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Well-posedness of the Hele–Shaw–Cahn–Hilliard system

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Abstract

We study the well-posedness of the Hele–Shaw–Cahn–Hilliard system modeling binary fluid flow in porous media with arbitrary viscosity contrast but matched density between the components. For initial data in H^s, $s > \frac{d}{2} + 1$, the existence and uniqueness of solution in $C([0, T]; H^s) \cap L^2(0, T; H^{s+2})$ that is global in time in the two dimensional case ($d = 2$) and local in time in the three dimensional case ($d = 3$) are established. Several blow-up criterions in the three dimensional case are provided as well. One of the tools that we utilized is the Littlewood–Paley theory in order to establish certain key commutator estimates.

Keywords: Hele–Shaw–Cahn–Hilliard; Well-posedness; Blow-up criterion

1. Introduction

The modeling and analysis of multi-phase fluid flow is a fascinating, challenging and important problem [18,4]. Well-known two phase fluid examples include the coupled atmosphere–ocean dynamical system with water and air being the two phases, as well as the system describing displacement of oil by water in oil reservoir (usually porous media) [5].

A common approach to two phase flows that are macroscopically immiscible is the sharp interface approach where the two phases are separated by a sharp interface $\Gamma(t)$. In the case of flow in porous media, the dynamics of the system is then governed by the two phase Hele–Shaw (Darcy) system (Muskat problem) [20,17,25] together with two interface boundary conditions: (1) continuity of the normal velocity; and (2) pressure jump proportional to the (mean) curvature. The normal velocity of the interface is set to be the normal velocity of the fluids. The local in time well-posedness of the sharp interface model with or without surface tension is known [2,3,13,11]. Global in time well-posedness with surface tension [14,10] and 2D without surface tension [26] is also known under the assumption that the initial data is a small perturbation of a flat interface or a sphere. Nevertheless, the sharp interface model encounters serious difficulty with physically important topological changes of the interface (possibly undefined curvature), especially in terms of pinchoff and reconnection that are important in applications [4,20].

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As an alternative approach, one could consider the so-called phase field models (or diffuse interface models) where an order parameter c is introduced and a capillary stress tensor is used to model the interface between the two fluids and the forces associated [4]. The sharp interface is then replaced by a thin transition layer and hence we avoid the difficulty of discontinuity. In this paper, we will consider phase field approach to two phase fluid flow with matched density in a Hele–Shaw cell or porous media. The dynamical equations are given by the following Hele–Shaw–Cahn–Hilliard system [20,12]:

$$\begin{cases} \nabla\cdot u = 0, \\ u = -\dfrac{1}{12\eta(c)}\left(\nabla p - \dfrac{1}{\mathbf{M}}\mu\nabla c\right), \\ c_t + u\cdot\nabla c = \dfrac{1}{\mathbf{Pe}}\Delta\mu, \\ c(0,x) = c_0(x), \end{cases} \tag{1.1}$$

where u is the fluid velocity, c is the order parameter which is related to the concentration of the fluid and usually takes values between -1 and 1, the chemical potential μ depends on the order parameter c and is given by

$$\mu(c) = f_0'(c) - \mathbf{C}\Delta c, \tag{1.2}$$

and $\mathbf{Pe}$ is the diffusion Péclet number, $\mathbf{C}$ is the Cahn number, and $\mathbf{M}$ is a Mach number. Furthermore, $\eta(c)$ is the kinematic viscosity coefficient satisfying

$$\eta \in C^\infty(\mathbf{R}^1), \quad 0 < \lambda \leqslant \eta(c) \leqslant \Lambda < \infty, \tag{1.3}$$

the Helmholtz free energy $f_0(c)$ is given by the classical double well potential

$$f_0(c) = (c^2-1)^2. \tag{1.4}$$

In the above system (1.1), p is not the physical pressure but the combination of certain generalized Gibbs free energy and the gravitational potential (see [20] for more details). This model can be also viewed as the Boussinesq approximation of a more general model with arbitrary viscosity and density contrast [20]. One may formally recover the sharp interface model by taking appropriate limit within the Hele–Shaw–Cahn–Hilliard system (1.1) [20]. We will assume that the fluid occupies the two or three dimensional torus $\mathbf{T}^d$, $d = 2, 3,$ for simplicity.

Besides applications in two phase flow in porous media and Hele–Shaw cell, certain simplified versions of this HSCH model have been also used in tumor growth study [30]. Moreover, unconditionally stable schemes have been developed [29] and the existence of certain type of weak solutions (without uniqueness) is also derived [15] for the case with matched density and viscosity.

The goal of this manuscript is to study the well-posedness of the matched density Hele–Shaw–Cahn–Hilliard system (1.1) with arbitrary viscosity contrast.

The Hele–Shaw–Cahn–Hilliard system can be formally viewed as an appropriate limit of the classical Navier–Stokes–Cahn–Hilliard system [4,20,16] which is a popular phase field model for two phase flow although no rigorous justification is known yet. There are a lot of works on the Navier–Stokes–Cahn–Hilliard system including local in time well-posedness in 2 and 3 dimensional and global in time well-posedness in 2D under various assumptions [1,7]. In fact the global in time well-posedness of the 2D Navier–Stokes–Cahn–Hilliard system is recently resolved [1] using a very different set of tools than employed here. Mathematically speaking, the difficulty associated with the Hele–Shaw–Cahn–Hilliard is about the same as those associated with the Navier–Stokes–Cahn–Hilliard: we gain the advantage of dropping the nonlinear advection term in the velocity equation but also lose the regularizing viscosity term; and their scaling behaviors are very similar. We refer to [21–23,4] and references therein for more related works on the Navier–Stokes–Cahn–Hilliard system.

The rest of the paper is organized as follows. We prove a key estimate on the “pressure” in the second section. This estimate is nontrivial due to the variable coefficient introduced with the mismatched viscosity. New estimates on certain commutator operators in fractional derivative spaces are needed and they are derived in Appendix A. In Section 3 we present the local in time well-posedness based on certain modified Galerkin approximation of the HSCH system and the “pressure” estimate from Section 2. In Section 4 we provide a Beale–Kato–Majda type blow-up criterion and prove that the system is global in time well-posed in the two dimensional case. We provide a refined blow-up criterion in the 3D case in Section 5.

2. The estimate of the pressure

In this section, we present the estimate of the modified pressure p. Taking the divergence for the second equation of (1.1), we find that

$$\operatorname{div}\left(\frac{1}{\eta(c)}\nabla p\right)=\operatorname{div}\left(\frac{1}{\mathbf{M}\eta(c)}\mu(c)\nabla c\right)\stackrel{\text{def}}{=}\operatorname{div}F(c). \tag{2.1}$$

This variable coefficient problem is dealt with utilizing commutator estimates that we derived in Appendix A. The commutator estimates themselves are derived utilizing Littlewood–Paley decomposition.

Proposition 2.1. *Let $s\geqslant 0$ and $c\in H^{s+2}(\mathbf{T}^d)$, and p be a smooth solution of (2.1). Then the solution p satisfies*

$$\|\nabla p\|_{H^s}\leqslant \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\big(1+\|c\|_{H^2}\big)^k\|c\|_{H^{s+2}}. \tag{2.2}$$

Here $k=[2s]+1$ and $\mathcal{F}$ is an increasing function on $\mathbf{R}^+$.

Proof. Thanks to (1.3) and (2.1), a straightforward energy estimate yields that

$$\|\nabla p\|_{L^2}\leqslant C\big\|\mu(c)\big\|_{L^2}\|\nabla c\|_{L^\infty}\leqslant C\big(1+\|c\|_{L^\infty}^2\big)\|\nabla c\|_{L^\infty}\|c\|_{H^2}. \tag{2.3}$$

Taking the operator $\langle D\rangle^s$ to (2.1) to obtain

$$\begin{aligned}\operatorname{div}\left(\frac{1}{\eta(c)}\nabla\langle D\rangle^s p\right)&=\operatorname{div}\langle D\rangle^s\left(\frac{1}{\mathbf{M}\eta(c)}\mu(c)\nabla c\right)-\operatorname{div}\left(\langle D\rangle^s\left(\frac{1}{\eta(c)}\nabla p\right)-\left(\frac{1}{\eta(c)}\nabla\langle D\rangle^s p\right)\right)\\&=\operatorname{div}(A+B),\end{aligned}$$

from which and from the energy estimate, we infer that

$$\|\nabla p\|_{H^s}\leqslant C\big(\|A\|_{L^2}+\|B\|_{L^2}\big).$$

Due to the definition of $\mu(c)$, we have

$$\frac{1}{\eta(c)}\mu(c)\nabla c=\frac{1}{\eta(c)}f_0'(c)\nabla c-\mathbf{C}\frac{1}{\eta(c)}\Delta c\nabla c=\nabla g_1(c)-\Delta c\nabla g_2(c),$$

for some g_1, g_2 with $g_1(0)=g_2(0)=0$. We have by Lemma A.3 that

$$\big\|\langle D\rangle^s\nabla g_1(c)\big\|_{L^2}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\|c\|_{H^{s+1}},$$

and using Bony's decomposition to write

$$\begin{aligned}\Delta c\nabla g_2(c)&=T_{\Delta c}\nabla g_2(c)+\widetilde{R}\big(\Delta c,\nabla g_2(c)\big)\\&=\partial_i T_{\partial_i c}\nabla g_2(c)-T_{\partial_i c}\partial_i\nabla g_2(c)+\widetilde{R}\big(\Delta c,\nabla g_2(c)\big),\end{aligned}$$

then from the proof of Lemma A.2, it is easy to see that

$$\big\|\langle D\rangle^s\Delta c\nabla g_2(c)\big\|_{L^2}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\|\nabla c\|_{L^\infty}\|c\|_{H^{s+2}}.$$

Thus we obtain

$$\|A\|_{L^2}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\|c\|_{H^{s+2}}$$

and by Lemmas A.4–A.3 and (2.3), for $s\in(0,1]$,

$$\|B\|_{L^2}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\|c\|_{H^{s+2}}\|\nabla p\|_{L^2}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\|\nabla c\|_{L^\infty}\|c\|_{H^2}\|c\|_{H^{s+2}}.$$

Thus we obtain that for $s\in(0,1]$,

$$\|\nabla p\|_{H^s}\leqslant\mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\big(1+\|c\|_{H^2}\big)\|c\|_{H^{s+2}}. \tag{2.4}$$

For general s, we will prove it by an induction argument. Let us assume that for $s \in (\frac{k-1}{2}, \frac{k}{2}]$, we have

$$\|\nabla p\|_{H^s} \leqslant \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\big(1+\|c\|_{H^2}\big)^k \|c\|_{H^{s+2}}.$$

Note that (2.4) means that the cases of $k = 1, 2$ hold. Now let us assume $s \in (\frac{k}{2}, \frac{k+1}{2}]$. We infer from Lemma A.4 and Lemma A.3 that

$$\|B\|_{L^2} \leqslant \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(\|c\|_{H^{s+2}}\|\nabla p\|_{L^2} + \|c\|_{H^2}\|\nabla p\|_{H^{s-\frac{1}{2}}}\big).$$

Then from (2.3) and the induction assumption, it follows that

$$\|B\|_{L^2} \leqslant \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\big(1+\|c\|_{H^2}\big)^{k+1} \|c\|_{H^{s+2}}.$$

Thus for $s \in (\frac{k}{2}, \frac{k+1}{2}]$, we have

$$\|\nabla p\|_{H^s} \leqslant \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)\big(1+\|c\|_{H^2}\big)^{k+1} \|c\|_{H^{s+2}}.$$

This completes the proof of Lemma 2.1. □

Remark. Instead of relying on the estimates from the appendix which depend on the Littlewood–Paley theory, classical energy method might work as well if we are content with less sharp and less general results. For instance, if $\nabla c \in L^\infty(\mathbf{T}^d)$ and $c \in H^k(\mathbf{T}^d)$ for $k \in \mathbf{Z}^+$, classical elliptic estimates may lead to

$$\|\nabla p\|_{H^k} \leqslant C\big\|F(c)\big\|_{H^k},$$

where C depends on $\|\nabla c\|_{L^\infty}$ and $\|c\|_{H^k}$. And a straightforward product estimate gives

$$\big\|F(c)\big\|_{H^k} \leqslant C\big(\|c\|_{L^\infty}\big)\big(\|c\|_{H^{k+1}} + \|\nabla c\|_{L^\infty}\|c\|_{H^{k+2}} + \|\Delta c\|_{L^\infty}\|c\|_{H^{k+1}}\big).$$

This estimate is enough to obtain the local well-posedness of the system (1.1) and global well-posedness in the 2D case in the space of

$$c \in C\big([0,T]; H^2\big(\mathbf{T}^d\big)\big) \cap L^2\big(0,T; H^4\big(\mathbf{T}^d\big)\big), \qquad u \in C\big([0,T]; L^2\big(\mathbf{T}^d\big)\big) \cap L^2\big(0,T; H^2\big(\mathbf{T}^d\big)\big)$$

when combined with the $L^\infty(H^2) \cap L^2(H^4)$ a priori estimates from (Theorem 4.1) for initial data in H^k, $k > 2$. However, in order to obtain the sharp blow-up criterion which in particular implies the global existence of the 2D system in general Sobolev spaces as specified in Theorem 3.1, we need to establish the refined pressure estimate (2.2). Notice that (2.2) is established for general (Hilbert) Sobolev spaces, and only a linear factor of $\|\nabla c\|_{L^\infty}$ appears in the estimate in contrast to pure energy estimates.

3. Local well-posedness

In this section we prove the local well-posedness of the Hele–Shaw–Cahn–Hilliard system. The procedure is mostly standard except for the pressure estimate.

Theorem 3.1. *Let $c_0(x) \in H^s(\mathbf{T}^d)$ for $s > \frac{d}{2} + 1$. Then there exists $T > 0$ such that the system (1.1) has a unique solution (c, u) in $[0, T]$ with*

$$c \in C\big([0,T]; H^s\big(\mathbf{T}^d\big)\big) \cap L^2\big(0,T; H^{s+2}\big(\mathbf{T}^d\big)\big), \qquad u \in C\big([0,T]; H^{s-2}\big(\mathbf{T}^d\big)\big) \cap L^2\big(0,T; H^s\big(\mathbf{T}^d\big)\big);$$

and satisfying the following energy estimate

$$\big\|c(t)\big\|_{H^s}^2 + \int_0^t \big\|c(\tau)\big\|_{H^{s+2}}^2\,d\tau \leqslant \|c_0\|_{H^s}\exp\left(\int_0^t G(\tau)\,d\tau\right), \tag{3.1}$$

for $t \in [0, T]$, where

$$G(t) = \mathcal{F}\big(\|c\|_{L^\infty}\big)\big(1+\|\nabla c\|_{L^\infty}\big)^2\big(\|\nabla c\|_{L^\infty} + \|c\|_{H^3}^{\frac{d-2}{2}}\big)^2\big(1+\|c\|_{H^2}\big)^{2([2s]+1)}.$$

Proof. We will use the energy method to prove Theorem 3.1.

Step 1. Construction of an approximate solution sequence.

The construction of the approximate solutions is based on the Galerkin method. Let us define the operator P_n by

$$P_n f(x) = \sum_{|k| \leqslant n} f_k e^{2\pi i k \cdot x}, \quad f_k = \int_{\mathbf{T}^d} f(x) e^{-2\pi i k \cdot x}\, dx.$$

Then we consider the following approximate system of (1.1):

$$\begin{cases} \nabla \cdot u_n = 0, \\ u_n = -\dfrac{1}{12\eta(P_n c_n)}\left(\nabla p_n - \dfrac{1}{\mathbf{M}}\mu(P_n c_n)\nabla P_n c_n\right), \\ \partial_t c_n + P_n(u_n \cdot \nabla P_n c_n) = \dfrac{1}{\mathbf{Pe}}\Delta P_n \mu(P_n c_n), \\ c_n(0,x) = P_n c_0(x). \end{cases} \tag{3.2}$$

It is easy to see that

$$\big\|\Delta P_n \mu\big(P_n c_n^1\big) - \Delta P_n \mu\big(P_n c_n^2\big)\big\|_{L^2} \leqslant C\big(n, \big\|c_n^1\big\|_{L^2}, \big\|c_n^2\big\|_{L^2}\big)\big\|c_n^1 - c_n^2\big\|_{L^2}.$$

Taking the divergence to the second equation in (3.2) gives

$$\operatorname{div}\left(\frac{1}{\eta(P_n c_n)}\nabla p_n\right) = \operatorname{div}\left(\frac{1}{\mathbf{M}\eta(P_n c_n)}\mu(P_n c_n)\nabla P_n c_n\right).$$

Thanks to (1.3), straightforward energy estimate yields that

$$\|\nabla p_n\|_{L^2} \leqslant C\big(n, \|c_n\|_{L^2}\big)\|c_n\|_{L^2},$$

from which, we can further deduce by making the energy estimate for the elliptic equation satisfied by $p_n^1 - p_n^2$ that

$$\big\|\nabla\big(p_n^1 - p_n^2\big)\big\|_{L^2} \leqslant C\big(n, \big\|c_n^1\big\|_{L^2}, \big\|c_n^2\big\|_{L^2}\big)\big\|c_n^1 - c_n^2\big\|_{L^2},$$

thus we infer from the second equation of (3.2) that

$$\big\|u_n^1 - u_n^2\big\|_{L^2} \leqslant C\big(n, \big\|c_n^1\big\|_{L^2}, \big\|c_n^2\big\|_{L^2}\big)\big\|c_n^1 - c_n^2\big\|_{L^2}.$$

Therefore, we have, since all norms are equivalent on a finite dimensional space,

$$\begin{aligned} \big\|P_n\big(u_n^1 \cdot \nabla P_n c_n^1\big) - P_n\big(u_n^2 \cdot \nabla P_n c_n^2\big)\big\|_{L^2} &\leqslant C\big(n, \big\|c_n^1\big\|_{L^2}, \big\|c_n^2\big\|_{L^2}\big)\big(\big\|u_n^1 - u_n^2\big\|_{L^2} + \big\|c_n^1 - c_n^2\big\|_{L^2}\big) \\ &\leqslant C\big(n, \big\|c_n^1\big\|_{L^2}, \big\|c_n^2\big\|_{L^2}\big)\big\|c_n^1 - c_n^2\big\|_{L^2}. \end{aligned}$$

Thus, the Cauchy–Lipschitz theorem (classical ODE existence and uniqueness result under local Lipschitz continuity assumption) ensures that there exists $T_n > 0$ such that the approximate system (3.2) has a unique solution $c_n \in C([0, T_n]; L^2(\mathbf{T}^d))$. Note that $P_n^2 = P_n$, $P_n c_n$ is also a solution of (3.2). So the uniqueness implies that $P_n c_n = c_n$. Thus, the approximate system (3.2) reduces to

$$\begin{cases} \nabla \cdot u_n = 0, \\ u_n = -\dfrac{1}{12\eta(c_n)}\left(\nabla p_n - \dfrac{1}{\mathbf{M}}\mu(c_n)\nabla c_n\right), \\ \partial_t c_n + P_n(u_n \cdot \nabla c_n) = \dfrac{1}{\mathbf{Pe}}\Delta P_n \mu(c_n), \\ c_n(0,x) = P_n c_0(x). \end{cases} \tag{3.3}$$

In what follows, we denote by T_n^* the maximal existence time of the solution c_n. Due to $P_n c_n = c_n$, the solution c_n is in fact smooth.

Step 2. Energy estimates.

Although the HSCH model (1.1) has a natural energy (which is somewhat equivalent to H^1 estimate, see [20,29] and Section 4 below), it is not sufficient for the strong solution. Therefore we have to derive estimates in Sobolev spaces with higher derivatives.

For this purpose we take the $H^s(\mathbf{T}^d)$ inner product of the third equation of (3.3) with c_n and obtain

$$\frac{1}{2}\frac{d}{dt}\|c_n\|_{H^s}^2 - \frac{1}{\mathbf{Pe}}\big(\Delta P_n\mu(c_n), c_n\big)_{H^s} = -(u_n\cdot\nabla c_n, c_n)_{H^s}. \tag{3.4}$$

Due to (1.2), we see that

$$-\big(\Delta P_n\mu(c_n), c_n\big)_{H^s} = \mathbf{C}\|\Delta c_n\|_{H^s}^2 - \big(\Delta f_0'(c_n), c_n\big)_{H^s}.$$

We deduce, thanks to Lemma A.3 that

$$\big|\big(\Delta f_0'(c_n), c_n\big)_{H^s}\big| \leqslant \big\|f_0'(c_n)\big\|_{H^s}\|\Delta c_n\|_{H^s} \leqslant C\big(1+\|c_n\|_{L^\infty}^2\big)\|c_n\|_{H^s}\|\Delta c_n\|_{H^s}, \tag{3.5}$$

and by Lemma A.2 with $\sigma = 1$,

$$\begin{aligned}\big|(u_n\cdot\nabla c_n, c_n)_{H^s}\big| &\leqslant \|u_n\cdot\nabla c_n\|_{H^s}\|c_n\|_{H^s}\\ &\leqslant C\big(\|u_n\|_{H^s}\|\nabla c_n\|_{L^\infty} + \|u_n\|_{H^{\frac{d}{2}-1}}\|\nabla c_n\|_{H^{s+1}}\big)\|c_n\|_{H^s}.\end{aligned} \tag{3.6}$$

Thanks to (3.3), we find that

$$\|u_n\|_{H^s} \leqslant C\left(\left\|\frac{1}{\eta(c_n)}\nabla p\right\|_{H^s} + \left\|\frac{1}{\eta(c_n)}\mu(c_n)\nabla c_n\right\|_{H^s}\right). \tag{3.7}$$

By Proposition 2.1, the first term on the right hand side of (3.7) is bounded by

$$\mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}},$$

and by Lemma A.2, Lemma A.3, the second term is bounded by

$$\mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\|c_n\|_{H^{s+2}}.$$

Thus we obtain

$$\|u_n\|_{H^s} \leqslant \mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}},$$

and in particular,

$$\|u_n\|_{H^{\frac{d}{2}-1}} \leqslant \mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(1+\|c_n\|_{H^2}\big)^{d-1}\|c_n\|_{H^{\frac{d}{2}+1}},$$

from which and (3.6), we infer that

$$\begin{aligned}\big|(u_n\cdot\nabla c_n, c_n)_{H^s}\big| &\leqslant \mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\\ &\quad\times\big(\|\nabla c_n\|_{L^\infty} + \|c_n\|_{H^3}^{\frac{d-2}{2}}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}}\|c_n\|_{H^s}.\end{aligned} \tag{3.8}$$

Here we used the following interpolation inequality:

$$\|c_n\|_{H^{\frac{d}{2}+1}} \leqslant \|c_n\|_{H^2}^{2-\frac{d}{2}}\|c_n\|_{H^3}^{\frac{d}{2}-1}.$$

Plugging (3.5) and (3.8) into (3.4) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_n\|_{H^s}^2 + \frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta c_n\|_{H^s}^2\\ &\quad\leqslant \mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(\|\nabla c_n\|_{L^\infty} + \|c_n\|_{H^3}^{\frac{d-2}{2}}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}}\|c_n\|_{H^s},\end{aligned}$$

which along with Young's inequality implies that

$$\frac{d}{dt}\|c_n\|_{H^s}^2+\|c_n\|_{H^{s+2}}^2$$
$$\leqslant \mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)^2\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)^2\big(1+\|c_n\|_{H^2}\big)^{2([2s]+1)}\|c_n\|_{H^s}^2.$$

Then Gronwall's inequality applied gives

$$E_n^s(t)\stackrel{\text{def}}{=}\big\|c_n(t)\big\|_{H^s}^2+\int_0^t\big\|c_n(\tau)\big\|_{H^{s+2}}^2\,d\tau\leqslant\|c_0\|_{H^s}^2\exp\left(\int_0^t G_n(\tau)\,d\tau\right)\tag{3.9}$$

for $t\in[0,T_n^*)$, where

$$G_n(t)=\mathcal{F}\big(\|c_n\|_{L^\infty}\big)\big(1+\|\nabla c_n\|_{L^\infty}\big)^2\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)^2\big(1+\|c_n\|_{H^2}\big)^{2([2s]+1)}.$$

Step 3. Uniform estimates and existence of the solution.

Let us define

$$\widetilde{T}_n^*\stackrel{\text{def}}{=}\sup\big\{t\in\big[0,T_n^*\big):\ E_n^s(\tau)\leqslant 2\|c_0\|_{H^s}^2\ \text{for}\ \tau\in[0,t]\big\}.$$

From (3.9) and Sobolev's embedding, we find that

$$E_n^s(t)\leqslant\|c_0\|_{H^s}^2\exp\left(\mathcal{A}\big(\|c_0\|_{H^s}\big)\int_0^t\big(1+\big\|c(\tau)\big\|_{H^3}^{d-2}\big)\,d\tau\right)$$
$$\leqslant\|c_0\|_{H^s}^2\exp\big(\mathcal{A}\big(\|c_0\|_{H^s}\big)\big(t+t^{\frac12}\big)\big),\quad t\in\big[0,\widetilde{T}_n^*\big).$$

Here $\mathcal{A}(\cdot)$ is some increasing function. Take T to be small enough such that

$$\exp\big(\mathcal{A}\big(\|c_0\|_{H^s}\big)\big(T+T^{\frac12}\big)\big)\leqslant\frac32.$$

Now we will show that $\widetilde{T}_n^*\geqslant T$. Otherwise, we have

$$E_n^s(t)\leqslant\frac32\|c_0\|_{H^s}^2\quad\text{for}\ t\in\big[0,\widetilde{T}_n^*\big],$$

which contradicts with the definition of $\widetilde{T}_n^*$. Thus the approximate solution (c_n,u_n) exists on $[0,T]$ and satisfies the following uniform estimate

$$\big\|c_n(t)\big\|_{H^s}^2+\int_0^t\big\|c_n(\tau)\big\|_{H^{s+2}}^2\,d\tau\leqslant 2\|c_0\|_{H^s}^2\tag{3.10}$$

for $t\in[0,T]$. On the other hand, it is easy to verify from the third equation of (3.3) that $\partial_t c_n$ is uniformly bounded in $L^2(0,T;H^{s-2}(\mathbf{T}^d))$. Thus, Lions–Aubin's compactness theorem (for example, see [27]) ensures that there exist a subsequence $(c_{n_k},u_{n_k})_k$ of $(c_n,u_n)_n$ and a function $c\in L^\infty(0,T;H^s(\mathbf{T}^d))\cap L^2(0,T;H^{s+2}(\mathbf{T}^d))$ and $u\in L^\infty(0,T;H^{s-2}(\mathbf{T}^d))\cap L^2(0,T;H^s(\mathbf{T}^d))$ such that

$$c_{n_k}\longrightarrow c,\quad\text{in}\ L^2\big(0,T;H^{s'+2}\big(\mathbf{T}^d\big)\big),$$
$$u_{n_k}\longrightarrow u,\quad\text{in}\ L^2\big(0,T;H^{s'}\big(\mathbf{T}^d\big)\big),$$

as $k\longrightarrow+\infty$, for any $s'<s$. Then passing to limit in (3.3), it is easy to see that (c,u) satisfies (1.1) in the weak sense and (c,u) satisfies (3.1).

Step 4. Continuity in time of the solution.

Let us claim that there holds the following better estimate for c_n (thus for c):

$$\|c_n\|_{\widetilde{L}^\infty(0,T;H^s)}^2\stackrel{\text{def}}{=}\sum_{j\geqslant -1}2^{2js}\|\Delta_j c\|_{L^\infty(0,T;L^2)}^2\leqslant C.$$

Indeed, we can deduce a frequency localized version of (3.4):

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j c_n\|_{L^2}^2+\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta\Delta_j c_n\|_{L^2}^2=-\big(\Delta_j(u_n\cdot\nabla c_n),\Delta_j c_n\big)_{L^2}+\frac{1}{\mathbf{Pe}}\big(\Delta_j\Delta f_0'(c_n),\Delta_j c_n\big)_{L^2},$$

which implies that

$$\sup_{\tau\in[0,t]}\big\|\Delta_j c_n(\tau)\big\|_{L^2}^2+\int_0^t\big\|\Delta\Delta_j c_n(\tau)\big\|_{L^2}^2\,d\tau$$
$$\leqslant C\int_0^t\big|\big(\Delta_j(u_n\cdot\nabla c_n),\Delta_j c_n\big)_{L^2}\big|\,d\tau+\int_0^t\big|\big(\Delta_j\Delta f_0'(c_n),\Delta_j c_n\big)_{L^2}\big|\,d\tau.$$

Multiplying 2^{2js} and taking summation on j on both sides, we get

$$\|c_n\|_{\tilde{L}^\infty(0,t;H^s)}^2+\int_0^t\big\|\Delta c_n(\tau)\big\|_{H^s}^2\,d\tau\leqslant C\int_0^t\|u_n\cdot\nabla c_n\|_{H^s}\|c_n\|_{H^s}+\big\|f_0'(c_n)\big\|_{H^s}\|\Delta c_n\|_{H^s}\,d\tau.$$

The rest of the proof is completely similar to Step 2. We omit the details.

Now we show that the claim will imply $c\in C([0,T];H^s(\mathbf{T}^d))$. In fact, for any $\varepsilon>0$, take N big enough such that

$$\sum_{j>N}2^{2js}\|\Delta_j c\|_{L^\infty(0,T;L^2)}^2\leqslant\frac{\varepsilon}{4}.$$

For any $t\in(0,T)$ and δ such that $t+\delta\in[0,T]$, we have

$$\begin{aligned}\big\|c(t+\delta)-c(t)\big\|_{H^s}^2&\leqslant\sum_{j=-1}^{N}2^{2js}\big\|\Delta_j c(t+\delta)-\Delta_j c(t)\big\|_{L^2}^2+\frac{\varepsilon}{2}\\&\leqslant\sum_{j=-1}^{N}2^{2js}|\delta|\|\partial_t c\|_{L^2(0,T;L^2)}^2+\frac{\varepsilon}{2}\\&\leqslant 2N2^{2N}\|\partial_t c\|_{L^2(0,T;L^2)}^2|\delta|+\frac{\varepsilon}{2}.\end{aligned}$$

Thus for $|\delta|$ small enough, we have

$$\big\|c(t+\delta)-c(t)\big\|_{H^s}^2\leqslant\varepsilon.$$

That is, $c(t)$ is continuous in $H^s(\mathbf{T}^d)$ at the time t, thus so is u.

Step 5. Uniqueness of the solution

Assume that (c_1,u_1) and (c_2,u_2) are two solutions of (1.1) with the same initial data. We introduce the difference of two solutions:

$$\delta_c=c_1-c_2,\qquad\delta_u=u_1-u_2.$$

Then (δ_c,δ_u) satisfies

$$\begin{cases}\partial_t\delta_c+u_1\cdot\nabla\delta_c+\delta_u\cdot\nabla c_2=\dfrac{1}{\mathbf{Pe}}\Delta\big(\mu(c_1)-\mu(c_2)\big),\\ \delta_u=\dfrac{\eta(c_1)-\eta(c_2)}{12\eta(c_1)\eta(c_2)}\Big(\nabla p_1-\dfrac{1}{\mathbf{M}}\mu(c_1)\nabla c_1\Big)-\dfrac{1}{12\eta(c_2)}\Big(\nabla(p_1-p_2)-\dfrac{1}{\mathbf{M}}\big(\mu(c_1)\nabla c_1-\mu(c_2)\nabla c_2\big)\Big),\\ \delta_c(0)=0.\end{cases}$$

Multiplying the first equation by δ_c, integrating, and then using the first equation of (1.1) yields that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta_c\|_{L^2}^2+\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta\delta_c\|_{L^2}^2&\leqslant\frac{1}{\mathbf{Pe}}\big(\Delta\big(f_0'(c_1)-f_0'(c_2),\delta_c\big)\big)_{L^2}-(\delta_u\cdot\nabla c_2,\delta_c)_{L^2}\\&\leqslant C\big(\|\Delta\delta_c\|_{L^2}+\|\delta_u\|_{L^2}\big)\|\delta_c\|_{L^2}.\end{aligned}$$

Note by (2.1) that

$$\operatorname{div}\left(\frac{1}{\eta(c_1)}\nabla(p_1-p_2)\right)=\operatorname{div}\bigl(F(c_1)-F(c_2)\bigr)-\operatorname{div}\left(\left(\frac{1}{\eta(c_1)}-\frac{1}{\eta(c_2)}\right)\nabla p_2\right),$$

so the energy estimate gives

$$\begin{aligned}\bigl\|\nabla(p_1-p_2)\bigr\|_{L^2}&\leqslant C\left(\bigl\|F(c_1)-F(c_2)\bigr\|_{L^2}+\left\|\left(\frac{1}{\eta(c_1)}-\frac{1}{\eta(c_2)}\right)\nabla p_2\right\|_{L^2}\right)\\&\leqslant C\bigl(\|\delta_c\|_{L^2}+\|\Delta\delta_c\|_{L^2}\bigr).\end{aligned}$$

Then we can deduce from the equation of δ_u, together with the Hölder inequality and the Sobolev type inequalities that

$$\begin{aligned}\|\delta_u\|_{L^2}&\leqslant C\bigl(\|\delta_c\|_{L^2}+\bigl\|\nabla(p_1-p_2)\bigr\|_{L^2}+\|\Delta\delta_c\|_{L^2}\bigr)\\&\leqslant C\bigl(\|\delta_c\|_{L^2}+\|\Delta\delta_c\|_{L^2}\bigr).\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\|\delta_c\|_{L^2}^2\leqslant C\|\delta_c\|_{L^2}^2,\qquad\bigl\|\delta_c(0)\bigr\|=0,$$

which along with Gronwall's inequality implies $\delta_c=0$, and the uniqueness follows. □

4. Blow-up criterion and global existence in 2D

In this section we prove a Beale–Kato–Majda type blow-up criterion [24] for the Hele–Shaw–Cahn–Hilliard system. As an application, we obtain the global well-posedness in 2D.

Theorem 4.1. *Let $c_0(x)\in H^s(\mathbf{T}^d)$ for $s>\frac{d}{2}+1$, and (c,u) be a solution of (1.1) stated in Theorem 3.1. Let T^* be the maximal existence time of the solution. If $T^*<+\infty$, then*

$$\int_0^{T^*}\bigl\|\nabla c(t)\bigr\|_{L^\infty}^4\,dt=+\infty. \tag{4.1}$$

In particular, this implies $T^*=+\infty$ for $d=2$. That is, the system (1.1) is globally well-posed in 2D.

Proof. First of all, we derive the basic energy law of the system. Multiplying by μ on both sides of the third equation of (1.1), we get by integration by parts that

$$\int_{\mathbf{T}^d}c_t\mu\,dx+\int_{\mathbf{T}^d}u\cdot\nabla c\mu\,dx=-\frac{1}{\mathbf{Pe}}\int_{\mathbf{T}^d}|\nabla\mu|^2\,dx.$$

Due to the definition of μ, we have

$$\int_{\mathbf{T}^d}c_t\mu\,dx=\frac{d}{dt}\left(\int_{\mathbf{T}^d}f_0(c)\,dx+\frac{\mathbf{C}}{2}\int_{\mathbf{T}^d}|\nabla c|^2\,dx\right),$$

and due to the first two equations in (1.1),

$$\int_{\mathbf{T}^d}u\cdot\nabla c\mu\,dx=12\mathbf{M}\int_{\mathbf{T}^d}\eta(c)|u|^2\,dx.$$

Thus we obtain the following classical energy equality [20]

$$\frac{d}{dt}\left(\int_{\mathbf{T}^d}f_0(c)\,dx+\frac{\mathbf{C}}{2}\int_{\mathbf{T}^d}|\nabla c|^2\,dx\right)+\frac{1}{\mathbf{Pe}}\int_{\mathbf{T}^d}|\nabla\mu|^2\,dx+12\mathbf{M}\int_{\mathbf{T}^d}\eta(c)|u|^2\,dx=0.$$

That is,

$$E(t)+\frac{1}{\mathbf{Pe}}\int_0^t\int_{\mathbf{T}^d}\left|\nabla\mu(\tau)\right|^2dx\,d\tau+12\mathbf{M}\int_0^t\int_{\mathbf{T}^d}\eta(c)\left|u(\tau)\right|^2dx\,d\tau=E(0),\tag{4.2}$$

where

$$E(t)\stackrel{\text{def}}{=}\int_{\mathbf{T}^d}f_0\big(c(t,x)\big)\,dx+\frac{\mathbf{C}}{2}\int_{\mathbf{T}^d}\left|\nabla c(t,x)\right|^2dx.$$

From the energy equality (4.2), it follows that

$$\left\|c(t)\right\|_{H^1}^2+\frac{1}{\mathbf{Pe}}\int_0^t\|\nabla\mu\|_{L^2}^2\,d\tau\leqslant E(0).$$

On the other hand, we have

$$\|\nabla\Delta c\|_{L^2}\leqslant C\big(\|\nabla\mu\|_{L^2}+\|\nabla c\|_{L^2}+\|c^2\nabla c\|_{L^2}\big),$$

and by Sobolev's inequality, interpolation and Young's inequality,

$$\begin{aligned}\left\|c^2\nabla c\right\|_{L^2}&\leqslant C\|c\|_{L^6}^2\|\nabla c\|_{L^6}\leqslant C\|c\|_{H^1}^2\|c\|_{H^2}\\&\leqslant C\|c\|_{H^1}^{\frac{5}{2}}\|c\|_{H^3}^{\frac{1}{2}}\leqslant C\|c\|_{H^1}^5+\frac{1}{2\widetilde{C}}\|c\|_{H^3},\end{aligned}$$

where $\widetilde{C}$ is the constant from the preceding inequality. This implies that

$$\|c\|_{H^3}\leqslant C\big(\|\nabla\mu\|_{L^2}+\|c\|_{H^1}+\|c\|_{H^1}^5\big).$$

Therefore we conclude that

$$\|c\|_{L^\infty(0,T;H^1)}+\|c\|_{L^2(0,T;H^3)}\leqslant C\big(T,\|c_0\|_{H^1}\big).\tag{4.3}$$

Next, we derive an H^2 energy estimate of the solution. By taking the Laplacian of the third equation of (1.1), multiplying them by Δc, and then integrating we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta c\|_{L^2}^2+\frac{\mathbf{C}}{\mathbf{Pe}}\left\|\Delta^2c\right\|_{L^2}^2&=-\big(u\cdot\nabla c,\Delta^2c\big)_{L^2}+\frac{1}{\mathbf{Pe}}\big(\Delta f_0'(c),\Delta^2c\big)\\&\leqslant\|u\|_{L^2}\|\nabla c\|_{L^\infty}\left\|\Delta^2c\right\|_{L^2}+\frac{1}{\mathbf{Pe}}\left\|\Delta f_0'(c)\right\|_{L^2}\left\|\Delta^2c\right\|_{L^2}.\end{aligned}\tag{4.4}$$

It is easy to verify that

$$\begin{aligned}\|u\|_{L^2}&\leqslant C\big(\|\nabla p\|_{L^2}+\left\|\mu(c)\nabla c\right\|_{L^2}\big)\\&\leqslant C\big(\|\nabla c\|_{L^\infty}\|\Delta c\|_{L^2}+\big(\|c\|_{L^3}+\|c\|_{L^9}^3\big)\|\nabla c\|_{L^6}\big)\\&\leqslant C\big(\|\nabla c\|_{L^\infty}+\|c\|_{L^3}+\|c\|_{L^9}^3\big)\|c\|_{H^2},\end{aligned}$$

and

$$\left\|\Delta f_0'(c)\right\|_{L^2}\leqslant C\big(1+\|c\|_{L^\infty}^2\big)\|c\|_{H^2}.$$

Plugging them into (4.4) yields that

$$\frac{d}{dt}\|\Delta c\|_{L^2}^2+\left\|\Delta^2c\right\|_{L^2}^2\leqslant C\big(1+\|\nabla c\|_{L^\infty}^4+\|c\|_{L^\infty}^4+\|c\|_{L^3}^4+\|c\|_{L^9}^{12}\big)\|c\|_{H^2}^2,$$

which along with Gronwall's inequality leads to

$$\|c\|_{H^2} \leqslant \|c_0\|_{H^2} \exp\left(C \int_0^t H(\tau)\, d\tau \right), \tag{4.5}$$

where $H(t) = 1 + \|\nabla c\|_{L^\infty}^4 + \|c\|_{L^\infty}^4 + \|c\|_{L^3}^4 + \|c\|_{L^9}^{12}$.

Now we are in position to prove the blow-up criterion. We will prove it by way of contradiction argument. Assume that $T^* < +\infty$ and

$$\int_0^{T^*} \left\|\nabla c(t)\right\|_{L^\infty}^4 dt < +\infty,$$

which together with (4.3) and Sobolev's inequality implies that

$$\int_0^{T^*} H(\tau)\, d\tau < +\infty,$$

for example,

$$\int_0^{T^*} \left\|c(t)\right\|_{L^9}^{12} dt \leqslant C \int_0^{T^*} \left\|c(t)\right\|_{H^1}^{11} \left\|c(t)\right\|_{H^3} dt < +\infty.$$

Then we infer from (4.5) that

$$\|c\|_{L^\infty(0,T^*;H^2)} < +\infty,$$

which implies that

$$\int_0^{T^*} G(t)\, dt < +\infty, \quad G(t) \text{ being as in Theorem 3.1.}$$

Then the energy inequality (3.1) ensures that

$$\sup_{t\in[0,T^*]} \left\|c(t)\right\|_{H^s}^2 + \int_0^{T^*} \left\|c(\tau)\right\|_{H^{s+2}}^2 d\tau < +\infty,$$

which means that the solution can be continued after $t = T^*$, and thus contradicts with the definition of T^*.

As an application of blow-up criterion, we can deduce the global existence in 2D. Indeed, in two dimensional case, we get by the Gagliardo–Nirenberg inequality and (4.3) that

$$\int_0^{T^*} \left\|\nabla c(t)\right\|_{L^\infty}^4 dt \leqslant C \int_0^{T^*} \left\|c(t)\right\|_{H^1}^2 \left\|c(t)\right\|_{H^3}^2 dt < +\infty,$$

which implies $T^* = +\infty$ by the blow-up criterion. □

5. A refined blow-up criterion in 3D

We first turn to a simple model relating to the Hele–Shaw–Cahn–Hilliard system:

$$\begin{cases} u = -\nabla p + \Delta c \nabla c, \quad \nabla \cdot u = 0, \\ c_t + u \cdot \nabla c + \Delta^2 c = 0. \end{cases} \tag{5.1}$$

For this system, we still have the energy equality:

$$\|\nabla c(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\Delta c(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2}^2\,d\tau = \|\nabla c_0\|_{L^2}.$$

Moreover, if c is a solution of (5.1), then $c_\lambda(t,x) \stackrel{\text{def}}{=} c(\lambda^4 t, \lambda x)$ is also a solution. It is easy to see that

$$\|\nabla c_\lambda(t,x)\|_{L^2} = \lambda^{\frac{d}{2}-1}\|\nabla c(\lambda^4 t, x)\|_{L^2}, \qquad \int_0^\infty \|\nabla\Delta c_\lambda(\tau)\|_{L^2}^2\,d\tau = \lambda^{2-d}\int_0^\infty \|\nabla\Delta c(\tau)\|_{L^2}^2\,d\tau.$$

Thus, the energy is scaling invariant for $d=2$. From this point of view, the 2D system is critical and the 3D system is supercritical like the 3D Navier–Stokes equations. Due to the bi-Laplacian Δ^2, there is no maximum principle for this system, which is the main obstacle to obtain the global existence in 3D case. For the 2D critical QG equation

$$\theta_t + (-\Delta)^{\frac{1}{2}}\theta + u\cdot\nabla\theta = 0, \quad u = \left(-(-\Delta)^{-\frac{1}{2}}\partial_{x_2}\theta, (-\Delta)^{-\frac{1}{2}}\partial_{x_1}\theta\right),$$

Caffarelli and Vasseur [8] proved the global regularity of weak solution. The key step of their proof is to prove the Hölder continuity of the solution by using the DeGiorgi method. Note that the quasigeostrophic equation has a maximum principle. For the 3D Hele–Shaw–Cahn–Hilliard system, we also show that the Hölder continuity of the solution will control the blow-up of the solution.

Theorem 5.1. *Let $\alpha\in(0,1)$ and $c_0(x)\in H^s(\mathbf{T}^3)$ for $s\geqslant 3$. Assume that (c,u) is the solution of (1.1) stated in Theorem 3.1. Let T^* be the maximal existence time of the solution. If $T^* < +\infty$, then*

$$\int_0^{T^*} \|c(t)\|_{C^\alpha}^{\frac{8}{\alpha}}\,dt = +\infty.$$

Proof. We will prove it by contradiction argument. Assume that $T^* < +\infty$ and

$$\int_0^{T^*} \|c(t)\|_{C^\alpha}^{\frac{8}{\alpha}}\,dt < +\infty. \tag{5.2}$$

Taking Δ_j to the third equation of (1.1) we obtain

$$\partial_t\Delta_j c + \frac{\mathbf{C}}{\mathbf{Pe}}\Delta^2\Delta_j c = -\Delta_j(u\cdot\nabla c) + \frac{1}{\mathbf{Pe}}\Delta\Delta_j f_0'(c).$$

Making an $L^2(\mathbf{T}^3)$ energy estimate, we get by Lemma A.1 that for $j\geqslant 0$,

$$\frac{d}{dt}\|\Delta_j c\|_{L^2}^2 + c2^{4j}\|\Delta_j c\|_{L^2}^2 \leqslant C\left(\|\Delta_j(u\cdot\nabla c)\|_{L^2} + \|\Delta f_0'(c)\|_{L^2}\right)\|\Delta_j c\|_{L^2}.$$

Dividing the above inequality by $\|\Delta_j c\|_{L^2}$ gives

$$\frac{d}{dt}\|\Delta_j c\|_{L^2} + c2^{4j}\|\Delta_j c\|_{L^2} \leqslant C\left(\|\Delta_j(u\cdot\nabla c)\|_{L^2} + \|\Delta f_0'(c)\|_{L^2}\right),$$

which implies that

$$\|\Delta_j c(t)\|_{L^2} \leqslant \|\Delta_j c_0\|_{L^2} + C\int_0^t e^{-c2^{4j}(t-\tau)}\left(\|\Delta_j(u\cdot\nabla c)(\tau)\|_{L^2} + \|\Delta f_0'(c(\tau))\|_{L^2}\right)d\tau. \tag{5.3}$$

We denote

$$\|c\|_{B^s_{2,\infty}} \stackrel{\text{def}}{=} \sup_{j\geqslant -1} 2^{js}\|\Delta_j c\|_{L^2}.$$

Using the definition of Sobolev space, it is easy to find that

$$\|c\|_{H^{s-\epsilon}}^2 \leqslant \sum_{j\geqslant -1} 2^{-2\varepsilon j}\|c\|_{B^s_{2,\infty}}^2 \leqslant C\|c\|_{B^s_{2,\infty}}^2, \quad \forall \varepsilon > 0.$$

It follows from (5.3) that

$$\begin{aligned}\|c(t)\|_{B^3_{2,\infty}} &\leqslant \|c(t)\|_{L^2} + \|c_0\|_{H^3}\\ &\quad + C\sup_{j\geqslant 0} 2^{3j}\int_0^t e^{-c2^{4j}(t-\tau)}\big(\|\Delta_j(u\cdot\nabla c)(\tau)\|_{L^2} + \|\Delta f_0'(c(\tau))\|_{L^2}\big)\,d\tau.\end{aligned} \tag{5.4}$$

Now we claim that

$$\|\Delta_j(u\cdot\nabla c)\|_{L^2} \leqslant C2^{j(1-\alpha)}\|u\|_{L^2}\|c\|_{C^\alpha}, \tag{5.5}$$

but we will show it later. Now we have

$$\begin{aligned}\|u\|_{L^2} &\leqslant C\|\mu(c)\nabla c\|_{L^2} \leqslant C\|c\|_{H^{3-\alpha}}\|c\|_{C^\alpha} + C\big(\|c\|_{L^3} + \|c\|_{L^6}^2\|c\|_{L^\infty}\big)\|\nabla c\|_{L^6}\\ &\leqslant C\big(1 + \|c\|_{H^1} + \|c\|_{H^1}^2\big)\|c\|_{C^\alpha}\|c\|_{B^3_{2,\infty}}.\end{aligned}$$

Here we used the product estimate

$$\|\Delta c\nabla c\|_{L^2} \leqslant C\|c\|_{H^{3-\alpha}}\|c\|_{C^\alpha} \leqslant C\|c\|_{B^3_{2,\infty}}\|c\|_{C^\alpha},$$

which can be proved as in Lemma A.2. And similarly we have

$$\|\Delta f_0'(c)\|_{L^2} \leqslant C\big(1 + \|c\|_{C^\alpha}^2\big)\|c\|_{H^2}.$$

Plugging the above estimates into (5.4) yields that

$$\begin{aligned}\|c(t)\|_{B^3_{2,\infty}} &\leqslant \|c(t)\|_{L^2} + \|c_0\|_{H^3}\\ &\quad + C\sup_{j\geqslant 0} 2^{j(4-\alpha)}\int_0^t e^{-c2^{4j}(t-\tau)}\big(1 + \|c\|_{H^1} + \|c\|_{H^1}^2\big)\big(1 + \|c\|_{C^\alpha}^2\big)\|c\|_{B^3_{2,\infty}}\,d\tau,\end{aligned}$$

which along with the Hölder inequality gives

$$\begin{aligned}\|c(t)\|_{L^\infty(0,t;B^3_{2,\infty})} &\leqslant \|c(t)\|_{L^\infty(0,t;L^2)} + \|c_0\|_{H^3}\\ &\quad + \big(1 + \|c\|_{L^\infty(0,t;H^1)} + \|c\|_{L^\infty(0,t;H^1)}^2\big)\big(t^{\frac{\alpha}{4}} + \|c\|_{L^{\frac{8}{\alpha}}(0,t;C^\alpha)}^2\big)\|c\|_{L^\infty(0,t;B^3_{2,\infty})}.\end{aligned}$$

The above argument is still valid on the interval $[T, T^*)$ for $T < T^*$. Thus we get by using (4.3) that

$$\begin{aligned}\|c(t)\|_{L^\infty(T,T^*;B^3_{2,\infty})} &\leqslant \|c_0\|_{H^1} + \|c_0(T)\|_{H^3}\\ &\quad + C\big(\|c_0\|_{H^1}\big)\big((T^*-T)^{\frac{\alpha}{4}} + \|c\|_{L^{\frac{8}{\alpha}}(T,T^*;C^\alpha)}^2\big)\|c\|_{L^\infty(T,T^*;B^3_{2,\infty})}.\end{aligned}$$

Due to (5.2), we can choose T such that

$$C\big(\|c_0\|_{H^1}\big)\big((T^*-T)^{\frac{\alpha}{4}} + \|c\|_{L^{\frac{8}{\alpha}}(T,T^*;C^\alpha)}^2\big) \leqslant \frac{1}{2}.$$

Then we obtain

$$\|c(t)\|_{L^\infty(T,T^*;B^3_{2,\infty})} \leqslant 2\big(\|c_0\|_{H^1} + \|c_0(T)\|_{H^3}\big),$$

which implies by $\|\nabla c\|_{L^\infty} \leqslant C\|c\|_{B^3_{2,\infty}}$ that

$$\int_0^{T^*} \|\nabla c(t)\|_{L^\infty}^4 \, dt < +\infty,$$

which is impossible by Theorem 4.1 if $T^* < +\infty$.

It remains to prove (5.5). As in the proof of Lemma A.2, we have

$$\begin{aligned}\Delta_j(u\cdot\nabla c) &= \Delta_j \sum_{|j-k|\leqslant 4} S_{k-1}u\cdot\nabla\Delta_k c + \Delta_j \sum_{|j-k|\leqslant 4} \Delta_k u\cdot\nabla S_{k-1}c + \Delta_j \sum_{|k-k'|\leqslant 1, k\geqslant j-3} \Delta_k u\cdot\nabla\Delta_{k'}c \\ &= A_1 + A_2 + A_3.\end{aligned}$$

We get by Lemma A.1 that

$$\|A_1\|_{L^2} \leqslant C \sum_{|j-k|\leqslant 4} \|S_{k-1}u\|_{L^2}\|\nabla\Delta_k c\|_{L^\infty} \leqslant C2^{j(1-\alpha)}\|u\|_{L^2}\|c\|_{C^\alpha},$$

and for A_2,

$$\begin{aligned}\|A_2\|_{L^2} &\leqslant C \sum_{|j-k|\leqslant 4} \|\Delta_k u\|_{L^2}\|\nabla S_{k-1}c\|_{L^\infty} \\ &\leqslant C\|u\|_{L^2} \sum_{|j-k|\leqslant 4} \sum_{\ell\leqslant k-2} 2^\ell \|\Delta_\ell c\|_{L^\infty} \\ &\leqslant C\|u\|_{L^2}\|c\|_{C^\alpha} \sum_{|j-k|\leqslant 4} \sum_{\ell\leqslant k-2} 2^{\ell(1-\alpha)} \leqslant C2^{j(1-\alpha)}\|u\|_{L^2}\|c\|_{C^\alpha},\end{aligned}$$

and due to $\nabla\cdot u = 0$,

$$\begin{aligned}\|A_3\|_{L^2} &\leqslant \Big\|\Delta_j \sum_{|k-k'|\leqslant 1, k\geqslant j-3} \nabla\cdot(\Delta_k u \Delta_{k'}c)\Big\|_{L^2} \\ &\leqslant C2^j \sum_{|k-k'|\leqslant 1, k\geqslant j-3} 2^{-k'\alpha}\|u\|_{L^2} 2^{k'\alpha}\|\Delta_{k'}c\|_{L^\infty} \\ &\leqslant C2^{j(1-\alpha)}\|u\|_{L^2}\|c\|_{C^\alpha}.\end{aligned}$$

Then the inequality (5.5) follows from the estimates of A_1, A_2 and A_3. The proof of Theorem 5.1 is completed. □

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Appendix A

Let us first recall some basic facts about the Littlewood–Paley theory. Let φ, χ be two functions in $C^\infty(\mathbf{T}^d)$ such that $\operatorname{supp}\widehat{\varphi} \subset \{\frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, $\operatorname{supp}\widehat{\chi} \subset \{|\xi| \leqslant \frac{4}{3}\}$ and

$$\widehat{\chi}(\xi) + \sum_{j\geqslant 0} \widehat{\varphi}\big(2^{-j}\xi\big) = 1.$$

Then the Littlewood–Paley operators are defined by

$$\Delta_j f = \varphi_j * f = \int_{\mathbf{T}^d} \varphi_j(x-y) f(y)\,dy, \qquad \varphi_j(x) = 2^{jd}\varphi\big(2^j x\big), \quad j \geqslant 0,$$

$$S_j f = \chi_j * f = \sum_{k=-1}^{j-1} \Delta_k f, \qquad \Delta_{-1} f = \chi * f.$$

Some classical spaces can be characterized in terms of Δ_j. Let $s \in \mathbf{R}$, then the Sobolev space $H^s(\mathbf{T}^d)$ is defined by

$$H^s\big(\mathbf{T}^d\big) \stackrel{\text{def}}{=} \Big\{ u \in \mathcal{D}'\big(\mathbf{T}^d\big) \colon \ \|u\|_{H^s}^2 \stackrel{\text{def}}{=} \sum_{j \geqslant -1} 2^{2js} \|\Delta_j u\|_{L^2}^2 < \infty \Big\}.$$

We denote by $(u, v)_{H^s}$ the inner product in $H^s(\mathbf{T}^d)$. And for $s \in (0,1)$, the Hölder space $C^s(\mathbf{T}^d)$ is defined by

$$C^s\big(\mathbf{T}^d\big) \stackrel{\text{def}}{=} \Big\{ u \in \mathcal{D}'\big(\mathbf{T}^d\big) \colon \ \|u\|_{C^s} \stackrel{\text{def}}{=} \sup_{j \geqslant -1} 2^{js} \|\Delta_j u\|_{L^\infty} \Big\}.$$

We refer to [28] for more details. Let us recall Bony's decomposition from [6]:

$$fg = T_f g + T_g f + R(f,g), \tag{A.1}$$

where

$$T_f g = \sum_{j \geqslant -1} S_{j-1} f \Delta_j g, \qquad R(f,g) = \sum_{|j-j'| \leqslant 1} \Delta_j f \Delta_{j'} g.$$

We also denote $\widetilde{R}(f,g) = T_g f + R(f,g)$.

Lemma A.1. *(See [9].) Let $k \in \mathbf{N}$, $1 \leqslant p \leqslant q \leqslant \infty$. Then there exists a positive constant C independent of j such that*

$$\big\|\partial^\alpha \Delta_j f\big\|_{L^q} + \big\|\partial^\alpha S_j f\big\|_{L^q} \leqslant C 2^{j|\alpha| + dj(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},$$
$$\|\Delta_j f\|_{L^p} \leqslant C 2^{-jk} \sup_{|\alpha| = k} \big\|\partial^\alpha \Delta_j f\big\|_{L^p}, \quad j \geqslant 0.$$

Lemma A.2. *Let $s \geqslant 0$. Then there holds*

$$\|fg\|_{H^s} \leqslant C\big(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}\big). \tag{A.2}$$

If $0 < \sigma \leqslant \frac{d}{2}$, then there holds

$$\|fg\|_{H^s} \leqslant C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}}\big). \tag{A.3}$$

Proof. The inequality (A.2) is classical, see [19]. Here we only present the proof of (A.3). Using Bony's decomposition (A.1) we write

$$\Delta_j(fg) = \Delta_j(T_f g) + \Delta_j(T_g f) + \Delta_j R(f,g).$$

Taking into consideration the support of Fourier transform of the term $T_f g$, we have

$$\Delta_j(T_f g) = \sum_{|j'-j| \leqslant 4} \Delta_j(S_{j'-1} f \Delta_{j'} g).$$

Due to $0 < \sigma \leqslant \frac{d}{2}$, this gives by Lemma A.1 that

$$\|S_j f\|_{L^\infty} \leqslant \begin{cases} C 2^{j\frac{d}{2}} \|f\|_{L^2}, & \text{if } \sigma = \frac{d}{2}, \\ C\sum_{k \leqslant j-1} 2^{k\frac{d}{2}} \|\Delta_k f\|_{L^2} \leqslant C 2^{j\sigma} \|f\|_{H^{\frac{d}{2}-\sigma}}, & \text{if } \sigma < \frac{d}{2}, \end{cases}$$

which implies that

$$\begin{aligned}\big\|\Delta_j(T_f g)\big\|_{L^2} &\leqslant C\sum_{|j'-j|\leqslant 4}\|S_{j'-1}f\|_{L^\infty}\|\Delta_{j'}g\|_{L^2}\\ &\leqslant C\|f\|_{H^{\frac{d}{2}-\sigma}}\sum_{|j'-j|\leqslant 4}2^{j'\sigma}\|\Delta_{j'}g\|_{L^2}\\ &\leqslant C2^{-js}c_j\|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}},\end{aligned}\tag{A.4}$$

here and hereafter $\{c_j\}_{j\geqslant -1}$ denotes a sequence satisfying $\|\{c_j\}_{j\geqslant -1}\|_{\ell^2}\leqslant 1$.

Similarly, we have

$$\begin{aligned}\big\|\Delta_j(T_g f)\big\|_{L^2} &\leqslant C\sum_{|j'-j|\leqslant 4}\|S_{j'-1}g\|_{L^\infty}\|\Delta_{j'}f\|_{L^2}\\ &\leqslant C\sum_{|j'-j|\leqslant 4}\|g\|_{L^\infty}\|\Delta_{j'}f\|_{L^2}\\ &\leqslant C2^{-js}c_j\|g\|_{L^\infty}\|f\|_{H^s}.\end{aligned}\tag{A.5}$$

Noticing that, after taking into account the support of the Fourier transforms,

$$\Delta_j R(f,g)=\sum_{j',j''\geqslant j-3;|j'-j''|\leqslant 1}\Delta_j(\Delta_{j'}f\Delta_{j''}g),$$

it follows from Lemma A.1 that

$$\begin{aligned}\big\|\Delta_j R(f,g)\big\|_{L^2} &\leqslant C\sum_{j',j''\geqslant j-3;|j'-j''|\leqslant 1}2^{j\frac{d}{2}}\|\Delta_{j'}f\|_{L^2}\|\Delta_{j''}g\|_{L^2}\\ &\leqslant C2^{-js}\sum_{j',j''\geqslant j-3;|j'-j''|\leqslant 1}2^{(j-j')(\frac{d}{2}+s)}2^{j'(\frac{d}{2}-\sigma)}\|\Delta_{j'}f\|_{L^2}2^{j''(s+\sigma)}\|\Delta_{j''}g\|_{L^2}\\ &\leqslant C2^{-js}c_j\|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}}.\end{aligned}\tag{A.6}$$

Thanks to the definition of Sobolev space, (A.3) follows from (A.4)–(A.6). □

Lemma A.3. *(See [28].) Let $s>0$. Assume that $F(\cdot)$ is a smooth function on $\mathbf{R}$ with $F(0)=0$. Then we have*

$$\big\|F(f)\big\|_{H^s}\leqslant C\big(1+\|f\|_{L^\infty}\big)^{\lfloor s\rfloor+1}\|f\|_{H^s},$$

where the constant C depends on $\sup_{k\leqslant\lfloor s\rfloor+2,|t|\leqslant\|f\|_{L^\infty}}\|F^{(k)}(t)\|_{L^\infty}$.

Lemma A.4. *Let $s>0$. Then there holds*

$$\big\|\langle D\rangle^s(fg)-f\langle D\rangle^s g\big\|_{L^2}\leqslant C\big(\|f\|_{H^{s+2}}\|g\|_{L^2}+\|f\|_{H^2}\|g\|_{H^{s-\frac{1}{2}}}\big).$$

If $s\in(0,1]$, then we have

$$\big\|\langle D\rangle^s(fg)-f\langle D\rangle^s g\big\|_{L^2}\leqslant C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

Here the Fourier multiplier $\langle D\rangle^s$ is defined by

$$\langle D\rangle^s f(x)=\sum_{k\in\mathbf{Z}^d}\big(1+|k|^2\big)^{\frac{s}{2}}e^{2\pi ik\cdot x}\widehat{f}(k).$$

Proof. Using Bony's decomposition (A.1) we write

$$\begin{aligned}&\langle D\rangle^s(fg)=\langle D\rangle^s(T_f g)+\langle D\rangle^s T_g f+\langle D\rangle^s R(f,g),\\ &f\langle D\rangle^s g=T_f\langle D\rangle^s g+T_{\langle D\rangle^s g}f+R\big(f,\langle D\rangle^s g\big).\end{aligned}$$

Thus we have

$$\langle D\rangle^s(fg) - f\langle D\rangle^s g = \langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g + \pi(f,g),$$

where

$$\pi(f,g) = \langle D\rangle^s T_g f + \langle D\rangle^s R(f,g) - T_{\langle D\rangle^s g} f - R\big(f, \langle D\rangle^s g\big).$$

As in the proof of (A.3), we can deduce by Lemma A.1 that

$$\big\|\pi(f,g)\big\|_{L^2} \leqslant C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

We illustrate the process by working out the estimate on the first term. Thanks to Lemma A.1, we have

$$\begin{aligned}\big\|\langle D\rangle^s T_g f\big\|_{L^2}^2 &= \sum_{j\geqslant -1}\big\|\Delta_j\langle D\rangle^s T_g f\big\|_{L^2}^2 \leqslant C\sum_{j\geqslant -1} 2^{2js}\|\Delta_j T_g f\|_{L^2}^2\\ &\leqslant C\sum_{|j-j'|\leqslant 4} 2^{2js}\|S_{j'-1}g\Delta_{j'}f\|_{L^2}^2\\ &\leqslant C\sum_{|j-j'|\leqslant 4} 2^{2js}\|S_{j'-1}g\|_{L^\infty}^2\|\Delta_{j'}f\|_{L^2}^2\\ &\leqslant C\sum_{|j-j'|\leqslant 4} 2^{2j(s+\frac{d}{2})}\|g\|_{L^2}^2\|\Delta_{j'}f\|_{L^2}^2\\ &\leqslant C\|g\|_{L^2}^2\|f\|_{H^{s+\frac{d}{2}}}^2 \leqslant C\|g\|_{L^2}^2\|f\|_{H^{s+2}}^2.\end{aligned}$$

Let $m(\xi_1,\xi_2)$ be the symbol of the paraproduct operator $T_f g$. Then $\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g$ has the symbol

$$m(\xi_1,\xi_2)\big(\langle\xi_1+\xi_2\rangle^s - \langle\xi_2\rangle^s\big),$$

which is supported in the region $|\xi_1+\xi_2| \sim |\xi_2|$. By the fundamental theorem of calculus we have

$$m(\xi_1,\xi_2)\big(\langle\xi_1+\xi_2\rangle^s - \langle\xi_2\rangle^s\big) = \int_0^1 \xi_1\cdot m(\xi_1,\xi_2)\nabla h^s(t\xi_1+\xi_2)\,dt, \qquad h^s(\xi) = \langle\xi\rangle^s.$$

It is easy to verify that $\langle\xi_1\rangle^\theta m(\xi_1,\xi_2)\nabla h^s(t\xi_1+\xi_2)\langle\xi_2\rangle^{1-\theta-s}$ with $\theta\in[0,1]$ is a Coifman–Meyer paraproduct uniformly for $t\in[0,1]$. Then we have

$$\big\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\big\|_{L^2} \leqslant C\big\|\langle D\rangle^{1-\theta} f\big\|_{L^p}\big\|\langle D\rangle^{s+\theta-1} g\big\|_{L^q}$$

for $\theta\in[0,1]$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $1<q<\infty$, see p. 106 in [31]. Taking $\theta=\frac{1}{2}$, $(p,q)=(\infty,2)$ for $d=2$, and $\theta=0$, $(p,q)=(6,3)$ for $d=3$, we obtain

$$\big\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\big\|_{L^2} \leqslant C\|f\|_{H^2}\|g\|_{H^{s-\frac{1}{2}}}.$$

In case of $s\in(0,1]$, taking $\theta = 1-s$ and $(p,q)=(\infty,2)$ we obtain

$$\big\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\big\|_{L^2} \leqslant C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

This completes the proof of Lemma A.4. □

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