\cdot(\xi-\eta) \geqslant \begin{cases} 2^{2-q}|\xi-\eta|^q & \text{if } q \geqslant 2, \\ (q-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-q}} & \text{if } 1 < q < 2. \end{cases} \tag{5.1}$$

In particular, we have

$$\left(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta\right)\cdot(\xi-\eta) > 0 \tag{5.2}$$

for all $\xi, \eta \in \mathbb{R}^n$ such that $\xi \neq \eta$ and $1 < p(x) < \infty$.

We begin with the following form of the comparison principle. Note that the second assumption holds if u is a weak subsolution, and v a weak supersolution. For this reason, the lemma is also the basis of the proof of the $p(x)$-superharmonicity of weak supersolutions, Theorem 3.3.

Lemma 5.1. *Let u and v be functions in $W^{1,p(x)}(\Omega)$ such that $(u-v)_+ \in W_0^{1,p(x)}(\Omega)$. If*

$$\int_\Omega |Du|^{p(x)-2}Du\cdot D\varphi\,\mathrm{d}x \leqslant \int_\Omega |Dv|^{p(x)-2}Dv\cdot D\varphi\,\mathrm{d}x$$

for all positive test functions $\varphi \in W_0^{1,p(x)}(\Omega)$, then $u \leqslant v$ almost everywhere in Ω.

Proof. By the assumption and (5.2), we see that

$$0 \leqslant \int_{\Omega} \big(|Du|^{p(x)-2}Du - |Dv|^{p(x)-2}Dv\big) \cdot D(u-v)_+ \,\mathrm{d}x \leqslant 0.$$

Thus $D(u-v)_+ = 0$, and since $(u-v)_+$ has zero boundary values, the claim follows. □

Lemma 5.2. *Let $u \in W^{1,p(x)}(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)} u = 0$$

in Ω, and u_ε a weak solution of

$$-\Delta_{p(x)} u = \varepsilon, \quad \varepsilon > 0,$$

such that $u - u_\varepsilon \in W_0^{1,p(x)}(\Omega)$. Then

$$u_\varepsilon \to u \quad \text{locally uniformly in } \Omega,$$

as $\varepsilon \to 0$.

Proof. We begin the proof by deriving a very rough estimate for $|Du - Du_\varepsilon|$ in $L^{p(x)}$. To this end, since u_ε minimizes the functional

$$v \mapsto \int_{\Omega} \left(\frac{1}{p(x)}|Dv|^{p(x)} - \varepsilon v\right) \mathrm{d}x,$$

using Hölder's inequality, Sobolev–Poincaré (Theorem 2.1), and the modular inequalities (2.2), we have

$$\begin{aligned}
\int_{\Omega} |Du_\varepsilon|^{p(x)} \,\mathrm{d}x &\leqslant C \int_{\Omega} \big(|Du|^{p(x)} + \varepsilon|u_\varepsilon - u|\big) \,\mathrm{d}x \\
&\leqslant C\left(\int_{\Omega} |Du|^{p(x)} \,\mathrm{d}x + \varepsilon \|1\|_{p'(x)} \|u_\varepsilon - u\|_{L^{p(x)}(\Omega)}\right) \\
&\leqslant C\left(\int_{\Omega} |Du|^{p(x)} \,\mathrm{d}x + \varepsilon \|Du_\varepsilon - Du\|_{L^{p(x)}(\Omega)}\right) \\
&\leqslant C\big(1 + \|Du\|_{L^{p(x)}(\Omega)}^{p^+} + \varepsilon\big(\|Du\|_{L^{p(x)}(\Omega)} + \|Du_\varepsilon\|_{L^{p(x)}(\Omega)}\big)\big).
\end{aligned}$$

Alternatively, we could start by testing the weak formulation for u_ε with $u - u_\varepsilon \in W_0^{1,p(x)}(\Omega)$, use Young's inequality, and then continue in the same way as above. Using (2.2) again, and absorbing one of the terms into the left gives

$$\|Du_\varepsilon\|_{L^{p(x)}(\Omega)} \leqslant C\big(1 + \|Du\|_{L^{p(x)}(\Omega)}^{p^+}\big)^{1/p^-} \tag{5.3}$$

with a constant C independent of ε for all $\varepsilon > 0$ small enough, and thus

$$\|Du - Du_\varepsilon\|_{L^{p(x)}(\Omega)} \leqslant C\big(1 + \|Du\|_{L^{p(x)}(\Omega)}^{p^+}\big). \tag{5.4}$$

Next we use $u - u_\varepsilon \in W_0^{1,p(x)}(\Omega)$ as a test-function in the weak formulations of $-\Delta_{p(x)} u = 0$ and $-\Delta_{p(x)} u_\varepsilon = \varepsilon$, and subtract the resulting equations. This yields

$$\int_{\Omega} \big(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\big) \cdot (Du - Du_\varepsilon) \,\mathrm{d}x = \varepsilon \int_{\Omega} (u_\varepsilon - u) \,\mathrm{d}x. \tag{5.5}$$

The right hand side can be estimated as above:

$$\varepsilon \int_{\Omega} (u_\varepsilon - u) \,\mathrm{d}x \leqslant C\varepsilon \|Du_\varepsilon - Du\|_{L^{p(x)}(\Omega)}.$$

In order to obtain a suitable lower bound for the left hand side of (5.5), we use the two inequalities in (5.1), and therefore we need to consider separately the subsets $\Omega^- := \{x \in \Omega:\ 1 < p(x) < 2\}$, and $\Omega^+ := \{x \in \Omega:\ p(x) \geqslant 2\}$.

Let us first concentrate on Ω^-. By using Hölder's inequality (2.3), and the modular inequalities (2.2), we have

$$\begin{aligned}\int_{\Omega^-} |Du - Du_\varepsilon|^{p(x)}\,dx &\leqslant C\left\| \frac{|Du_\varepsilon - Du|^{p(x)}}{(|Du| + |Du_\varepsilon|)^{\frac{p(x)}{2}(2-p(x))}}\right\|_{L^{\frac{2}{p(x)}}(\Omega^-)} \\ &\quad\times \left\|\left(|Du| + |Du_\varepsilon|\right)^{\frac{p(x)}{2}(2-p(x))}\right\|_{L^{\frac{2}{2-p(x)}}(\Omega^-)} \\ &\leqslant C \max_{p\in\{\hat p^+,\hat p^-\}}\left(\int_{\Omega^-} \frac{|Du_\varepsilon - Du|^2}{(|Du| + |Du_\varepsilon|)^{2-p(x)}}\,dx\right)^{p/2} \\ &\quad\times\left(1 + \int_{\Omega^-}\left(|Du| + |Du_\varepsilon|\right)^{p(x)}\,dx\right)^{1/2},\end{aligned}$$

where $\hat p^- = \inf_{\Omega^-} p(x)$ and $\hat p^+ = \sup_{\Omega^-} p(x)$. The vector inequality (5.1), Young's inequality, and the fact that $1 < \hat p^-, \hat p^+ \leqslant 2$ imply

$$\begin{aligned}&\max_{p\in\{\hat p^+,\hat p^-\}}\left(\int_{\Omega^-} \frac{|Du_\varepsilon - Du|^2}{(|Du| + |Du_\varepsilon|)^{2-p(x)}}\,dx\right)^{p/2} \\ &\leqslant \max_{p\in\{\hat p^+,\hat p^-\}} C\left(\int_{\Omega^-}\left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du - Du_\varepsilon)\,dx\right)^{p/2} \\ &\leqslant C\left(\delta^{\frac{2}{2-\hat p^-}} + \delta^{-\frac{2}{\hat p^-}}\int_{\Omega^-}\left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du - Du_\varepsilon)\,dx\right)\end{aligned}$$

for any $0 < \delta < 1$, to be chosen later. Combining this with (5.3), which can be used to bound the term $(1 + \int_{\Omega^-}(|Du| + |Du_\varepsilon|)^{p(x)}\,dx)$, we obtain

$$\int_{\Omega^-} |Du - Du_\varepsilon|^{p(x)}\,dx \leqslant C\delta^{\frac{2}{2-\hat p^-}} + C\delta^{-\frac{2}{\hat p^-}}\int_{\Omega^-}\left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du - Du_\varepsilon)\,dx \tag{5.6}$$

for $0 < \delta < 1$ and for a constant C depending on u but independent of ε and δ.

For $\Omega^+ = \{x \in \Omega:\ p(x) \geqslant 2\}$, (5.1) gives

$$\int_{\Omega^+} |Du - Du_\varepsilon|^{p(x)}\,dx \leqslant C\int_{\Omega^+}\left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du - Du_\varepsilon)\,dx,$$

and summing up this with (5.6) and using (5.5) yields

$$\begin{aligned}\int_{\Omega} |Du - Du_\varepsilon|^{p(x)}\,dx &\leqslant C\left(\delta^{\frac{2}{2-\hat p^-}} + \delta^{-\frac{2}{\hat p^-}}\int_{\Omega}\left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du - Du_\varepsilon)\,dx\right) \\ &\leqslant C\left(\delta^{\frac{2}{2-\hat p^-}} + \delta^{-\frac{2}{\hat p^-}}\varepsilon\|Du - Du_\varepsilon\|_{L^{p(x)}(\Omega)}\right).\end{aligned}$$

Now we choose $\delta = \varepsilon^{\frac{(2-\hat p^-)\hat p^-}{4}}$, and have

$$\int_{\Omega} |Du - Du_\varepsilon|^{p(x)}\,dx \leqslant C\varepsilon^{\hat p^-/2}\left(1 + \|Du - Du_\varepsilon\|_{L^{p(x)}(\Omega)}\right).$$

Owing to (5.4) and the modular inequalities, we obtain

$$\|Du - Du_\varepsilon\|_{L^{p(x)}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows from this and another application of Poincaré's inequality as well as inequalities (2.2) that

$$u_\varepsilon \to u \quad \text{in } W^{1,p(x)}(\Omega) \text{ as } \varepsilon \to 0. \tag{5.7}$$

Then we choose ε_1 and ε_2 such that $\varepsilon_1 \geqslant \varepsilon_2$ and subtract the corresponding equations to get

$$\int_\Omega \left(|Du_{\varepsilon_1}|^{p(x)-2} Du_{\varepsilon_1} - |Du_{\varepsilon_2}|^{p(x)-2} Du_{\varepsilon_2}\right) \cdot D\varphi \,\mathrm{d}x = (\varepsilon_1 - \varepsilon_2) \int_\Omega \varphi \,\mathrm{d}x \geqslant 0$$

for positive φ. According to Lemma 5.1, we have $u_{\varepsilon_1} \geqslant u_{\varepsilon_2}$ almost everywhere. This together with (5.7) implies that

$$u_\varepsilon \to u \quad \text{almost everywhere in } \Omega.$$

The claim about the locally uniform convergence follows from C^α_{loc}-estimates for u_ε which are uniform in ε due to the results in [11, Section 4]. □

We use the next lemma to deal with the singularity in the equation in the region where $1 < p(x) < 2$.

Lemma 5.3. *Let $v_\varepsilon \in W^{1,p(x)}(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)} v = \varepsilon. \tag{5.8}$$

Suppose that $\varphi \in C^2(\Omega)$ is such that $v_\varepsilon(x_0) = \varphi(x_0)$, $v_\varepsilon(x) > \varphi(x)$ for $x \neq x_0$, and that either x_0 is an isolated critical point of φ, or $D\varphi(x_0) \neq 0$. Then

$$\limsup_{\substack{x \to x_0 \\ x \neq x_0}} \left(-\Delta_{p(x)} \varphi(x)\right) \geqslant \varepsilon.$$

Proof. We may, of course, assume that $x_0 = 0$. Let us suppose that the claim does not hold. Then there is a radius $r > 0$ such that

$$D\varphi(x) \neq 0 \quad \text{and} \quad -\Delta_{p(x)} \varphi(x) < \varepsilon$$

for $0 < |x| < r$.

We aim at showing first that φ is a weak subsolution of (5.8) in $B_r = B(0, r)$. Let $0 < \rho < r$. For any positive $\eta \in C_0^\infty(B_r)$, we have

$$-\int_{|x|=\rho} \eta |D\varphi|^{p(x)-2} D\varphi \cdot \frac{x}{\rho} \,\mathrm{d}S = \int_{\rho<|x|<r} |D\varphi|^{p(x)-2} D\varphi \cdot D\eta \,\mathrm{d}x + \int_{\rho<|x|<r} (\Delta_{p(x)}\varphi)\eta \,\mathrm{d}x$$

by the divergence theorem. The left hand side tends to zero as $\rho \to 0$, since

$$\left| \int_{|x|=\rho} \eta |D\varphi|^{p(x)-2} D\varphi \cdot \frac{x}{\rho} \,\mathrm{d}S \right| \leqslant C \|\eta\|_\infty \max\left\{ \|D\varphi\|_\infty^{p^+-1}, \|D\varphi\|_\infty^{p^--1} \right\} \rho^{n-1}.$$

By the counterassumption,

$$\int_{\rho<|x|<r} \eta \Delta_{p(x)} \varphi \,\mathrm{d}x \geqslant -\varepsilon \int_{\rho<|x|<r} \eta \,\mathrm{d}x \geqslant -\varepsilon \int_{B_r} \eta \,\mathrm{d}x.$$

Letting ρ tend to zero, we see that

$$\int_{B_r} |D\varphi|^{p(x)-2} D\varphi \cdot D\eta \,\mathrm{d}x \leqslant \varepsilon \int_{B_r} \eta \,\mathrm{d}x,$$

which means that φ is indeed a weak subsolution.

Now a contradiction follows from the comparison principle in a similar fashion as in the first part of the proof of Theorem 4.1. Indeed, we have $m = \inf_{\partial B_r}(v_\varepsilon - \varphi) > 0$. Then $\tilde{\varphi} = \varphi + m$ is a weak subsolution such that $\tilde{\varphi} \leqslant v_\varepsilon$ on ∂B_r, but $\tilde{\varphi}(0) > v_\varepsilon(0)$. □

6. The comparison principle

As seen in Section 4, Proposition 4.3 is the core of the proof of the equivalence of weak and viscosity solutions. To prepare for its proof, we write $\Delta_{p(x)}\varphi(x)$ in a more convenient form. For a vector $\xi \neq 0$, $\xi \otimes \xi$ is the matrix with entries $\xi_i\xi_j$. Let

$$A(x,\xi) = |\xi|^{p(x)-2}\left(I + \big(p(x)-2\big)\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right),$$
$$B(x,\xi) = |\xi|^{p(x)-2}\log|\xi|\xi \cdot Dp(x),$$

and

$$F(x,\xi,X) = \operatorname{trace}\big(A(x,\xi)X\big) + B(x,\xi)$$

for $x \in \Omega$, $\xi \in \mathbb{R}^n$, and X a symmetric $n \times n$ matrix. Then we may write

$$\begin{aligned}\Delta_{p(x)}\varphi(x) &= F\big(x, D\varphi(x), D^2\varphi(x)\big)\\ &= \operatorname{trace}\big(A\big(x, D\varphi(x)\big)D^2\varphi(x)\big) + B\big(x, D\varphi(x)\big)\end{aligned}$$

if $D\varphi(x) \neq 0$.

We prove the claim about viscosity subsolutions in Proposition 4.3, the case of supersolutions then following by symmetry. For the convenience of the reader, we repeat the statement before proceeding with the proof.

Proposition 6.1. *Let Ω be a bounded domain, and suppose that u is a viscosity subsolution to the $p(x)$-Laplace equation, and v is a locally Lipschitz continuous weak solution of*

$$-\Delta_{p(x)}v = \varepsilon, \quad \varepsilon > 0, \tag{6.1}$$

in Ω such that

$$u \leqslant v \quad \text{on } \partial\Omega.$$

Then

$$u \leqslant v \quad \text{in } \Omega.$$

Proof. The argument follows the usual outline of proving a comparison principle for viscosity solutions to second order elliptic equations. We argue by contradiction and assume that $u - v$ has a strict interior maximum, that is,

$$\sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v). \tag{6.2}$$

We proceed by doubling the variables; consider the functions

$$w_j(x,y) = u(x) - v(y) - \Psi_j(x,y), \quad j = 1,2,\ldots,$$

where

$$\Psi_j(x,y) = \tfrac{j}{q}|x-y|^q,$$

and

$$q > \max\left\{2, \frac{p^-}{p^- - 1}\right\}, \qquad p^- = \inf_{x\in\Omega} p(x) > 1.$$

Let (x_j, y_j) be a maximum of w_j relative to $\overline{\Omega} \times \overline{\Omega}$. By (6.2), we see that for j sufficiently large, (x_j, y_j) is an interior point. Moreover, up to selecting a subsequence, $x_j \to \hat{x}$ and $y_j \to \hat{x}$ as $j \to \infty$ and $\hat{x}$ is a maximum point for $u - v$ in Ω. Finally, since

$$u(x_j) - v(x_j) \leqslant u(x_j) - v(y_j) - \frac{j}{q}|x_j - y_j|^q,$$

and v is locally Lipschitz, we have

$$\frac{j}{q}|x_j - y_j|^q \leqslant v(x_j) - v(y_j) \leqslant C|x_j - y_j|,$$

and hence dividing by $|x_j - y_j|^{1-\delta}$ we get

$$j|x_j - y_j|^{q-1+\delta} \to 0 \quad \text{as } j \to \infty \text{ for any } \delta > 0. \tag{6.3}$$

Observe that although u is, in general, an extended real valued function, it follows from Definition 3.5 that u is finite at x_j.

In what follows, we will need the fact that $x_j \neq y_j$. To see that this holds, let us denote

$$\varphi_j(y) = -\Psi_j(x_j, y) + v(y_j) + \Psi_j(x_j, y_j),$$

and observe that since

$$u(x) - v(y) - \Psi_j(x, y) \leqslant u(x_j) - v(y_j) - \Psi_j(x_j, y_j)$$

for all $x, y \in \Omega$, we obtain by choosing $x = x_j$ that

$$v(y) \geqslant -\Psi_j(x_j, y) + v(y_j) + \Psi_j(x_j, y_j)$$

for all $y \in \Omega$. That is, φ_j touches v at y_j from below, and thus

$$\limsup_{y \to y_j} \bigl(-\Delta_{p(x)}\varphi_j(y)\bigr) \geqslant \varepsilon \tag{6.4}$$

by Lemma 5.3. On the other hand, a calculation yields

$$\begin{aligned}\Delta_{p(x)}\varphi_j(y) = j^{p(y)-1}|x_j - y|^{(q-1)(p(y)-2)+q-2}\bigl[&n + q - 2 + \bigl(p(y) - 2\bigr)(q - 1)\\ &+ \log\bigl(j|x_j - y|^{q-1}\bigr)(x_j - y) \cdot Dp(y)\bigr],\end{aligned}$$

where

$$(q - 1)\bigl(p(y) - 2\bigr) + q - 2 = q\bigl(p(y) - 1\bigr) - p(y) > 0$$

by the choice of q. Hence if $x_j = y_j$, we would have

$$\limsup_{y \to y_j} \bigl(-\Delta_{p(x)}\varphi_j(y)\bigr) = 0,$$

contradicting (6.4). Thus $x_j \neq y_j$ as desired.

For equations that are continuous in all the variables, viscosity solutions may be equivalently defined in terms of the closures of super- and subjets. The next aim is to exploit this fact, together with the maximum principle for semicontinuous functions, see [7,8,18]. Since (x_j, y_j) is a local maximum point of $w_j(x, y)$, the aforementioned principle implies that there exist symmetric $n \times n$ matrices X_j, Y_j such that

$$\begin{aligned}&\bigl(D_x\Psi_j(x_j, y_j), X_j\bigr) \in \overline{J}^{2,+}u(x_j),\\ &\bigl(-D_y\Psi_j(x_j, y_j), Y_j\bigr) \in \overline{J}^{2,-}v(y_j),\end{aligned}$$

where $\overline{J}^{2,+}u(x_j)$ and $\overline{J}^{2,-}v(y_j)$ are the closures of the second order superjet of u at x_j and the second order subjet of v at y_j, respectively. Further, writing $z_j = x_j - y_j$, the matrices X_j and Y_j satisfy

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leqslant D^2\Psi_j(x_j,y_j) + \frac{1}{j}\big[D^2\Psi_j(x_j,y_j)\big]^2$$
$$= j\big(|z_j|^{q-2} + 2|z_j|^{2q-4}\big)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$
$$+ j(q-2)\big(|z_j|^{q-4} + 2q|z_j|^{2q-6}\big)\begin{pmatrix} z_j\otimes z_j & -z_j\otimes z_j \\ -z_j\otimes z_j & z_j\otimes z_j \end{pmatrix}, \tag{6.5}$$

where

$$D^2\Psi_j(x_j,y_j) = \begin{pmatrix} D_{xx}\Psi_j(x_j,y_j) & D_{xy}\Psi_j(x_j,y_j) \\ D_{yx}\Psi_j(x_j,y_j) & D_{xx}\Psi_j(x_j,y_j) \end{pmatrix}$$
$$= j|z_j|^{q-2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + j(q-2)|z_j|^{q-4}\begin{pmatrix} z_j\otimes z_j & -z_j\otimes z_j \\ -z_j\otimes z_j & z_j\otimes z_j \end{pmatrix}.$$

Observe that (6.5) implies

$$X_j\xi\cdot\xi - Y_j\zeta\cdot\zeta \leqslant j\big[|z_j|^{q-2} + 2|z_j|^{2q-4}\big]|\xi-\zeta|^2 + j(q-2)\big[|z_j|^{q-4} + 2q|z_j|^{2q-6}\big]\big(z_j\cdot(\xi-\zeta)\big)^2$$
$$\leqslant j\big[(q-1)|z_j|^{q-2} + 2(q-1)^2|z_j|^{2(q-2)}\big]|\xi-\zeta|^2 \tag{6.6}$$

for all $\xi,\zeta\in\mathbb{R}^n$.

Now, u is a viscosity subsolution of the $p(x)$-Laplace equation, and v a viscosity solution of $-\Delta_{p(x)}v = \varepsilon$. By the equivalent definition in terms of jets, we obtain that

$$-\operatorname{trace}\big(A(x_j,\eta_j)X_j\big) - B(x_j,\eta_j) \leqslant 0$$

and

$$-\operatorname{trace}\big(A(y_j,\eta_j)Y_j\big) - B(y_j,\eta_j) \geqslant \varepsilon.$$

Here it is crucial that

$$\eta_j = D_x\Psi_j(x_j,y_j) = -D_y\Psi_j(x_j,y_j) = j|x_j-y_j|^{q-2}(x_j-y_j)$$

is nonzero as observed above. This guarantees that the $p(x)$-Laplace equation is non-singular at the neighborhoods of (x_j,η_j,X_j) and (y_j,η_j,Y_j), which in turn allows us to use jets. Notice also that since v is locally Lipschitz, there is a constant $C>0$ such that $|\eta_j|\leqslant C$ for at least large j's, for the reason that $(\eta_j,Y_j)\in \overline{J}^{2,-}v(y_j)$.

Since $\eta_j\neq 0$, $A(\cdot,\cdot)$ is positive definite, so that its matrix square root exists. We denote $A^{1/2}(x_j) = A(x_j,\eta_j)^{1/2}$ and $A^{1/2}(y_j) = A(y_j,\eta_j)^{1/2}$. Observe that the matrices X_j, Y_j as well as $A(\cdot,\cdot)$, and $A^{1/2}(\cdot)$ are symmetric. We use matrix calculus to obtain

$$\operatorname{trace}\big(A(x_j,\eta_j)X_j\big) = \operatorname{trace}\big(A^{1/2}(x_j)A^{1/2}(x_j)X_j\big)$$
$$= \operatorname{trace}\big(A^{1/2}(x_j)^T X_j A^{1/2}(x_j)\big)$$
$$= \sum_{k=1}^n X_j A_k^{1/2}(x_j)\cdot A_k^{1/2}(x_j),$$

where $A_k^{1/2}(x_j)$ denotes the kth column of $A^{1/2}(x_j)$. This together with (6.6) implies

$$0<\varepsilon \leqslant B(x_j,\eta_j) - B(y_j,\eta_j)$$
$$+ \sum_{k=1}^n X_j A_k^{1/2}(x_j)\cdot A_k^{1/2}(x_j) - \sum_{k=1}^n Y_j A_k^{1/2}(y_j)\cdot A_k^{1/2}(y_j)$$
$$\leqslant B(x_j,\eta_j) - B(y_j,\eta_j) + Cj|x_j-y_j|^{q-2}\big\|A^{1/2}(x_j) - A^{1/2}(y_j)\big\|_2^2$$
$$\leqslant B(x_j,\eta_j) - B(y_j,\eta_j)$$
$$+ \frac{Cj|x_j-y_j|^{q-2}}{(\lambda_{\min}(A^{1/2}(x_j)) + \lambda_{\min}(A^{1/2}(y_j)))^2}\big\|A(x_j,\eta_j) - A(y_j,\eta_j)\big\|_2^2.$$

The last inequality is the local Lipschitz continuity of $A \mapsto A^{1/2}$, see [14, p. 410], and $\lambda_{\min}(M)$ denotes the smallest eigenvalue of a symmetric $n \times n$ matrix M.

Since $p(x) \in C^1(\mathbb{R}^n)$, and

$$\begin{aligned}\big||\eta_j|^{p(x_j)-2} - |\eta_j|^{p(y_j)-2}\big| &= \big|\exp\big(\log\big(|\eta_j|^{p(x_j)-2}\big)\big) - \exp\big(\log\big(|\eta_j|^{p(y_j)-2}\big)\big)\big| \\ &\leqslant \left|\frac{\partial \exp((s-2)\log|\eta_j|)}{\partial s}\right|\big|p(x_j) - p(y_j)\big| \\ &= \big|\log\big(|\eta_j|\big)\big||\eta_j|^{s-2}\big|p(x_j) - p(y_j)\big|,\end{aligned}$$

for some $s \in [p(x_j), p(y_j)]$, we have

$$\begin{aligned}B(x_j,\eta_j) - B(y_j,\eta_j) &= |\eta_j|^{p(x_j)-2}\log|\eta_j|\eta_j \cdot Dp(x_j) - |\eta_j|^{p(y_j)-2}\log|\eta_j|\eta_j \cdot Dp(y_j) \\ &\leqslant |\eta_j|^{p(x_j)-1}\big|\log|\eta_j|\big|\big|Dp(x_j) - Dp(y_j)\big| \\ &\quad + |\eta_j|\big|\log|\eta_j|\big|\big|Dp(y_j)\big|\big||\eta_j|^{p(x_j)-2} - |\eta_j|^{p(y_j)-2}\big| \\ &\leqslant |\eta_j|^{p(x_j)-1}\big|\log|\eta_j|\big|\big|Dp(x_j) - Dp(y_j)\big| \\ &\quad + C|\eta_j|^{s-1}\log^2|\eta_j|\big|p(x_j) - p(y_j)\big|.\end{aligned}$$

Moreover,

$$\begin{aligned}\big\|A(x_j,\eta_j) - A(y_j,\eta_j)\big\|_2 &\leqslant \big||\eta_j|^{p(x_j)-2} - |\eta_j|^{p(y_j)-2}\big| \\ &\quad + \big||\eta_j|^{p(x_j)-2}\big(p(x_j)-2\big) - |\eta_j|^{p(y_j)-2}\big(p(y_j)-2\big)\big| \\ &\leqslant \max\big\{3 - p(x_j), p(x_j) - 1\big\}\big|\log|\eta_j|\big||\eta_j|^{s-2}\big|p(x_j) - p(y_j)\big| \\ &\quad + |\eta_j|^{p(y_j)-2}\big|p(x_j) - p(y_j)\big| \\ &\leqslant C\big(\big(p^+ + 1\big)\big|\log|\eta_j|\big||\eta_j|^{s-2} + |\eta_j|^{p(y_j)-2}\big)|x_j - y_j|,\end{aligned}$$

and

$$\begin{aligned}\lambda_{\min}\big(A^{1/2}(x_j)\big) &= \big(\lambda_{\min}\big(A(x_j,\eta_j)\big)\big)^{1/2} = \Big(\min_{|\xi|=1} A(x_j,\eta_j)\xi\cdot\xi\Big)^{1/2} \\ &\geqslant \min\big\{1, \sqrt{p(x_j)-1}\big\}|\eta_j|^{\frac{p(x_j)-2}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}0 < \varepsilon &\leqslant |\eta_j|^{p(x_j)-1}\big|\log|\eta_j|\big|\big|Dp(x_j) - Dp(y_j)\big| + C|\eta_j|^{s-1}\log^2|\eta_j|\big|p(x_j) - p(y_j)\big| \\ &\quad + \frac{C((p^+ + 1)|\log|\eta_j||\,|\eta_j|^{s-2} + |\eta_j|^{p(y_j)-2})^2}{\min\{1, p^- - 1\}\big(|\eta_j|^{\frac{p(x_j)-2}{2}} + |\eta_j|^{\frac{p(y_j)-2}{2}}\big)^2} j|x_j - y_j|^q.\end{aligned} \tag{6.7}$$

The first two terms on the right hand are easily shown to converge to 0 as $j \to \infty$. Indeed, since $x_j \to \hat{x}$, $p(\hat{x}) > 1$ and $|\eta_j| \leqslant C$, we have that $|\eta_j|^{p(x_j)-1}|\log|\eta_j||$ remains bounded as $j \to \infty$. Thus, owing to the continuity of $x \mapsto Dp(x)$, the first term converges to zero as $j \to \infty$. The second term is treated in a similar way.

Finally, we deal with the third term. Recalling that $|\eta_j| = j|x_j - y_j|^{q-1}$, we have

$$\begin{aligned}&\left(\frac{|\log|\eta_j||\,|\eta_j|^{s-2}}{|\eta_j|^{\frac{p(x_j)-2}{2}} + |\eta_j|^{\frac{p(y_j)-2}{2}}}\right)^2 j|x_j - y_j|^q \\ &\quad \leqslant \log^2|\eta_j|\,|\eta_j|^{2s-p(x_j)-2} j|x_j - y_j|^q \\ &\quad \leqslant \log^2|\eta_j|\, j^{2s-p(x_j)-1}|x_j - y_j|^{q+(q-1)(2s-p(x_j)-2)} \\ &\quad = \log^2\big(j|x_j - y_j|^{q-1}\big)\big[j|x_j - y_j|^{q-1}\big]^{2s-p(x_j)-1}|x_j - y_j| \\ &\quad = \log^2\big(j|x_j - y_j|^{q-1}\big)\big[j|x_j - y_j|^{q-1+\delta}\big]^{2s-p(x_j)-1}|x_j - y_j|^{1-\delta(2s-p(x_j)-1)}.\end{aligned}$$

Now, since $2s - p(x_j) - 1 \to p(\hat{x}) - 1 > 0$ and $j|x_j - y_j|^{q-1+\delta} \to 0$ as $j \to \infty$, we see that

$$\left[j|x_j - y_j|^{q-1+\delta}\right]^{2s-p(x_j)-1} \to 0 \quad \text{as } j \to \infty.$$

Further, we write

$$\begin{aligned} &\log^2\left(j|x_j - y_j|^{q-1}\right)|x_j - y_j|^{1-\delta(2s-p(x_j)-1)} \\ &\quad = \log^2\left(j|x_j - y_j|^{q-1}\right)\left(j|x_j - y_j|^{q-1}\right)^{\frac{1-\delta(2s-p(x_j)-1)}{q-1}}\left(\frac{1}{j}\right)^{\frac{(1-\delta(2s-p(x_j)-1))}{q-1}}, \end{aligned}$$

and note that choosing $\delta > 0$ so small that $1 - \delta(p^+ - 1) > 0$ suffices for the third term to converge to zero as $j \to \infty$. A contradiction has been reached. □

Remark 6.2. The $p(x)$-Laplace equation occasionally appears in the literature also in the form

$$-\operatorname{div}\left(p(x)|Du|^{p(x)-2}Du(x)\right) = 0. \tag{6.8}$$

This version is the Euler–Lagrange equation of the functional

$$I(u) = \int_{\Omega} |Du|^{p(x)}\,\mathrm{d}x,$$

and as far as distributional weak solutions are concerned, the known results for (6.8) and (3.1) are virtually identical. In particular, the obvious counterparts of Lemmas 5.1 and 5.2 for (6.8) hold with essentially the same proofs, as do all the regularity results we have needed above.

If we compute the divergence in (6.8), the result is the equation

$$-p(x)|Du|^{p(x)-2}\left[\Delta u + \left(p(x) - 2\right)\Delta_\infty u + \left(\frac{1}{p(x)} + \log|Du|\right)Dp(x)\cdot Du\right] = 0.$$

Compared to (1.4), the only difference is that the multiplier of $Dp(x)\cdot Du$ has changed from $\log|Du|$ to $\frac{1}{p(x)} + \log|Du|$, which, in view of the assumptions on $p(x)$, does not cause any additional difficulties. Thus one can proceed almost as above to show that also for the version (6.8) of the $p(x)$-Laplace equation the weak and viscosity solutions coincide.

7. An application: a Radó type removability theorem

The classical theorem of Radó says that, if the continuous complex function f is analytic when $f(z) \neq 0$, then it is analytic in its domain of definition. This result has been extended for solutions of various partial differential equations, including the Laplace equation and the p-Laplace equation, see the references in [16]. Here we prove a corresponding removability result for $p(x)$-harmonic functions.

Theorem 7.1. *If a function $u \in C^1(\Omega)$ is a weak solution of* (3.1) *in $\Omega \setminus \{x\colon u(x) = 0\}$, then u is a weak solution in the entire domain Ω.*

Proof. A key step in the proof is to observe that if $u \in C^1(\Omega)$ is a weak solution in

$$\Omega \setminus \left\{x\colon u(x) = 0\right\},$$

then it is a weak solution in $\Omega \setminus U$, where

$$U := \left\{x\colon u(x) = 0 \text{ and } Du(x) \neq 0\right\}.$$

This readily follows from Corollary 4.2, because in the definition of a viscosity solution we ignore the test functions with $D\varphi(x_0) = 0$. Since $u \in C^1(\Omega)$, it follows that if $Du(x_0) = 0$, then $D\varphi(x_0) = 0$.

Thus the original problem has been reduced to proving the removability of U, which is locally a C^1-hypersurface. There are at least two ways to accomplish this. One option is to apply [16, Theorem 2.2], which means using viscosity solutions and an argument similar to Hopf's maximum principle. The second alternative is to use a coordinate transformation and map U to a hyperplane, and then prove the removability of a hyperplane by a relatively simple computation. The price one has to pay in this approach is that the equation changes, but fortunately this is allowed in [23, Lemma 2.22]. □

Remark 7.2. We do not know how to prove Theorem 7.1 without using viscosity solutions, not even in the simpler case when $p(x)$ is constant. In particular, the removability of a level set is an open question for the weak solutions of (3.1) when $p(x)$ is, say, only continuous.

Theorem 7.1 fails if u is assumed to be only Lipschitz continuous. A simple example showing this is $u(x) = |x_1|$, which is $p(x)$-harmonic (for any $p(x) \in C^1$) in $\mathbb{R}^n \setminus \{u = 0\}$ but not in $\mathbb{R}^n$.

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