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Green bundles, Lyapunov exponents and regularity along the supports of the minimizing measures

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Abstract

In this article, we study the minimizing measures of the Tonelli Hamiltonians. More precisely, we study the relationships between the so-called Green bundles and various notions as:

- the Lyapunov exponents of minimizing measures;
- the weak KAM solutions.

In particular, we deduce that the support of every minimizing measure μ, all of whose Lyapunov exponents are zero, is C^1-regular μ-almost everywhere.

Résumé

Dans cet article, on étudie les mesures minimisantes de Hamiltoniens de Tonelli. Plus précisément, on explique quelles relations existent entre les fibrés de Green et différentes notions comme :

- les exposants de Lyapunov des mesures minimisantes ;
- les solutions KAM faibles.

On en déduit par exemple que si tous les exposants de Lyapunov d'une mesure minimisante μ sont nuls, alors le support de cette mesure est C^1-régulier en μ-presque tout point.

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Keywords: Minimizing orbits and measures; Lyapunov exponents; Weak KAM theory; Green bundles; Regularity of solutions to Hamilton–Jacobi equations

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1. Introduction

In this article, M is a closed n-dimensional manifold and $\pi : T^*M \to M$ its cotangent bundle. We consider a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$, i.e. a C^2 function that is strictly C^2-convex and superlinear in the fiber. The Hamiltonian flow associated with such a function is denoted by $(\varphi_t)_{t\in\mathbb{R}}$ or $(\varphi_t^H)_{t\in\mathbb{R}}$. To such a Hamiltonian, there corresponds a Lagrangian function $L : TM \to \mathbb{R}$ that has the same regularity as H and is also superlinear and strictly convex in the fiber. The corresponding Euler–Lagrange flow is denoted by $(f_t)_{t\in\mathbb{R}}$.

For such a Hamiltonian system, it is usual to study its "minimizing objects"; more precisely, a piece of orbit $(\varphi_t(q,p))_{t\in[a,b]} = (q_t, p_t)_{t\in[a,b]}$ is minimizing if the arc $(q_t)_{t\in[a,b]}$ minimizes the action functional A_L defined by $A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\,dt$ among the C^2-arcs joining q_a to q_b. More generally, if I is an interval and $(\varphi_t)_{t\in I} = (q_t, p_t)_{t\in I}$ is an orbit piece, we say that it is minimizing if for every segment $[a,b] \subset I$, its restriction to $[a,b]$ is minimizing. Then we call the set of points of T^*M whose (complete) orbit is minimizing the *Mañé set*. We denote it by $\mathcal{N}^*(H)$ and its projection, the *projected Mañé set*, is denoted by: $\mathcal{N}(H) = \pi(\mathcal{N}^*(H))$. The Mañé set is non-empty, compact and invariant by the Hamiltonian flow (see [10]). The first proof of the non-emptiness of the Mañé set is due to J. Mather: he proved in the 90's in [19] the existence of minimizing measures.

We are interested in invariant subsets of the Mañé set, i.e. subsets that are the union of some minimizing orbits. More precisely, we would like to know if we can say something about the regularity of such subsets (we will be more precise very soon. This is a kind of differentiability) and particularly if there is a link between the dynamics of the flow restricted to such a set and the regularity of the set.

The oldest result in this direction concerns the time-dependent case: considering a symplectic twist map of the annulus $T^*\mathbb{S}$, G. Birkhoff proved in the 1920's that any essential invariant curve is the graph of a Lipschitz map (see [5] or [14]). It is easy to prove that such a curve is action minimizing. In the case of higher dimensions, M. Herman proved in [15] that any C^0-Lagrangian graph of $T^*\mathbb{T}^n$ that is invariant by a symplectic twist map is, in fact, the graph of a Lipschitz map. A related result in the autonomous case is that any C^1-Hamilton–Jacobi solution of a Tonelli Hamiltonian is, in fact, $C^{1,1}$ (see [11]). As Rademacher's theorem says to us that any Lipschitz function is differentiable Lebesgue almost everywhere, these results are a kind of regularity result.

In [1], we did, in fact, improve these results of regularity in the autonomous case, proving that if a C^0-Lagrangian graph is invariant by a Tonelli flow, and if one of the two following hypotheses is satisfied:

- $\dim M = 2$ and all the singularities of H are non-degenerate;
- the dynamics of the restriction of the flow to the invariant graph is Lipschitz conjugate to a translations' flow;

then the invariant graph is, in fact, C^1 almost everywhere (this is stronger than just differentiable). Let us point out that any of the two previous hypotheses implies that the dynamics of the restricted flow to the graph is soft on a certain sense (our arguments are not very precise, but we only want to give a certain intuition of the forthcoming result); indeed, when $\dim M = 2$, if we reduce the dynamics modulo the vector field, we obtain a 1-dimension dynamics, and it is known at least in the differentiable case that the Lyapunov exponents of a dynamics on the circle are zero. The same is true for any dynamics that is Lipschitz conjugate to a translation.

We gave a similar results for the invariant curves of the twist maps of the annulus in [2], proving that Birkhoff's result can be improved: any essential invariant curve of a symplectic twist map of the annulus $T^*\mathbb{S}$ is the graph of a Lipschitz map that is C^1 Lebesgue almost everywhere.

Hence, it seems reasonable to try to find a relationship between the Lyapunov exponents of any minimizing measure and the regularity of its support, where an invariant measure is *minimizing* if its support is in the Mañé set.

For a twist map of the annulus $T^*\mathbb{S}$, we studied the ergodic minimizing measures in [3] and proved that the C^1-regularity (we will be more precise very soon) of its support is equivalent to the fact that the Lyapunov exponents are zero. Hence, in a certain way, in this case, "C^1-irregularity" is equivalent to non-vanishing Lyapunov exponents.

The question that we ask now ourselves is the following: what can we say for higher dimensions? Is the irregularity (in a sense we will soon specify) of the support of a minimizing ergodic measure equivalent to non-vanishing exponents?

A first and obvious answer is: no. Indeed, let us consider the following example: (ψ_t) is an Anosov flow defined on the cotangent bundle $T^*\mathcal{S}$ of a closed surface $\mathcal{S}$. Let $\mathcal{N} = T_1^*\mathcal{S}$ be its unitary cotangent bundle, which is a 3-manifold invariant by (ψ_t). Then a method due to Mañé (see [17]) allows us to define a Tonelli Hamiltonian H on $T^*\mathcal{N}$ such that the restriction of its flow (φ_t) to the zero section $\mathcal{N}$ is (ψ_t): the Lagrangian L associated with H is defined by: $L(q, v) = \frac{1}{2}\|\dot\psi(q) - v\|^2$ where $\|.\|$ is any Riemannian metric on $\mathcal{N}$. In this case, the zero section is very regular (even C^∞), but the Lyapunov exponents of every invariant measure whose support is contained in $\mathcal{N}$ are non-zero (except two, the one corresponding to the flow direction and the one corresponding to the energy direction). Hence, it may happen that some exponents are non-zero and the support of the measure is very regular...

In fact, the other implication is true: we will see that the nullity of the Lyapunov exponents implies the regularity of the support of the considered measure.

Let us now explain in a detailed way in which kind of regularity we are interested:

Definition. Let A be a subset of a manifold M and let a belong to A. The contingent cone to A at a is the set of the tangent vectors $v \in T_a M$ such that there exist a sequence (a_n) of elements of A and a sequence (λ_n) of positive real numbers such that (we write everything in a chart, but this is independent of the chosen chart):

$$\lim_{n\to\infty} \frac{1}{\lambda_n}(a_n - a) = v.$$

We denote it by: $\mathcal{C}_a A$.

This notion of contingent cone is due to Bouligand (see [7]). The contingent cone is never empty (it always contains the null vector), and it is equal to the null vector if and only if a is an isolated point of A.

We will see later that the sets in which we are interested are contained in some (weak) Lagrangian manifolds. Our definitions of 1-regularity and C^1-regularity seems very natural for such sets:

Definition. Let A be a subset of a symplectic manifold M and let a belong to A. We say that A is 1-regular at a if the contingent cone to A at a is contained in a Lagrangian subspace of $T_a M$.

We say that A is C^1-regular at a if there exists a Lagrangian subspace $\mathcal{L}$ of $T_a M$ such that: for every sequence $(a_n, v_n \in \mathcal{C}_{a_n} A)$ such that $\lim_{n\to\infty} a_n = a$ and the sequence (v_n) converges to an element v of $T_a M$, then $v \in \mathcal{L}$.

Let us notice that this notion of C^1-regularity is slightly different from the ones given in [2,1,3]: the notions given in these former articles are a little stronger. This notion of C^1-regularity is stronger than the notion of 1-regularity, which is nothing else but the notion of differentiability for the C^0-Lagrangian graphs (see [1] for a definition of C^0-Lagrangian graphs).

The measures that we study are the minimizing ones, that is the ones that are invariant and whose supports are contained in the Mañé set. Then we prove:

Theorem 1. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and let μ be an ergodic minimizing probability measure all of whose Lyapunov are zero. Then, at μ-almost every point of the support* $\operatorname{supp}(\mu)$ *of μ, the set* $\operatorname{supp}(\mu)$ *is C^1-regular.*

Hence:

- we succeed in proving that a kind of "soft dynamics" implies some C^1-regularity;
- we know that we can have simultaneously a strong dynamics (for example hyperbolic) and a C^∞-regularity.

In fact, we obtain more precise results than this theorem; for example, an interesting question is: what happens if there are simultaneously some zero and non-zero exponents?

To explain what happens, we need to introduce some other notions. Let us begin by recalling what the Green bundles are. These Lagrangian bundles were introduced by L. Green in 1958 in [13] for geodesic flows to prove some rigidity results. For the existence and the construction of these bundles, the reader is referred to [1,8] or [16]. We recall:

Definition. Here, $V(x) = \ker D\pi(x)$ designates the linear vertical.

Let $(\varphi_t(q,p))_{t\in]-\infty,0]}$ be a minimizing negative orbit; then the positive Green bundle G_+ is defined along this orbit by: $G_+(x) = \lim_{t\to+\infty} D\varphi_t.V(\varphi_{-t}x)$.

Let $(\varphi_t(q,p))_{t\in[0,+\infty[}$ be a minimizing positive orbit; then the negative Green bundle G_- is defined along this orbit by: $G_-(x) = \lim_{t\to+\infty} D\varphi_{-t}.V(\varphi_t x)$.

Hence, at every point of the Mañé set, the two Green bundles are defined.

Let us recall that the two Green bundles are Lagrangian, invariant under the linearized flow $D\varphi_t$, transverse to the vertical, that they depend semi-continuously on the considered point (see [1] for the definition of semi-continuity of Lagrangian subspaces transverse to the vertical), that $G_- \leqslant G_+$ (see [1] for the definition of the order between two Lagrangian subspaces that are transverse to the vertical; in coordinates, this corresponds to the usual order on the set of symmetric matrices whose Lagrangian subspaces are the graphs). Hence, if μ is an ergodic minimizing probability measure, the integer $\dim(G_-(x)\cap G_+(x))$ is constant μ-almost everywhere.

We obtain a result linking the dimension of the intersection of the two Green bundles to the number of non-zero Lyapunov exponents:

Theorem 2. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and let μ be an ergodic minimizing probability measure. Then the two following assertions are equivalent*:

- *at μ-almost every point,* $\dim(G_-(x)\cap G_+(x)) = p$;
- *μ has exactly $2p$ zero Lyapunov exponents, $n-p$ positive ones and $n-p$ negative ones.*

Let us mention some former related results:

- in [8], the authors prove that the transversality of the two Green bundles along an energy level implies that the restriction of the flow to this level is Anosov; they use some ideas about quasi-Anosov dynamics due to R. Mañé that are contained in [18]; in [9], P. Eberlein gives the same statement for the geodesic flows;
- we proved in [3] that any quasi-hyperbolic symplectic cocycle above a compact set is hyperbolic; we can apply this result to any minimizing compact invariant subset K contained in an energy level $\mathcal{E}$ without singularity: considering the restricted/reduced dynamical system to the energy level $\mathcal{E}$ modulo the vector-field (see [1, p. 899] for the construction), we deduce that the transversality of the Green bundles in the energy level above K is equivalent to the partial hyperbolicity of the linearized flow along K with a center bundle's dimension equal to 2;
- concerning the non-uniform case (i.e. the case of minimizing measures), the only known result was a formula giving the entropy due to A. Freire and R. Mañé (see [12]). Roughly speaking, by integrating some functional along one of the two Green bundles, they compute the sum of the positive Lyapunov exponents. This formula was generalized in [8] to any Tonelli Hamiltonian. But this formula doesn't say to us how many non-zero Lyapunov exponents exist: it only gives the sum of the positive Lyapunov exponents. Let us mention too that G. Knieper gives a nicer formula in his (non-published) thesis.

To prove Theorem 1, we recall in Section 3 some points of the recent weak KAM theory developed by A. Fathi in [10]. In this section too, we give some statements concerning the relationships between weak KAM solutions and the Green bundles. We don't give them in the introduction because we would need all the notions that will be defined in Section 3, but the interested reader can go to Section 3. Roughly speaking, the theorem asserts that along the support of the minimizing measures, the contingent cones to the weak KAM pseudographs is not far from some cone delimited by the two Green bundles.

Theorem 2 is proved in Section 2. The statement concerning the relationships between the weak KAM solutions and the Green bundles are contained in Section 3 and the proofs are in Section 4.

2. Green bundles and Lyapunov exponents

In this section, we prove Theorem 2. We consider an ergodic minimizing measure μ that is not the Dirac measure at a critical point and we denote the integer such that we have μ-almost everywhere: $\dim G_- \cap G_+ = p$ by p. Let us recall the dynamical criterion that is proved in [1]:

Proposition 3 *(Dynamical criterion). Let (x_t) be a minimizing and relatively compact orbit. Let $v \in T_{x_0}(T^*M)$. Then:*

- *if $v \notin G_-(x_0)$, then $\lim_{t\to+\infty} \|D\pi \circ D\varphi_t.v\| = +\infty$;*
- *if $v \notin G_+(x_0)$, then $\lim_{t\to+\infty} \|D\pi \circ D\varphi_{-t}.v\| = +\infty$.*

and some direct consequences of this criterion:

Remark. 1) We deduce from the dynamical criterion that the Hamiltonian vector-field X_H belongs to the two Green bundles. This implies that $p \geqslant 1$. Because these two Green bundles are Lagrangian, this implies that G_+ and G_- are tangent to the Hamiltonian levels $\{H = c\}$.

2) Moreover, we deduce also that if there is an Oseledet splitting (this will be precisely defined very soon) $T(T^*M) = E^s \oplus E^c \oplus E^u$ above a minimizing compact set K, then $E^s \subset G_-$ and $E^u \subset G_+$. Because the flow is symplectic, E^u and E^s are isotropic and orthogonal to E^c for the symplectic form (see [6]). Moreover, $E^{s\perp} = E^s \oplus E^c$ (where $\perp$ designates the orthogonal subspace for the symplectic form) and $E^{u\perp} = E^u \oplus E^c$; we deduce that: $G_-(x) = G_-(x)^\perp \subset E^{s\perp} = E^s \oplus E^c$ and similarly that $G_+(x) \subset E^u(x) \oplus E^c(x)$. Hence, finally:

$$E^s(x) \subset G^-(x) \subset E^s(x) \oplus E^c(x) \quad \text{and} \quad E^u(x) \subset G^+(x) \subset E^u(x) \oplus E^c(x)$$

and then: $G_-(x) \cap G_+(x) \subset E^c(x)$. Hence, $G_- \cap G_+$ being an isotropic subspace of the symplectic subspace E^c, we obtain: $\dim E^c \geqslant 2\dim(G_- \cap G_+)$. The dimension of the intersection of the two Green bundles gives a lower bound to the number of zero Lyapunov exponents. Theorem 2 says to us that this inequality is, in fact, an equality. Let us notice that when $p = n$, we directly have the conclusion of the theorem because $\dim E^c \geqslant 2\dim M$ implies that $\dim E^c = 2n$.

We have the same results for a hyperbolic or partially hyperbolic dynamics. Let us notice that in the hyperbolic case, G_- (resp. G_+) is nothing else but the stable (resp. unstable) bundle E^s (resp. E^u).

3) Let us consider the case of a KAM torus that is a graph (when $M = \mathbb{T}^n$): the dynamics on this torus is C^1 conjugated to a flow of irrational translations on the torus $\mathbb{T}^n$; M. Herman proved in [15] that such a torus is Lagrangian, and it is well-known that any invariant Lagrangian graph is locally minimizing. Then the orbit of every vector tangent to the KAM torus is bounded, and belongs to $G_- \cap G_+$. In this case, the two Green bundles are equal to the tangent space to the invariant torus.

Let us introduce some notations:

Notations. Oseledet's theorem implies that there exist an invariant subset N of T^*M with full μ-measure, some real numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_q$ and a (measurable) splitting with constant dimensions above N:

$$T_x\big(T^*M\big) = E_1^s(x) \oplus E_2^s(x) \oplus \cdots \oplus E_q^s(x) \oplus E^c(x) \oplus E_1^u(x) \oplus E_2^u(x) \oplus \cdots \oplus E_q^u(x)$$

such that:

- for every $v \in E_j^s(x)\backslash\{0\}$; $\lim_{t\to\pm\infty} \frac{1}{t}\log(\|D\varphi_t(x)v\|) = -\lambda_j$;
- for every $v \in E^c(x)\backslash\{0\}$; $\lim_{t\to\pm\infty} \frac{1}{t}\log(\|D\varphi_t(x)v\|) = 0$;
- for every $v \in E_j^u(x)\backslash\{0\}$; $\lim_{t\to\pm\infty} \frac{1}{t}\log(\|D\varphi_t(x)v\|) = +\lambda_j$.
  We may ask, too, that: $\forall x \in N$, $\dim(G_-(x) \cap G_+(x)) = p$.

Let us recall that the stable bundle $E^s(x) = E_1^s(x) \oplus E_2^s(x) \oplus \cdots \oplus E_q^s(x)$ and the unstable one $E^u(x) = E_1^u(x) \oplus E_2^u(x) \oplus \cdots \oplus E_q^u(x)$ are isotropic (for the symplectic form) and that $E^c(x)$ is a symplectic subspace of $T_x(T^*M)$ that is orthogonal (for ω) to $E^s(x) \oplus E^u(x)$. Moreover, we have: $\dim E_i^s = \dim E_i^u$.

2.1. Reduction of the problem

As in the statement of Theorem 2, we assume that μ is a minimizing ergodic measure whose support is not reduced to a point and that $p \in [1, n]$ is so that at μ-almost every point x, the intersection of the Green bundles $G_+(x)$ and $G_-(x)$ is p-dimensional. We deduce from the previous remark that for every $x \in N$: $G_+(x) \cap G_-(x) \subset E^c(x)$ and $E^s(x) \oplus E^u(x) = (E^c(x))^\perp \subset G_+(x)^\perp + G_-(x)^\perp = G_-(x) + G_+(x)$.

Notations. We introduce the two notations: $E(x) = G_-(x) + G_+(x)$ and $R(x) = G_-(x) \cap G_+(x)$. We denote the reduced space: $F(x) = E(x)/R(x)$ by $F(x)$ and we denote the canonical projection $p : E \to F$ by p. As G_- and G_+ are invariant by the linearized flow $D\varphi_t$, we may define a reduced cocycle $M_t : F \to F$. But (M_t) is not continuous, because G_- and G_+ don't vary continuously.

Moreover, we introduce the notation: $\mathcal{V}(x) = V(x) \cap E(x)$ is the trace of the linearized vertical on $E(x)$ and $v(x) = p(\mathcal{V}(x))$ is the projection of $\mathcal{V}(x)$ on $F(x)$. We introduce a notation for the images of the reduced vertical $v(x)$ by M_t: $g_t(\varphi_t x) = M_t v(x)$.

The subspace $E(x)$ of $T_x(T^*M)$ is co-isotropic with $E(x)^\perp = R(x)$. Hence $F(x)$ is nothing else than the symplectic space that is obtained by symplectic reduction of $E(x)$. We denote its symplectic form by Ω. Hence we have: $\forall (v, w) \in E(x)^2$, $\Omega(p(v), p(w)) = \omega(v, w)$. Moreover, (M_t) is a symplectic cocycle.

We can notice, too, that $\dim E(x) = \dim(G_-(x) + G_+(x)) = \dim G_-(x) + \dim G_+(x) - \dim(G_-(x) \cap G_+(x)) = 2n - p$ and deduce that $\dim F(x) = \dim E(x) - \dim(G_-(x) \cap G_+(x)) = 2(n - p)$.

Notations. If L is any Lagrangian subspace of $T_x(T^*M)$, we denote $(L \cap E(x)) + R(x)$ by $\tilde{L}$ and $p(\tilde{L})$ by l.

Lemma 4. *If $L \subset T_x(T^*M)$ is Lagrangian, then $\tilde{L}$ is also Lagrangian and $l = p(\tilde{L}) = p(L \cap E(x))$ is a Lagrangian subspace of $F(x)$. Moreover, $p^{-1}(l) = \tilde{L}$. In particular, $v(x)$ is a Lagrangian subspace of $F(x)$ and $p^{-1}(v(x)) = \mathcal{V}(x) + R(x)$.*

Proof. We just have to prove that $\tilde{L}$ is Lagrangian, the other assertions being easy consequences of this fact.

We begin by proving that $\tilde{L}$ is isotropic. If $u, u' \in L \cap E(x)$ and $v, v' \in R(x)$, then $\omega(u + v, u' + v') = 0$ because L is Lagrangian and then $\omega(u, u') = 0$ and because $R(x) \subset E(x)^\perp$.

Let us determine $\dim \tilde{L}$. Let L' be such that: $L = (E(x) \cap L) \oplus L'$. Then the dimension of $L \cap R(x) = (L + E(x))^\perp$ is: $2n - \dim(L + E(x)) = 2n - (2n - p + \dim L') = p - \dim L'$. We deduce: $\dim \tilde{L} = \dim(L \cap E(x)) + \dim R(x) - \dim(L \cap R(x)) = \dim(L \cap E(x)) + p - (p - \dim L') = \dim(L \cap E(x)) + \dim L' = \dim L$. □

Lemma 5. *The subspace $v(x)$ is a Lagrangian subspace of $F(x)$. Moreover, for every $t \neq 0$, $g_t(\varphi_t x) = M_t v(x)$ is transverse to $v(\varphi_t(x))$.*

Proof. The first sentence is contained in Lemma 4.

Let us consider $t \neq 0$ and let us assume that $M_t v(x) \cap v(\varphi_t x) \neq \{0\}$. We may assume that $t > 0$ (or we replace x by $\varphi_t(x)$ and t by $-t$).

Then there exists $v \in \mathcal{V}(x)\backslash\{0\}$ such that $D\varphi_t(x)v \in \mathcal{V}(\varphi_t x) + (G_-(\varphi_t x) \cap G_+(\varphi_t x))$. Let us write $D\varphi_t(x)v = w + g$ with $w \in \mathcal{V}(\varphi_t x)$ and $g \in R(\varphi_t x)$. We know that the orbit has no conjugate vectors (because the measure is minimizing); hence $g \neq 0$.

Moreover, we proved in [1] that $D\varphi_t V(x)$ is strictly above $G_-(\varphi_t x)$, i.e. that:

$$\forall h \in G_-(\varphi_t x), \forall k \in V(\varphi_t x), \quad h + k \in D\varphi_t V(x)\backslash\{0\} \quad \Rightarrow \quad \omega(h, h + k) > 0.$$

We deduce that: $\omega(g, w + g) > 0$.

This contradicts: $D\varphi_t(x)v \in E(\varphi_t x) = (G_+(\varphi_t x) \cap G_-(\varphi_t x))^\perp \subset (\mathbb{R}g)^\perp$. □

As in [1], we ask ourselves what the order between the different Lagrangian subspaces $g_t(x) = M_t v(\varphi_{-t} x)$ is. Let us recall how we define this order:

Definition. Let g_1 and g_2 be two subspaces of $F(x)$ that are transverse to the (reduced) vertical $v(x)$. Let $f(x) = F(x)/v(x)$ be the reduced space and $P(x): F(x) \to f(x)$ the canonical projection. Then to every $w \in f(x)$, we can associate a unique $\ell_1(w) \in g_1$ (resp. $\ell_2(w) \in g_2$) such that: $P(\ell_1(w)) = w$ (resp. $P(\ell_2(w)) = w$). We then define the altitude of g_2 above g_1, which is a quadratic form defined on $f(x)$, by: $q(g_1, g_2)(w) = \Omega(\ell_1(w), \ell_2(w))$.

We say that g_2 is above (resp. strictly above) g_1 when $q(g_1, g_2)$ is positive semi-definite (resp. positive definite). We write $g_1 \leqslant g_2$ (resp. $g_1 < g_2$).

Lemma 6. *Let L_1, L_2 be two Lagrangian subspaces of $T_x(T^*M)$ transverse to $V(x)$ such that at least one of them is contained in $E(x)$. Then, if $L_1 < L_2$ (resp. $L_1 \leqslant L_2$), we have: l_1 and l_2 are transverse to $v(x)$ and $l_1 < l_2$ (resp. $l_1 \leqslant l_2$). We deduce that $p(G_-) < p(G_+)$.*

Proof. We assume that $L_2 \subset E(x)$ and that $L_1 < L_2$. Let $v_1 \in L_1 \cap E(x)$ be a non-zero vector of $L_1 \cap E(x)$. As L_1 and L_2 are transverse to $V(x)$, there exists a unique $v_2 \in L_2$ such that $v_2 - v_1 \in V(x)$. Moreover, as $v_1, v_2 \in E(x)$, we have $v_2 - v_1 \in \mathcal{V}(x)$ and $p(v_2) - p(v_1) \in v(x)$. Hence:

$$\Omega\big(p(v_1), p(v_2)\big) = \omega(v_1, v_2) > 0.$$

This means exactly that $l_1 < l_2$.

To deduce the assertion for $\leqslant$, we can use a limit.

As $G_- \leqslant G_+$, we deduce that $p(G_-) \leqslant p(G_+)$. Because of the definition of $E(x)$, $R(x)$ and $F(x)$, $p(G_-)$ and $p(G_+)$ are transverse and then $p(G_-) < p(G_+)$. □

Lemma 7. *If μ is a minimizing measure, for every $x \in \operatorname{supp}\mu$, for all $0 < t < s$, we have:*

$$g_{-t}(x) < g_{-s}(x) < g_s(x) < g_t(x).$$

Proof. The map $(t \in \mathbb{R}^* \to g_t(x))$ is continuous; moreover, we know by Lemma 5 that if $t \neq s$, then $g_t(x)$ is transverse to $g_s(x)$. Hence, the index of $q(g_s(x), g_t(x))$ is constant for $(s,t) \in \mathcal{E}$ where $\mathcal{E}$ is one of the sets: $\{(s,t);\ 0 < s < t\}$; $\{(s,t);\ s < 0 < t\}$, $\{(s,t);\ s < t < 0\}$. Hence, we only have to determine this index for one point (s,t) of each of these three sets.

We prove the result only for the first set, the other inequalities being very similar. Let us fix $s > 0$ and introduce the notation $G_s(x) = D\varphi_s V(\varphi_{-s}x)$. Then $\tilde{G}_s(x)$ is a Lagrangian subspace of $E(x)$ that is transverse to the vertical because $\tilde{G}_s(x) \cap V(x) = \tilde{G}_s(x) \cap \mathcal{V}(x) = (\tilde{G}_s(x) \cap \tilde{V}(x)) \cap \mathcal{V}(x) = p^{-1}(g_s(x) \cap v(x)) \cap \mathcal{V}(x) = R(x) \cap \mathcal{V}(x) = \{0\}$. We assume that $t > 0$ is very small and we work in a chart, with symplectic coordinates defined in [1] (p. 897) such that the "horizontal" subspace of $T_x(T^*M)$ is $G_-(x)$. A vector of $G_t(x) = D\varphi_t(\varphi_{-t}x)V(\varphi_{-t}x)$ is $(h, S_t^+(x)h)$ and it is proved in [1] (p. 894) that $S_t^+(x) \sim \frac{1}{t}D$ where D is a fixed positive definite matrix. Hence, for $t > 0$ small enough, we have $\tilde{G}_s < G_t$. We deduce from Lemma 6 that $g_s = p(\tilde{G}_s) < p(G_t) = g_t$. □

Definition. As in [1], when t tends to $\pm\infty$, we find two M_t-invariant Lagrangian sub-bundles of $F(x)$ that are: $g_-(x) = \lim_{t\to-\infty} g_t(x)$ and $g_+(x) = \lim_{t\to+\infty} g_t(x)$; they are transverse to $v(x)$ and satisfy: $g_-(x) \leqslant g_+(x)$. We call them the reduced Green bundles.

Remark. Then we have: $\forall t > 0$, $g_{-t}(x) < g_-(x) \leqslant g_+(x) < g_t(x)$. If we use the notations $\tilde{G}_\pm(x) = p^{-1}(g_\pm(x))$, then $\tilde{G}_\pm$ are transverse to the vertical because $\tilde{G}_\pm(x) \cap V(x) = \tilde{G}_\pm(x) \cap \mathcal{V}(x) = (\tilde{G}_\pm \cap \tilde{V}(x)) \cap \mathcal{V}(x) = p^{-1}(g_\pm(x) \cap v(x)) \cap \mathcal{V}(x) = R(x) \cap \mathcal{V}(x) = \{0\}$. Moreover, $\tilde{G}_-(x) \leqslant \tilde{G}_+(x)$ and the two bundles $\tilde{G}_-$, $\tilde{G}_+$, are invariant by the linearized flow $(D\varphi_t)$. Theorem 3.11 of [1] asserts that any invariant Lagrangian bundle that is transverse to the vertical is between the two Green bundles. We deduce that $G_-(x) \leqslant \tilde{G}_-(x) \leqslant \tilde{G}_+(x) \leqslant G_+(x)$. We can then use Lemma 6 and we obtain: $p(G_-(x)) \leqslant g_-(x) \leqslant g_+(x) \leqslant p(G_+(x))$.

Lemma 8. *We have: $\forall x \in \operatorname{supp}\mu$, $g_-(x) = p(G_-(x)) < p(G_+(x)) = g_+(x)$.*

Proof. Because of the last remark, we just have to prove that on $\operatorname{supp}\mu$: $g_- \leqslant p(G_-) < p(G_+) \leqslant g_+$. Because of Lemma 6, we just have to prove that $g_- \leqslant p(G_-)$ and $p(G_+) \leqslant g_+$. But $p(G_\pm)$ is a Lagrangian subspace of $F(x)$

whose orbit is transverse to the vertical. Proposition 3.11 of [1] asserts that any invariant Lagrangian sub-bundle along an orbit that is transverse to the vertical has to be between the two Green bundles. A similar argument in the reduced case implies the inequalities. □

Hence we have proved that $\tilde{G}_\pm = G_\pm$, the notation $\tilde{G}_\pm$ will disappear from now on.

2.2. Reduced Green bundles and Lyapunov exponents

We have to be careful because the bundles that we consider are not continuous and, as this is noted in [1], we don't use a continuous change of coordinates, but just a bounded one when we say that G_- or G_+ is the horizontal subspace (the matrix P that is necessary to change the coordinates is uniformly bounded, as P^{-1}).

We choose at every point $x \in N$ some (linear) symplectic coordinates (Q, P) of $F(x)$ such that $v(x)$ has for equation: $Q = 0$ and $g_+(x)$ has for equation $P = 0$. We will be more precise on this choice later. Then the matrix of $M_t(x)$ in these coordinates is a symplectic matrix: $M_t(x) = \begin{pmatrix} a_t(x) & b_t(x) \\ 0 & d_t(x) \end{pmatrix}$. As $M_t(x)v(x) = g_t(\varphi_t x)$ is a Lagrangian subspace of $E(\varphi_t x)$ that is transverse to the vertical, then $\det b_t(x) \neq 0$ and there exists a symmetric matrix $s_t^+(\varphi_t x)$ whose graph is $g_t(\varphi_t x)$, i.e.: $d_t(x) = s_t^+(\varphi_t(x))b_t(x)$. Moreover, the family $(s_t^+(x))_{t>0}$ being decreasing and tending to zero (because by hypothesis the horizontal is g_+), the symmetric matrix $s_t^+(\varphi_t x)$ is positive definite. Moreover, the matrix $M_t(x)$ being symplectic, we have:

$$\big(M_t(x)\big)^{-1} = \begin{pmatrix} {}^t d_t(x) & -{}^t b_t(x) \\ 0 & {}^t a_t(x) \end{pmatrix}$$

and by definition of $g_{-t}(x)$, if it is the graph of the matrix $s_t^-(x)$ (that is negative definite), then: ${}^t a_t(x) = -s_t^-(x)^t b_t(x)$ and finally:

$$M_t(x) = \begin{pmatrix} -b_t(x)s_t^-(x) & b_t(x) \\ 0 & s_t^+(\varphi_t x)b_t(x) \end{pmatrix}.$$

Let us be now more precise in the way we choose our coordinates; we may associate an almost complex structure J and then a Riemannian metric $(.,.)_x$ defined by: $(v, u)_x = \omega(x)(v, Ju)$ with the symplectic form ω of T^*M; from now on, we work with this fixed Riemannian metric of T^*M. We choose on $G_+(x) = p^{-1}(g_+(x))$ an orthonormal basis whose last vectors are in $R(x)$ and complete it in a symplectic base whose last vectors are in $V(x)$. We denote the associated coordinates of $T_x(T^*M)$ by $(q_1, \dots, q_n, p_1, \dots, p_n)$. These (linear) coordinates don't depend in a continuous way on the point x (because G_+ doesn't), but in a bounded way. Then $G_-(x) = p^{-1}(g_-(x))$ is the graph of a symmetric matrix whose kernel is $R(x)$ and then on $G_-(x)$, we have: $p_{n-p+1} = \dots = p_n = 0$. An element of $E(x)$ has coordinates such that $p_{n-p+1} = \dots = p_n = 0$, and an element of $F(x) = E(x)/R(x)$ may be identified with an element with coordinates $(q_1, \dots, q_{n-p}, 0, \dots, 0, p_1, \dots, p_{n-p}, 0, \dots, 0)$. We then use on $F(x)$ the norm $\sum_{i=1}^{n-p}(q_i^2 + p_i^2)$, which is the norm for the Riemannian metric of the considered element of $F(x)$. Then this norm depends in a measurable way on x.

Let us now notice the following fact: μ being ergodic for the flow (φ_t), there exists a dense G_δ subset A of $\mathbb{R}$ such that, for every $t \in A$, the diffeomorphism φ_t is ergodic. As it is simpler for us to work with a diffeomorphism instead of a flow, we fix such a $t \in A$. We assume that $t = 1$ (if not we replace H by $\frac{1}{t}H$).

Lemma 9. *For every $\varepsilon > 0$, there exists a measurable subset J_ε of N such that*:

- $\mu(J_\varepsilon) \geqslant 1 - \varepsilon$;
- *on J_ε, (s_n^+) and (s_n^-) converge uniformly*;
- *there exist two constants $\beta = \beta(\varepsilon) > \alpha = \alpha(\varepsilon) > 0$ such that*: $\forall x \in J_\varepsilon$, $\beta \mathbf{1} \geqslant -s_-(x) \geqslant \alpha \mathbf{1}$ *where g_- is the graph of s_-*.

Proof. This is a consequence of the Egorov theorem and of the fact that on N, g_+ and g_- are transverse and then $-s_-$ is positive definite. □

We deduce:

Lemma 10. *Let J_ε be as in the previous lemma. On the set $\{(n,x)\in\mathbb{N}\times J_\varepsilon, \varphi_n(x)\in J_\varepsilon\}$, the sequence of conorms $(m(b_n(x)))$ converges uniformly to $+\infty$, where $m(b_n)=\|b_n^{-1}\|^{-1}$.*

Proof. Let n, x be as in the lemma.

The matrix $M_n(x)=\begin{pmatrix} -b_n(x)s_n^-(x) & b_n(x)\\ 0 & s_n^+(\varphi_n x)b_n(x)\end{pmatrix}$ being symplectic, we have: $-s_n^-(x)^t b_n(x)s_n^+(\varphi_n x)b_n(x)=\mathbf{1}$ and thus $-b_n(x)s_n^-(x)^t b_n(x)s_n^+(\varphi_n x)=\mathbf{1}$ and $b_n(x)s_n^-(x)^t b_n(x)=-(s_n^+(\varphi_n x))^{-1}$.

We know that on J_ε, (s_n^+) converges uniformly to zero. Hence, for every $\delta>0$, there exists $N=N(\delta)$ such that: $n\geqslant N\Rightarrow \|s_n^+(\varphi_n x)\|\leqslant\delta$. Moreover, we know that $\|s_n^-(x)\|\leqslant\beta$. Hence, if we choose $\delta'=\frac{\delta^2}{\beta}$, for every $n\geqslant N=N(\delta')$ and $x\in J_\varepsilon$ such that $\varphi_n x\in J_\varepsilon$, we obtain:

$$\forall v\in\mathbb{R}^p,\quad \beta\left\|{}^t b_n(x)v\right\|^2={}^t v b_n(x)(\beta\mathbf{1})^t b_n(x)v\geqslant -{}^t v b_n(x)s_n^-(x)^t b_n(x)v={}^t v\bigl(s_n^+(\varphi_n x)\bigr)^{-1}v$$

and we have: ${}^t v(s_n^+(\varphi_n x))^{-1}v\geqslant\frac{\beta}{\delta^2}\|v\|^2$ because $s_n^+(\varphi_n x)$ is a positive definite matrix that is less than $\frac{\delta^2}{\beta}\mathbf{1}$. We finally obtain: $\|{}^t b_n(x)v\|\geqslant\frac{1}{\delta}\|v\|$ and then the result that we wanted. □

From now we fix a small constant $\varepsilon>0$, associate a set J_ε with ε via Lemma 9 and two constants $0<\alpha<\beta$; then there exists $N\geqslant 0$ such that

$$\forall x\in J_\varepsilon,\ \forall n\geqslant N,\quad \varphi_n(x)\in J_\varepsilon\quad\Rightarrow\quad m\bigl(b_n(x)\bigr)\geqslant\frac{2}{\alpha}.$$

Lemma 11. *Let J_ε be as in Lemma 9. For μ-almost every point x in J_ε, there exists a sequence of integers $(j_n)=(j_n(x))$ tending to $+\infty$ such that*:

$$\forall n\in\mathbb{N},\quad m\bigl(b_{j_n}(x)s_{j_n}(x)\bigr)\geqslant\bigl(2^{\frac{1-\varepsilon}{2N}}\bigr)^{j_n}.$$

Proof. As μ is ergodic for φ_1, we deduce from the Birkhoff ergodic theorem that for almost every point $x\in J_\varepsilon$, we have:

$$\lim_{\ell\to+\infty}\frac{1}{\ell}\sharp\bigl\{0\leqslant k\leqslant\ell-1;\ \varphi_k(x)\in J_\varepsilon\bigr\}=\mu(J_\varepsilon)\geqslant 1-\varepsilon.$$

We introduce the notation: $N(\ell)=\sharp\{0\leqslant k\leqslant\ell-1;\ \varphi_k(x)\in J_\varepsilon\}$.

For such an x and every $\ell\in\mathbb{N}$, we find a number $n(\ell)$ of integers:

$$0=k_1\leqslant k_1+N\leqslant k_2\leqslant k_2+N\leqslant k_3\leqslant k_3+N\leqslant\cdots\leqslant k_{n(\ell)}\leqslant\ell$$

such that $\varphi_{k_i}(x)\in J_\varepsilon$ and $n(\ell)\geqslant[\frac{N(\ell)}{N}]\geqslant\frac{N(\ell)}{N}-1$. In particular, we have: $\frac{n(\ell)}{\ell}\geqslant\frac{1}{N}(\frac{N(\ell)}{\ell}-\frac{N}{\ell})$, the right term converging to $\frac{\mu(J_\varepsilon)}{N}\geqslant\frac{1-\varepsilon}{N}$ when ℓ tends to $+\infty$. Hence, for ℓ large enough, we find: $n(\ell)\geqslant 1+\ell\frac{1-\varepsilon}{2N}$.

As $\varphi_{k_i}(x)\in J_\varepsilon$ and $k_{i+1}-k_i\geqslant N$, we have: $m(b_{k_{i+1}-k_i}(\varphi_{k_i}(x)))\geqslant\frac{2}{\alpha}$. Moreover, we have: $m(s^-_{k_{i+1}-k_i}(\varphi_{k_i}x))\geqslant\alpha$; hence:

$$m\bigl(b_{k_{i+1}-k_i}(\varphi_{k_i}x)s^-_{k_{i+1}-k_i}(\varphi_{k_i}x)\bigr)\geqslant 2.$$

But the matrix $-b_{k_{n(\ell)}}(x)s^-_{k(n(\ell))}(x)$ is the product of $n(\ell)-1$ such matrices. Hence:

$$m\bigl(b_{k_{n(\ell)}}(x)s^-_{k(n(\ell))}(x)\bigr)\geqslant 2^{n(\ell)-1}\geqslant 2^{\ell\frac{1-\varepsilon}{2N}}\geqslant\bigl(2^{\frac{1-\varepsilon}{2N}}\bigr)^{k_{n(\ell)}}.\qquad\square$$

Let us now come back to the whole tangent space $T_x(T^*M)$ with a slight change in the coordinates that we use. We defined the symplectic coordinates $(q_1,\dots,q_n,p_1,\dots,q_n)$ and now we use the non-symplectic ones: $(Q_1,\dots,Q_n,P_1,\dots,P_n)=(q_{n-p+1},\dots,q_n,q_1,\dots,q_{n-p},p_1,\dots,p_n)$. Then:

- $(Q_1,\dots,Q_p)$ are coordinates in $R(x)$;
- $(Q_1,\dots,Q_n)$ are coordinates in $G_+(x)$;
- $(Q_1,\dots,Q_n,P_1,\dots,P_{n-p})$ are coordinates of $E(x)=G_+(x)+G_-(x)$.

We write then the matrix of $D\varphi_t(x)$ in these coordinates $(Q_1,\dots,Q_n,P_1,\dots,P_n)$ (which are not symplectic):

$$\begin{pmatrix} A_t^1(x) & A_t^2(x) & A_t^3(x) & A_t^4(x)\\ 0 & b_t(x)s_t^-(x) & b_t(x) & A_t^5(x)\\ 0 & 0 & s_t^+(\varphi_t x)b_t(x) & A_t^6(x)\\ 0 & 0 & 0 & A_t^9(x)\end{pmatrix}$$

where the blocks correspond to the decomposition $T_x(T^*M)=E_1(x)\oplus E_2(x)\oplus E_3(x)\oplus E_4(x)$ with $\dim E_1(x)=\dim E_4(x)=p$ and $\dim E_2(x)=\dim E_3(x)=n-p$.

We have noticed that $E_1(x)=E(x)\subset E^c(x)$ and that $G_+(x)=E_1(x)\oplus E_2(x)$.

If $x\in J_\varepsilon$, we have found a sequence (j_n) of integers tending to $+\infty$ so that:

$$\forall n\in\mathbb{N},\quad m\big(b_{j_n}(x)s_{j_n}^-(x)\big)\geqslant\big(2^{\frac{1-\varepsilon}{2N}}\big)^{j_n}.$$

We deduce:

$$\forall v\in E_2(x)\backslash\{0\},\quad \frac{1}{j_n}\log\big(\big\|b_{j_n}(x)s_{j_n}^-(x)v\big\|\big)\geqslant\frac{1-\varepsilon}{2N}\log 2+\frac{\|v\|}{j_n};$$

and because $E_1(x)\subset E^c(x)$:

$$\forall v\in G_+(x)\backslash E_1(x),\quad \liminf_{n\to\infty}\frac{1}{n}\log\big\|D\varphi_n(x)v\big\|\geqslant\frac{1-\varepsilon}{2N}\log 2.$$

Hence there are at least $n-p$ Lyapunov exponents bigger than $\frac{1-\varepsilon}{2N}\log 2$ and then bigger than 0 for the linearized flow. Because this flow is symplectic, we deduce that it has at least $n-p$ negative Lyapunov exponents (see [6]). As we noticed that the linearized flow has at least $2p$ zero Lyapunov exponents, we deduce that μ has exactly $n-p$ positive Lyapunov exponents, exactly $n-p$ negative Lyapunov exponents and exactly $2p$ zero Lyapunov exponents.

This finishes the proof of Theorem 2.

Remark. Let us notice that we proved too that for $x\in N$ (i.e. generic in the Oseledet's sense), we have: $E^u(x)\subset G_+(x)$, and then $G_+(x)=E^u(x)\oplus R(x)$.

3. Weak KAM solutions and Green bundles

In this section, we recall the weak KAM theory and give a relationship between some tangent cones to the pseudographs of the weak KAM solutions and the Green bundles. These results imply Theorem 1. The proofs are given in Section 4.

3.1. Weak KAM theory

We don't give any proof in this section, but all the results that we give are proved in [10] or [4].

Notations. If $t>0$, the function $A_t: M\times M\to\mathbb{R}$ is defined by:

$$A_t(q_0,q_1)=\inf_\gamma\int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,ds=\min_\gamma\int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,ds$$

where the infimum is taken on the set of C^2 curves $\gamma:[0,t]\to M$ such that $\gamma(0)=q_0$ and $\gamma(t)=q_1$.

Definition.

1. A function $v: V\to\mathbb{R}$ defined on a subset V of $\mathbb{R}^d$ is K-*semi-concave* if for every $Q\in V$, there exists a linear form ψ_Q defined on $\mathbb{R}^d$ so that:

   $$\forall Q'\in V,\quad v(Q')\leqslant v(Q)+\psi_Q(Q'-Q)+K\|Q'-Q\|^2.$$

   Then we say that ψ_Q is a *K-super-differential* of v at Q.

2. Let us fix a finite atlas $\mathcal{A}$ of the manifold M; a function $u : M \to \mathbb{R}$ is *K-semi-concave* if for every chart (U, ϕ) belonging to $\mathcal{A}$, $u \circ \phi^{-1}$ is K-semi-concave. Then a *K-super-differential* of u at q is $\psi_Q \circ D\phi(q)$ where ψ_Q is a K-super-differential of $u \circ \phi^{-1}$ at $Q = \phi(q)$.

A semi-concave function is always Lipschitz and then differentiable almost everywhere and for such a function, we define its pseudograph: a *pseudograph* is the graph $\mathcal{G}(du)$ of du, where $u : M \to \mathbb{R}$ is a semi-concave function.

A function $u : M \to \mathbb{R}$ is K-semi-convex if $-u$ is K-semi-concave. We have a notion of sub-differential and the anti-pseudograph of a semi-convex function u is $\mathcal{G}(du)$.

It is proved in [4] that A_t is semi-concave and that for every minimizing curve $\gamma : [0, t] \to M$ between q_0 and q_1, $(-\frac{\partial L}{\partial v}(\gamma(0), \dot\gamma(0)), \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t)))$ is a super-differential of A_t at (q_0, q_1). It is proved, too, that $A_t(., q_1)$ is differentiable at q_0 if and only if $A_t(q_0, .)$ is differentiable at q_1 if and only if there exists a unique minimizing curve $\gamma : [0, t] \to M$ joining q_0 to q_1.

We denote the two Lax–Oleinik semi-groups associated with L by $(T_t)_{t>0}$ and $(\breve{T}_t)_{t>0}$; for $u \in C^0(M, \mathbb{R})$, they are defined by:

$$T_t u(q) = \min_{q' \in M}\big(u(q') + A_t(q', q)\big) \quad \text{and} \quad \breve{T}_t u(q) = \max_{q' \in M}\big(u(q') - A_t(q, q')\big).$$

A function $u : M \to \mathbb{R}$ is a negative (resp. positive) weak KAM solution if there exists $c \in \mathbb{R}$ such that: $\forall t > 0$, $T_t u = u - ct$ (resp. $\forall t > 0$, $\breve{T}_t u = u + ct$).

Then there exist at least one positive and one negative weak KAM solutions (see [10] or [4]). The constant c is unique and is called Mañé's critical value. If u_- is a negative weak KAM solution and u_+ a positive one, then u_- is semi-concave and u_+ is semi-convex. Let us introduce the Mather set:

Definition. The Mather set, denoted by $\mathcal{M}^*(H)$, is the union of the supports of the minimizing measures. The projected Mather set is $\mathcal{M}(H) = \pi(\mathcal{M}^*(H))$.

J. Mather proved that $\mathcal{M}^*(H)$ is compact, non-empty and that it is a Lipschitz graph above a compact part of the zero-section of T^*M.

A. Fathi proved in [10] that if u_- is a negative weak KAM solution, there exists a unique positive weak KAM solution u_+ such that $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)}$. Such a pair (u_-, u_+) is called a pair of conjugate weak KAM solutions. For such a pair, we have:

- $\forall q \in \mathcal{M}(H), u_-(q) = u_+(q)$; let us denote the set of equality: $\mathcal{I}(u_-, u_+) = \{q;\ u_-(q) = u_+(q)\}$ by $\mathcal{I}(u_-, u_+)$; then $\mathcal{M}(H) \subset \mathcal{I}(u_-, u_+)$;
- u_- and u_+ are differentiable at every point $q \in \mathcal{I}(u_-, u_+)$; for such a q we have $(q, du_-(q)) \in \mathcal{N}^*(H)$; when $q \in \mathcal{M}(H)$ and $(q, p) \in \mathcal{M}^*(H)$ is its lift to $\mathcal{M}^*(H)$, then $du_-(q) = du_+(q) = p$;
- $u_+ \leqslant u_-$.

Moreover, it is proved in [4] that if q is a point of differentiability of $T_t u$ (resp. $\breve{T}_t u$), then the minimum (resp. maximum) in the definition of $T_t u(q)$ (resp. $\breve{T}_t u$) is attained at a unique q' and there is a unique curve $\gamma : [0, t] \to M$ minimizing between q' and q (resp. q and q'); in this case: $\frac{\partial L}{\partial v}(q, \dot\gamma(t)) = dT_t u(q)$ (resp. $\frac{\partial L}{\partial v}(q, \dot\gamma(0)) = d\breve{T}_t u(q)$).

3.2. Comparison between the weak KAM solutions and the Green bundles

If (u_-, u_+) is a pair of conjugate weak KAM solutions, if $q \in \mathcal{I}(u_-, u_+)$, we have seen that $(q, du_-(q)) = (q, du_+(q)) \in \mathcal{N}^*(H)$. Hence, the two Green subspaces $G_-(q, du_-(q))$ and $G_+(q, du_+(q))$ exist.

We always have $G_- \leqslant G_+$, the bundle G_- is lower semi-continuous and the bundle G_+ is upper semi-continuous, hence they are continuous at the points where $G_- = G_+$.

Notations. If the orbit of x is minimizing, $G_-(x)$ is the graph of a symmetric matrix $s_-(x)$ and $G_+(x)$ the graph of a symmetric matrix $s_+(x)$. If $\Delta s(x) = s_+(x) - s_-(x)$, then $\Delta s(x)$ is positive semi-definite.

Moreover, if s is a positive semi-definite matrix, we will denote by proj_s the orthogonal projection on its image $\mathrm{Im}(s)$ and by $\|s\|$ its greatest eigenvalue.

Let us recall that if $x \in A \subset T^*M$, $\mathcal{C}_x A$ designates the contingent cone to A at x, that was defined in the introduction.

Theorem 12. *Let (u_-, u_+) be a pair of conjugate weak KAM solutions and let q belong to $\mathcal{I}(u_-, u_+)$. Then we have*: $\forall (Q, P) \in \mathcal{C}_{(q,du_-(q))}\mathcal{G}(du_-)$,

$$\big\|P - s_-\big(q, du_-(q)\big)Q\big\| \leqslant 2\big\|\Delta s\big(q, du_-(q)\big)\big\|.\big\|\mathrm{proj}_{\Delta s(q,du_-(q))}(Q)\big\|$$

and: $\forall (Q, P) \in \mathcal{C}_{(q,du_+(q))}\mathcal{G}(du_+)$,

$$\big\|P - s_+\big(q, du_+(q)\big)Q\big\| \leqslant 2\big\|\Delta s\big(q, du_+(q)\big)\big\|.\big\|\mathrm{proj}_{\Delta s(q,du_+(q))}(Q)\big\|.$$

We postpone the proof of this theorem to Section 4.
As $\mathcal{M}^*(H) \subset \mathcal{G}(du_-) \cap \mathcal{G}(du_+)$, we deduce:

Corollary 13. *If x is an element of $\mathcal{M}^*(H)$, then we have*: $\forall (Q, P) \in \mathcal{C}_x\mathcal{M}^*(H)$,

$$\max\big\{\big\|P - s_-(x)Q\big\|, \big\|P - s_+(x)Q\big\|\big\} \leqslant 2\big\|\Delta s(x)\big\|.\big\|\mathrm{proj}_{\Delta s(x)}(Q)\big\|.$$

We now prove Theorem 1. We first use Theorem 2: if μ is an ergodic minimizing measure whose Lyapunov exponents are zero, then we have μ-almost everywhere: $G_- = G_+$ i.e. $\Delta s = 0$. We deduce from Corollary 13 that $\mathcal{C}_x(\mathrm{supp}\,\mu) \subset G_-(x) = G_+(x)$ at μ-almost every point. This implies that $\mathrm{supp}\,\mu$ is 1-regular at x, and even that it is C^1-regular at x. Indeed, if (x_n) is a sequence of points of $\mathrm{supp}(\mu)$ that converges to x and $v_n = (Q_n, P_n) \in \mathcal{C}_{x_n}(\mathrm{supp}\,\mu)$ converges to $v = (Q, P)$, we have for every n:

$$\big\|P_n - s_-(x_n)Q_n\big\| \leqslant 2\Delta s(x_n)\big\|\mathrm{proj}_{\Delta s(x_n)}(Q_n)\big\|.$$

As $G_-(x) = G_+(x)$, s_- and Δs are continuous at x. We deduce that $\|P - s_-(x)Q\| = 0$ and then $(Q, P) \in G_-(x)$. We have then proved:

Corollary 14. *If μ is an ergodic minimizing measure all of whose Lyapunov exponents are zero, then,* $\mathrm{supp}\,\mu$ *is C^1-regular at μ-almost every point.*

This is exactly Theorem 1.

Remark. Corollary 13 implies a result more precise than Theorem 1. It implies that above $D\pi(G_-(x) \cap G_+(x))$ for every $x \in \mathcal{M}^*(H)$, the points of the contingent of $\mathcal{M}^*(H)$ at x are contained in $G_-(x) \cap G_+(x)$:

$$\mathcal{C}_x\mathcal{M}^*(H) \cap D\pi^{-1}\big(D\pi\big(G_-(x) \cap G_+(x)\big)\big) \subset G_-(x) \cap G_+(x).$$

4. Proof of the results of Section 3

In this section, we use the images of the physical verticals to obtain a control of the weak KAM solutions. More precisely, we can choose a graph in the image of a vertical, the graph of da for a certain function a, and prove a certain inequality between a and the considered weak KAM solution u. Then we deduce an inequality along some subset of the Mañé set between the "second derivatives" of a and u. This gives a relationship between the Green bundles and the Bouligand's contingent cones to the pseudograph of any weak KAM solution along some subset of the Mañé set.

4.1. Selection of some graphs in the images of the verticals

Notations.

- If $q \in M$, we denote the (physical) vertical $\pi^{-1}(\{q\})$ by $\mathcal{V}(q) \subset T^*M$.
- If $t > 0$, the function $A_t : M \times M \to \mathbb{R}$ is defined by:

$$A_t(q_0, q_1) = \inf_\gamma \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds = \min_\gamma \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds$$

  where the infimum is taken on the set of C^2 curves $\gamma : [0, t] \to M$ such that $\gamma(0) = q_0$ and $\gamma(t) = q_1$.
- if $u : M \to \mathbb{R}$ is a Lipschitz function, then by Rademacher's theorem, it is differentiable (Lebesgue) almost everywhere and the graph of its derivative is denoted by:

$$\mathcal{G}(du) = \big\{\big(q, du(q)\big);\ u \text{ is differentiable at } q\big\}.$$

Tonelli's theorem asserts that for every $t \neq 0$, $\pi \circ \varphi_t(\mathcal{V}(q)) = M$ (i.e. for every $q' \in M$ there exists a solution to the Euler–Lagrange equations γ such that $\gamma(0) = q$ and $\gamma(t) = q'$); but in general $\varphi_t(\mathcal{V}(q))$ is not a graph. To select a graph in $\varphi_t(\mathcal{V}(q))$, we prove:

Proposition 15. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and $L : TM \to \mathbb{R}$ be the associated Lagrangian. Then for every $t > 0$ and every $q \in M$, the function $v_q^t = A_t(q, .)$ and $v_q^{-t} = A_t(., q)$ are semi-concave, and satisfy*:

$$\mathcal{G}\big(dv_q^t\big) \subset \varphi_t\big(\mathcal{V}(q)\big) \quad \textit{and} \quad \mathcal{G}\big(-dv_q^{-t}\big) \subset \varphi_{-t}\big(\mathcal{V}(q)\big).$$

Proof. Because A_t is semi-concave, the two functions v_q^t and v_q^{-t} are semi-concave and then Lipschitz. By Rademacher's theorem they are differentiable almost everywhere.

Moreover, if q_0 is a point where v_q^t is differentiable, then v_q^t has exactly one super-differential at this point, there is only one minimizing arc γ joining $(0, q)$ to (t, q_0), and we have:

- $dv_q^t(q_0) = \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))$;
- $(\gamma(0), \frac{\partial L}{\partial v}(\gamma(0), \dot\gamma(0))) = (q, \frac{\partial L}{\partial v}(\gamma(0), \dot\gamma(0))) \in \mathcal{V}(q)$;
- $\varphi_t(q, \frac{\partial L}{\partial v}(\gamma(0), \dot\gamma(0))) = (\gamma(t), \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))) = (q_0, dv_q^t(q_0))$.

Then we have proved that: $\varphi_t(\mathcal{V}(q)) \supset \mathcal{G}(dv_q^t)$. Hence, we have selected a pseudograph in the image $\varphi_t(\mathcal{V}(q))$ of the vertical.

In a very similar way, we may see that the anti-pseudograph of the semi-convex function $-v_q^{-t}$ is a subset of $\varphi_{-t}(\mathcal{V}(q))$: $\mathcal{G}(-dv_q^{-t}) \subset \varphi_{-t}(\mathcal{V}(q))$. □

4.2. Local smoothness of some of these graphs

Notations. For every $x \in T^*M$, we denote the *linear* vertical at x by $V(x)$: $V(x) = \ker D\pi(x) = T_x\mathcal{V}(\pi(x)) \subset T_x(T^*M)$.

The images of the linear vertical are denoted by: $G_t(x) = D\varphi_t V(\varphi_{-t}x)$.

We recall that an orbit piece $(\varphi_t(x))_{t\in[a,b]}$ has no conjugate vectors if:

$$\forall s \neq t \in [a, b], \quad G_{t-s}(\varphi_t x) \cap V(\varphi_t x) = D\varphi_{t-s}\Big(V\big(\varphi_s(x)\big)\Big) \cap V\big(\varphi_t(x)\big) = \{0\}.$$

Notations. Let us now fix a minimizing arc $\gamma : [-t, 0] \to M$ such that:

- there is only one minimizing arc between $(-t, \gamma(-t))$ and $(0, \gamma(0))$ (then it is γ);
- the orbit piece $(\gamma(\tau), \frac{\partial L}{\partial v}(\gamma(\tau), \dot\gamma(\tau)))_{\tau\in[-t,0]}$ has no conjugate vectors.

Let us notice that when $(q, p) \in \mathcal{N}^*(H)$, then any piece of the curve $(t \to \pi \circ \varphi_t(q, p))$ satisfies the previous hypotheses.

We define a function $a_t^+ : M \to \mathbb{R}$ by: $a_t^+(q) = v^t_{\gamma(-t)}(q) = A_t(\gamma(-t), q)$ (this function depends on γ). In a similar way, we can consider $x_0 = (q_0, p_0)$ such that the orbit $(\varphi_s(x_0))_{s\in[0,t]}$ has no conjugate points and so that there is only one minimizing arc $\gamma : [0, t] \to M$ joining q_0 to q_t. We define a function $a_t^- : M \to \mathbb{R}$ by: $a_t^- = -v^{-t}_{q_t}(q) = -A_t(q, q_t)$.

Proposition 16. *Let $\gamma : [-t, 0] \to M$ (resp. $\gamma : [0, t] \to M$) be a minimizing arc such that*:

- *γ is the only minimizing arc joining its two ends*;
- *the orbit piece $(\gamma, \frac{\partial L}{\partial v}(\gamma, \dot\gamma))$ has no conjugate vectors.*

Then there exists a neighborhood V_0 of $q_0 = \gamma(0)$ in M such that $a^+_{t|V_0}$ (resp. $a^-_{t|V_0}$) is as regular as H is (then at least C^2).

Proof. We have seen that: $\mathcal{G}(da_t^+) \subset \varphi_t(V(q_{-t}))$. Let us now prove that a_t^+ is smooth near q_0.

We use now the so-called "a priori compactness lemma" (see [10]) that says to us that there exists a constant $K_t = K > 0$ such that the velocities $(\dot\gamma(s))_{s\in[0,t]}$ of any minimizing arc between any points $q \in M$ and $q' \in M$ are bounded by K; hence if we denote the set of the minimizing arcs that are parametrized by $[0, t]$ by $\mathcal{K}$, $\mathcal{K}$ is a compact set for the C^1 topology because it is the image by the projection π of a closed set of bounded orbits. Let us denote the set of $\gamma \in \mathcal{K}$ such that $\gamma(0) = q_{-t}$ by $\mathcal{K}_0$; then $\mathcal{K}_0$ is compact. Let us introduce another notation: $\mathcal{K}(q) = \{\gamma \in \mathcal{K}_0; \gamma(t) = q\}$. Then $\mathcal{K}(q_0) = \{\gamma_0\}$ and hence, because $\mathcal{K}_0$ is closed, for q close enough to q_0, all the elements of $\mathcal{K}(q)$ are C^1 close to γ_0.

Moreover, $\varphi_t(\mathcal{V}(q_{-t}))$ is a sub-manifold of M that contains $(q_0, \frac{\partial L}{\partial v}(q_0, \dot\gamma_0(0))) = (q_0, p_0)$. Its tangent space at (q_0, p_0) is $G_t(q_0, p_0)$, which is transverse to the vertical because $(q_s, p_s)_{s\in[-t,0]}$ has no conjugate vectors. Hence, the manifold $\varphi_t(\mathcal{V}(q_{-t}))$ is, in a neighborhood U_0 of (q_0, p_0), the graph of a C^1 section of T^*M defined on a neighborhood V_0 of q_0 in M. Moreover, because this sub-manifold is Lagrangian (indeed, $\mathcal{V}(q_{-t})$ is Lagrangian and φ_t is symplectic), it is the graph of du_0 where $u_0 : V_0 \to \mathbb{R}$ is a C^2 function.

Now, if q is close enough to q_0, we know that all the elements γ of $\mathcal{K}(q)$ are C^1 close to γ_0, and then that $(q, \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t)))$ belongs to the neighborhood U_0 of $(q_0, p_0) = (q_0, \frac{\partial L}{\partial v}(\gamma_0(t), \dot\gamma_0(t)))$ and to $\varphi_t(\mathcal{V}(q_{-t}))$. Because $\varphi_t(\mathcal{V}(q_{-t})) \cap U_0$ is a graph, this element is unique: $\mathcal{K}(q)$ has only one element and a_t^+ is differentiable at q, with $da_t^+(q) = \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t)) = du_0(q)$. We deduce that near q_0, on the set of differentiability of a_t^+, da_t^+ is equal to du_0; because a_t^+ and u_0 are Lipschitz on V_0 and their differentials are equal almost everywhere, we deduce that on V_0, $a_t^+ - u_0$ is constant. Hence, on a neighborhood V_0 of q_0, a_t^+ is C^2.

In a similar way, using the fact that $\mathcal{G}(da_t^-) \subset \varphi_{-t}(V(q_t))$, we obtain that a_t^- is C^2 near q_0. □

Remark. If $x_0 = (q_0, p_0)$ is a point of the Mañé set, $(q_t, p_t)_{t\in\mathbb{R}} = (\varphi_t(q_0, p_0))_{t\in\mathbb{R}}$ has no conjugate vectors and for every $t < \tau$, there is only one minimizing arc $\gamma : [t, \tau] \to M$ joining q_t to q_τ, hence for every $t > 0$ the two functions $a^+_{x_0,t}$ and $a^-_{x_0,t}$ are smooth near q_0 (of course the neighborhood of q_0 where they are smooth depends on t).

4.3. Comparison between the weak KAM solutions and the maps a_t^+ and a_t^-

Lemma 17. *We assume that u_- is a negative weak KAM solution and that u_+ is a positive weak KAM solution. Let $q_0 \in M$ be a point of differentiability of u_- (resp. u_+) and a_t^+ (resp. a_t^-) be the function built in the previous subsection for the arc $\gamma = (\pi \circ \varphi_s(q_0, du_-(q_0)))_{s\in[-t,0]}$ (resp. $\gamma = (\pi \circ \varphi_s(q_0, du_+(q_0)))_{s\in[0,t]}$). Then, in a chart*: $u_-(q) - u_-(q_0) - du_-(q_0)(q - q_0) \leqslant a_t^+(q) - a_t^+(q_0) - da_t^+(q_0)(q - q_0)$ (*resp.* $a_t^-(q) - a_t^-(q_0) - da_t^-(q_0)(q - q_0) \leqslant u_+(q) - u_+(q_0) - du_+(q_0)(q - q_0)$).

Proof. Let us consider q_0 in M that is a point of differentiability of a weak KAM solution u_- and let us denote the point above q_0 on the pseudograph $\mathcal{G}(du_-)$ of u_- by x_0: $x_0 = (q_0, du_-(q_0))$. Then, for every $t > 0$, because $T_t u_- = u_- - ct$ is differentiable at q_0, there is only one point $q \in M$ such that $u_-(q_0) = T_t u_-(q_0) + ct = u_-(q) + A_t(q, q_0) +$

ct and only one minimizing arc $\gamma : [-t, 0] \to M$ joining q to q_0. We introduce the notation: $x_t = (q_t, p_t) = \varphi_t(x_0)$. Then: $T_t u_-(q_0) = u_-(q_{-t}) + A(q_{-t}, q_0)$; moreover: $T_t u_-(q) \leqslant u_-(q_{-t}) + A(q_{-t}, q) = T_t u_-(q_0) + A(q_{-t}, q) - A(q_{-t}, q_0)$. Finally: $u_-(q) - u_-(q_0) \leqslant a_t^+(q) - a_t^+(q_0)$. Because these two maps a_t^+ and u_- are differentiable at q_0, they have the same differential at this point and we obtain (in chart): $u_-(q) - u_-(q_0) - du_-(q_0)(q - q_0) \leqslant a_t^+(q) - a_t^+(q_0) - da_t^+(q_0)(q - q_0)$.

Using the same argument for u_+, we obtain: if q_0 is a point of differentiability of u_+:

$$a_t^-(q) - a_t^-(q_0) - da_t^-(q_0)(q - q_0) \leqslant u_+(q) - u_+(q_0) - du_+(q_0)(q - q_0). \qquad \square$$

Now we would like to use these inequalities at different points q_0; we have to be careful, because a_t^+ and a_t^- depend on the point q_0 that we choose. That is why we change now our notation, replacing a_t^+ by $a_{q_0,t}^+$ if the considered point is $(q_0, du_-(q_0))$ and a_t^- by $a_{q_0,t}^-$ if the considered point is $(q_0, du_+(q_0))$.

Proposition 18. *We assume that (u_-, u_+) is a pair of conjugate weak KAM solutions. Let $q \in \mathcal{I}(u_-, u_+)$ be a point, (q_n) be a sequence of points of M converging to q, and (λ_n) be a sequence of positive real numbers so that the two limits (written in charts) $\lim_{n\to\infty} \frac{q_n - q}{\lambda_n} = Q$ and $P = \lim_{n\to\infty} \frac{du_-(x_n) - du_-(y)}{\lambda_n}$ (resp. $\lim_{n\to\infty} \frac{du_+(q_n) - du_+(q)}{\lambda_n}$) exist. Then we have*:

$$\forall k \in \mathbb{R}^n, \quad P.k \leqslant \frac{1}{2}\big(d^2 a_{q,t}^+(q)(k, k) + d^2 a_{q,t}^+(q)(Q, Q) - d^2 a_{q,t}^-(q)(Q - k, Q - k)\big)$$

(*resp.*:

$$\forall k \in \mathbb{R}^n, \quad \frac{1}{2}\big(d^2 a_{q,t}^-(q)(k, k) + d^2 a_{q,t}^-(q)(Q, Q) - d^2 a_{q,t}^+(q)(k - Q, k - Q)\big) \leqslant P.k).$$

Proof. We work in a chart, and we have, if $q \in \mathcal{I}(u_-, u_+)$ and q' is a point of differentiability of u_-:

- $u_-(q' + h) - u_-(q') - du_-(q')h \leqslant a_{q',t}^+(q' + h) - a_{q',t}^+(q') - da_{q',t}^+(q')h$;
- $u_-(q') - u_-(q) - du_-(q)(q' - q) \leqslant a_{q,t}^+(q') - a_{q,t}^+(q) - da_{q,t}^+(q)(q' - q)$;
- $a_{q,t}^-(q' + h) - a_{q,t}^-(q) - da_{q,t}^-(q)(q' + h - q) \leqslant u_+(q' + h) - u_+(q) - du_+(q)(q' + h - q)$.

Hence, by adding these three inequalities and using that $u_-(q) = u_+(q)$, $du_-(q) = du_+(q)$ and $u_+ \leqslant u_-$:

$$\begin{aligned}\big(du_-(q) - du_-(q')\big)h \leqslant{}& a_{q',t}^+\big(q' + h\big) - a_{q',t}^+\big(q'\big) - da_{q',t}^+\big(q'\big)h + a_{q,t}^+\big(q'\big) - a_{q,t}^+(q)\\ &- da_{q,t}^+(q)\big(q' - q\big) - a_{q,t}^-\big(q' + h\big) + a_{q,t}^-(q) + da_{q,t}^-(q)\big(q' + h - q\big).\end{aligned}$$

We now need to precise the regularity of the maps: $q' \to da_{q',t}^-$ and $q' \to da_{q',t}^+$. To do that, we prove a lemma. We fix a finite atlas of M to write that u_- is K-semi-concave and that u_+ is K-semi-convex. The proof is very similar to the one given by A. Fathi in [10] to prove that the Aubry set is a Lipschitz graph.

Lemma 19. *There exists a constant $K > 0$ such that, for every $q \in \mathcal{I}(u_-, u_+)$ and every $q' \in M$ where u_- (resp. u_+) is differentiable, then $\|du_-(q) - du_-(q')\| \leqslant K\|q - q'\|$ (resp. $\|du_+(q) - du_+(q')\| \leqslant K\|q - q'\|$). In particular, du_- and du_+ are continuous at every point of $\mathcal{I}(u_-, u_+)$.*

Proof. Because $u_+ \leqslant u_-$, u_- is semi-concave and u_+ is semi-convex, then u_- is K-semi-convex at every point of $\mathcal{I}(u_-, u_+)$; hence:

- $u_-(q' + h) - u_-(q') - du_-(q')h \leqslant K\|h\|^2$;
- $u_-(q') - u_-(q) - du_-(q)(q' - q) \leqslant K\|q' - q\|^2$;
- $-K\|q' + h - q\|^2 \leqslant u_-(q' + h) - u_-(q) - du_-(q)(q' + h - q)$.

Adding these three inequalities, we obtain:

$$\big(du_-(q) - du_-(q')\big)h \leqslant K\|h\|^2 + K\big\|q' - q\big\|^2 + K\big\|q' + h - q\big\|^2.$$

We choose h such that $\|h\| = \|q' - q\|$:

$$\big(du_-(q) - du_-(q')\big)\frac{h}{\|h\|} \leqslant 6K\|q' - q\|$$

and then: $\|du_-(q') - du_-(q)\| \leqslant 6K\|q' - q\|$. We have found a constant for q close to q', this is enough to conclude because $\mathcal{I}(u_-, u_+)$ is compact and du_- is bounded on M. $\square$

Let us now fix $q \in \mathcal{I}(u_-, u_+)$. For q' close to q that is a point of differentiability of u_-, we have:

- $a^+_{q',t}(z) = A_t(\pi \circ \varphi_{-t}(q', du_-(q')), z)$;
- $\{(z, da^+_{q',t}(z))\} = \mathcal{V}(z) \cap \varphi_t(\mathcal{V}_{\mathrm{loc}}(\pi \circ \varphi_{-t}(q', du_-(q'))))$;
- $\operatorname{graph}(d^2 a^+_{q',t}(z)) = T_{(z, da^+_{q',t}(z))} D\varphi_t(\mathcal{V}(\pi \circ \varphi_{-t}(q', du_-(q')))) = G_t(q', du_-(q'))$ and then the previous intersection is transverse.

These three quantities depend on q' and z; because du_- is continuous at q, we have: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|q' - q\| < \delta$ and z is in the chart near q: $\|d^2 a^+_{q',t}(z) - d^2 a^+_{q,t}(z)\| \leqslant \varepsilon$.

Moreover, by the Taylor–Lagrange inequality, we have:

$$\left\| a^+_{q',t}(q' + h) - a^+_{q',t}(q') - da^+_{q',t}(q')h - \frac{1}{2} d^2 a^+_{q',t}(q')(h, h) \right\| \leqslant \max_{z \in [q', q'+h]} \left\| d^2 a^+_{q',t}(z) - d^2 a^+_{q',t}(q') \right\| \|h\|^2.$$

Hence, if q' is close enough to q and h small enough:

$$\left\| a^+_{q',t}(q' + h) - a^+_{q',t}(q') - da^+_{q',t}(q')h - \frac{1}{2} d^2 a^+_{q,t}(q')(h, h) \right\| \leqslant \varepsilon \|h\|^2.$$

We have of course a similar result for $a^-_{q',t}$ and q' any differentiability point of u_+.

Let us now consider a sequence (q_n) of points of differentiability of u_- that converges to q so that: $\forall n$, $q_n \neq q$, a vector k with fixed norm $\|k\| = \mu > 0$ and $(h_n) = (\lambda_n k)$ where (λ_n) is a sequence of positive numbers tending to 0. We have proved that:

$$\begin{aligned}\big(du_-(q) - du_-(q_n)\big)h_n \leqslant{}& a^+_{q_n,t}(q_n + h_n) - a^+_{q_n,t}(q_n) - da^+_{q_n,t}(q_n)h_n + a^+_{q,t}(q_n) - a^+_{q,t}(q)\\ &- da_{q,t}(q)(q_n - q) - a^-_{q,t}(q_n + h_n) + a^-_{q,t}(q) + da^-_{q,t}(q)(q_n + h_n - q).\end{aligned}$$

We assume that $\lim_{n\to\infty} \frac{q_n - q}{\lambda_n} = Q$ and $P = \lim_{n\to\infty} \frac{du_-(q_n) - du_-(q)}{\lambda_n}$.

We divide by λ_n^2 the previous inequality and take the limit when n tends to $+\infty$ and we obtain:

$$-P.k \leqslant \frac{1}{2}\big(d^2 a^+_{q,t}(q)(k, k) + d^2 a^+_{q,t}(q)(Q, Q) - d^2 a^-_{q,t}(q)(Q + k, Q + k)\big)$$

changing k into $-k$, this gives the wanted result. In a similar way we obtain for u_+:

$$\forall k \in \mathbb{R}^n, \quad \frac{1}{2}\big(d^2 a^-_{q,t}(q)(k, k) + d^2 a^-_{q,t}(q)(Q, Q) - d^2 a^+_{q,t}(q)(k - Q, k - Q)\big) \leqslant P.k. \qquad \square$$

4.4. Links between the Green bundles and the weak KAM solutions

Notations. Near every point $q \in M$, we choose some coordinates $(q_1, \ldots, q_n)$ of M and associate to them their dual coordinates $(p_1, \ldots, p_n)$ such that $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are symplectic coordinates on T^*M. Then we can associate to these coordinates their infinitesimal coordinates $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$.

Then any Lagrangian subspace G of $T_x(T^*M)$ that is transverse to the vertical is the graph of a linear map whose matrix s in the coordinates $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ is symmetric. We can then associate to G the unique quadratic form $\mathcal{Q}$ whose matrix (as a quadratic form) in coordinates $(Q_1, \ldots, Q_n)$ is s.

For example, if $q \in M$ is a point of differentiability of u_- (resp. u_+) then the Green bundle $G_+(q, du_-(q))$ (resp. $G_-(q, du_+(q))$) is well defined and transverse to the vertical. We denote by $\mathcal{Q}_-$ (resp. $\mathcal{Q}_+$) its associated quadratic form and by s_- (resp. s_+) its matrix.

Let us recall that if $x \in A \subset T^*M$, $\mathcal{C}_x A$ designates the contingent cone to A at x, that was defined in the introduction.

Proposition 20. *We assume that (u_-, u_+) is a pair of conjugate weak KAM solutions. Let $q \in \mathcal{I}(u_-, u_+)$ be a point and $(Q, P) \in \mathcal{C}_{(q,du_-(q))}\mathcal{G}(du_-)$. Then we have*:

$$\forall k \in \mathbb{R}^n, \quad P.k \leqslant \frac{1}{2}\big(\mathcal{Q}_+(k,k) + \mathcal{Q}_+(Q,Q) - \mathcal{Q}_-(Q-k, Q-k)\big)$$

and if $(Q, P) \in \mathcal{C}_{(q,du_+(q))}\mathcal{G}(du_+)$:

$$\forall k \in \mathbb{R}^n, \quad \frac{1}{2}\big(\mathcal{Q}_-(k,k) + \mathcal{Q}_-(Q,Q) - \mathcal{Q}_+(Q-k, Q-k)\big) \leqslant P.k.$$

Proof. We know that $G_+(q,p) = \lim_{t\to+\infty} G_t(q,p)$ (resp. $G_-(q,p) = \lim_{t\to-\infty} G_t(q,p)$). Hence, if q is a point of differentiability of u_-, we have: $\mathcal{Q}_+(q, du_-(q)) = \lim_{t\to+\infty} d^2 g^+_{q,t}(q)$ and if q is a point of differentiability of u_+: $\mathcal{Q}_-(q, du_+(q)) = \lim_{t\to+\infty} d^2 g^-_{q,t}(q)$. If we use the inequalities that we proved in Proposition 18, we obtain for the (Q, P) that were described in this proposition:

$$\forall k \in \mathbb{R}^n, \quad P.k \leqslant \frac{1}{2}\big(\mathcal{Q}_+(Q,Q) + \mathcal{Q}_+(k,k) - \mathcal{Q}_-(Q-k, Q-k)\big).$$

Let us now look for the contingent cone to the pseudograph $\mathcal{G}(du_-)$ at $(q, du_-(q)) \in \mathcal{I}(u_-, u_+)$. Working in a chart, we assume that $(Q, P) \in \mathcal{C}_{(q,du_-(q))}\mathcal{G}(du_-)$ is not the null vector. Hence, there exist a sequence (λ_n) of positive numbers that converges to 0^+ and a sequence (q_n) of points of differentiability of u_- that converges to q so that:

$$(Q, P) = \lim_{n\to\infty} \frac{1}{\lambda_n}\big(q_n - q, du_-(q_n) - du_-(q)\big).$$

This corresponds exactly to the limit that we computed in Proposition 18. Hence, we proved:

If $q \in \mathcal{I}(u_-, u_+)$, if (Q, P) is a vector of the contingent cone to $\mathcal{G}(du_-)$ at $(q, du_-(q))$, then:

$$\forall k \in \mathbb{R}^n, \quad P.k \leqslant \frac{1}{2}\big(\mathcal{Q}_+(k,k) + \mathcal{Q}_+(Q,Q) - \mathcal{Q}_-(Q-k, Q-k)\big).$$

In a similar way, we obtain:

If $q \in \mathcal{I}(u_-, u_+)$, if (Q, P) is a vector of the contingent cone to $\mathcal{G}(du_+)$ at $(q, du_+(q))$, then:

$$\forall k \in \mathbb{R}^n, \quad \frac{1}{2}\big(\mathcal{Q}_-(k,k) + \mathcal{Q}_-(Q,Q) - \mathcal{Q}_+(Q-k, Q-k)\big) \leqslant P.k. \qquad \square$$

4.5. Proof of Theorem 12

Let (u_-, u_+) be a pair of conjugate weak KAM solutions and let q belong to $\mathcal{I}(u_-, u_+)$. We want to prove that: $\forall (Q, P) \in \mathcal{C}_{(q,du_-(q))}\mathcal{G}(du_-)$,

$$\begin{aligned}\big\|P - s_-\big(q, du_-(q)\big)Q\big\| &\leqslant 2\sqrt{\big\|\Delta s\big(q, du_-(q)\big)\big\|}.\sqrt{\Delta s\big(q, du_-(q)\big)(Q,Q)} \\ &\leqslant 2\big\|\Delta s\big(q, du_-(q)\big)\big\|.\big\|\mathrm{proj}_{\Delta s(q,du_-(q))}(Q)\big\|.\end{aligned}$$

We denote the quadratic form associated with G_- (resp. G_+) by $\mathcal{Q}_-$ (resp. $\mathcal{Q}_+$). Let $(Q, P) \in \mathcal{C}_{(q,du_-(q))}\mathcal{G}(du_-)$ be a vector of the contingent cone. We have proved that:

$$\forall k \in \mathbb{R}^n, \quad P.k \leqslant \frac{1}{2}\big(\mathcal{Q}_+(k,k) + \mathcal{Q}_+(Q,Q) - \mathcal{Q}_-(Q-k, Q-k)\big).$$

Then we write: $P = {}^t\mathcal{Q}_+ Q + \Delta P$ and $\Delta\mathcal{Q} = \mathcal{Q}_+ - \mathcal{Q}_-$. The previous inequality can be rewritten as follows:

$$\forall k \in \mathbb{R}^n, \quad \Delta P.k \leqslant \frac{1}{2}\Delta\mathcal{Q}(Q-k, Q-k). \tag{$*$}$$

We have the following splitting: $\mathbb{R}^n = \ker {}^t\Delta\mathcal{Q} \oplus \mathrm{Im}^t\Delta\mathcal{Q}$ and $\Delta P = P_1 + P_2$ with $P_1 \in \ker {}^t\Delta\mathcal{Q}$ and $P_2 \in \mathrm{Im}^t\Delta\mathcal{Q}$. We deduce from $(*)$:

$$\forall k \in \ker {}^t\Delta\mathcal{Q}, \quad P_1.k \leqslant \frac{1}{2}\Delta\mathcal{Q}(Q, Q).$$

This implies: $P_1 = \vec{0}$. Hence $\Delta P = P_2 \in \mathrm{Im}\,{}^t\Delta\mathcal{Q}$.

Let $Q_1 \in \ker {}^t\Delta\mathcal{Q}$ and $Q_2 \in \mathrm{Im}^t\Delta\mathcal{Q}$ be such that $Q = Q_1 + Q_2$. Then $(*)$ is equivalent to:

$$\forall k \in \mathrm{Im}\,{}^t\Delta\mathcal{Q}, \quad P_2.k \leqslant \frac{1}{2}\Delta\mathcal{Q}(Q_2 - k, Q_2 - k). \tag{$**$}$$

If $Q_2 = \vec{0}$, then $P_2 = \vec{0}$. If not, we have for $k := \|Q_2\|\vec{u}$ with $\vec{u}$ unitary vector such that $P_2 = \|P_2\|\vec{u}$:

- $\|Q_2 - k\| \leqslant \|Q_2\| + \|k\| = 2\|Q_2\|$;
- replacing in $(**)$:

$$\|P_2\|.\|Q_2\| \leqslant \frac{1}{2}\Delta\mathcal{Q}(Q_2 - k, Q_2 - k) \leqslant \frac{1}{2}\|\Delta\mathcal{Q}\|\|Q_2 - k\|^2 \leqslant 2\|\mathcal{Q}\|\|Q_2\|^2.$$

We deduce: $\|P_2\| \leqslant 2\|\mathcal{Q}\|\|Q_2\|$ that is exactly:

$$\left\|P - {}^t\mathcal{Q}Q\right\| \leqslant 2\|\Delta\mathcal{Q}\|.\left\|\mathrm{proj}_{\Delta\mathcal{Q}}(Q)\right\|. \qquad \square$$

Remark. It would be nice to deduce that in some sense, the contingent cone is between G_- and G_+. We would like to have some statement that says that every vector of the contingent cone is contained in some Lagrangian subspace (that depends on the considered vector) that is between G_- and G_+.

Even if we do not know any counter-example to that, it is easy to construct in $\mathbb{R}^2$ an example of:

- two quadratic form $\mathcal{Q}_- \leqslant \mathcal{Q}_+$;
- two vectors Q, P;

so that:

- the inequalities of Proposition 20 are satisfied;
- there exists no quadratic form $\mathcal{Q}$ such that $\mathcal{Q}_- \leqslant \mathcal{Q} \leqslant \mathcal{Q}_+$ and $P = {}^t\mathcal{Q}Q$.

This implies that we have to obtain some estimates more precise that those contained in Proposition 20 to hope to prove such a result.

Let us describe our example: $\mathcal{Q}_- = 0$, $\mathcal{Q}_+$ is the standard Euclidean metric, $Q = (1, 0)$ and $P = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The fact that this example satisfies the inequalities of Proposition 20 is straightforward. Let us assume that S is the matrix of a quadratic form $\mathcal{Q}$ such that $\mathcal{Q}_- \leqslant \mathcal{Q} \leqslant \mathcal{Q}_+$ and $P = {}^t\mathcal{Q}Q$. Then $S = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & b \end{pmatrix} \geqslant 0$ and $I_2 - S = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1-b \end{pmatrix} \geqslant 0$. This implies that $b \in [0, 1]$, $\frac{b}{2} - \frac{3}{4} \geqslant 0$ and $\frac{1-b}{2} - \frac{3}{4} \geqslant 1$, which is impossible.

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