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Conformal deformations of the Ebin metric and a generalized Calabi metric on the space of Riemannian metrics

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Abstract

We consider geometries on the space of Riemannian metrics conformally equivalent to the widely studied Ebin L^2 metric. Among these we characterize a distinguished metric that can be regarded as a generalization of Calabi's metric on the space of Kähler metrics to the space of Riemannian metrics, and we study its geometry in detail. Unlike the Ebin metric, its geodesic equation involves non-local terms, and we solve it explicitly by using a constant of the motion. We then determine its completion, which gives the first example of a metric on the space of Riemannian metrics whose completion is strictly smaller than that of the Ebin metric.

1. Introduction

Let M be an n-dimensional compact closed manifold, and consider the infinite-dimensional space $\mathcal{M}$ of all smooth Riemannian metrics on M. The space $\mathcal{M}$ is endowed with a natural L^2-type Riemannian structure, the Ebin metric [14],

$$g_{\mathrm{E}}(h,k)|_g := (h,k)_{\mathrm{E}} := \int_M \operatorname{tr}\bigl(g^{-1}hg^{-1}k\bigr)\,dV_g, \tag{1}$$

where $g \in \mathcal{M}$, $h, k \in T_g\mathcal{M}$, $T_g\mathcal{M}$ may be identified with the space $\Gamma(S^2T^*M)$ of smooth symmetric $(0,2)$-tensor fields on M, and $g^{-1}h$ represents the $(1,1)$-tensor dual to h with respect to g. This metric has received much attention since being introduced in the 1960s [14], see, e.g., [17,18,7,11], and has found various applications, for example in the Weil–Petersson geometry of moduli spaces of Riemann surfaces [16,25] and in the study of the moduli space $\mathcal{M}/\mathrm{Diff}(M)$ of Riemannian structures (e.g., [14,15,3]). A related pseudo-Riemannian metric, the DeWitt metric [13,23], has been used in the Hamiltonian formulation of general relativity.

Recently, the metric completion of $\overline{\mathcal{M}_{\mathrm{E}}}$ of $(\mathcal{M}, g_{\mathrm{E}})$ has been determined [9], and it was shown by means of examples that convergence in $\overline{\mathcal{M}_{\mathrm{E}}}$ is too weak to control any geometric quantities or to imply geometric convergence of any sort (e.g., Gromov–Hausdorff convergence) [10]. Therefore, it seems natural to look for other metrics on $\mathcal{M}$

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with the property that their metric completions are strictly contained in $\overline{\mathcal{M}_{\mathrm{E}}}$. In other words, metrics for which certain types of degenerations are excluded along convergent sequences. One purpose of this article is to take a first step in this direction by studying conformal deformations of the Ebin metric in the search for metrics with this and other distinguished properties.

Our first observation (Proposition 3.1) is that there is a distinguished metric in the conformal class characterized by the property that the tautological vector field $X|_g = g$ on $\mathcal{M}$ is parallel. This metric, which we call the *generalized Calabi metric* (or sometimes the *normalized Ebin metric*), is given by

$$g_{\mathrm{N}} := \frac{1}{V_g} g_{\mathrm{E}}, \quad g \in \mathcal{M},$$

where $V_g := \mathrm{Vol}(M, g)$ is the volume function on $\mathcal{M}$. We then restrict attention to conformal factors that depend on the volume, i.e., metrics on $\mathcal{M}$ of the form $e^{2f(V_g)} g_{\mathrm{E}}$, with f a smooth function on $\mathbb{R}_{>0}$, and mostly to the metrics $g_p := g_{\mathrm{E}}/V^p$, which serve as the basic models within this family, as they capture the possible degenerations of manifolds in terms of either volume collapse or blow-up. By studying this family of metrics, we then show that g_{N} has the smallest metric completion (Theorem 5.3), and in particular one that is smaller than that of the Ebin metric. This provides the first example of an L^2-type metric on $\mathcal{M}$ whose metric completion is strictly smaller than that of the Ebin metric.

An additional motivation for introducting g_{N} comes from the study of the subspace of Kähler metrics $\mathcal{H} \subset \mathcal{M}$ in a fixed Kähler class (when M admits a Kähler structure). In our previous work [12], we studied the intrinsic and extrinsic geometry of $\mathcal{H}$ in $\mathcal{M}$. We observed that the Ebin metric induces the so-called Calabi geometry on $\mathcal{H}$, and that this embedding is essentially as far from being totally geodesic as possible. It then seems natural to ask whether there exists a metric on $\mathcal{M}$ that still induces the Calabi geometry on $\mathcal{H}$ but with the property that $\mathcal{H}$ is totally geodesic. As before, it is natural to restrict to conformal deformations depending on the volume, this time since the volume is an invariant of the Kähler class, and so any such metric will induce the Calabi geometry on $\mathcal{H}$. We then show that to the extent possible, g_{N} is the unique metric with the aforementioned property. In particular, $\mathcal{H}$ is totally geodesic in the case that M is a Riemann surface. In general $\mathcal{H}$ is not totally geodesic, but by the Calabi–Yau Theorem it is isometric to the "Riemannian Kähler spaces" $\mathcal{P}g \cap \mathcal{M}_v$, consisting of metrics of fixed volume in a fixed conformal class, which are totally geodesic in $(\mathcal{M}, g_{\mathrm{N}})$ (Corollary 4.4).

One further possible application of the metric g_{N} is to the Ricci flow. Recently, we showed that in the Kähler setting there is a connection between the existence of Einstein metrics, the smooth convergence of the normalized Ricci flow, and the metric geometry of $(\mathcal{M}, g_{\mathrm{E}})$. Namely, a Kähler–Einstein metric exists on a Fano manifold if and only if the Kähler–Ricci flow converges in the metric completion of $(\mathcal{M}, g_{\mathrm{E}})$, and in particular if and only if the flow path has finite length [12, Theorem 6.3, Corollary 6.9]. It would be very interesting to find analogous results for other classes of Riemannian manifolds, perhaps ones for which the singularities of the Ricci flow can be understood fairly well. In studying this problem, it might prove useful to use the metric g_{N}, for which the submanifold $\mathcal{M}_v \subset \mathcal{M}$ of metrics of fixed volume v—which is preserved by the normalized Ricci flow—is also totally geodesic (Corollary 3.3).

Motivated by these and possible other applications of the metric g_{N} to geometric problems, we thus study the geometry of $(\mathcal{M}, g_{\mathrm{N}})$ in detail. Under the conformal change, the geodesic equation becomes substantially more difficult since it contains non-local terms involving integration over the whole manifold. The solution of the geodesic equation is obtained in several steps, building upon the work of Freed and Groisser for g_{E} [17]. A key extra ingredient here is an invariant of the g_{N}-geodesic flow, or a 'constant of motion' (Corollary 3.2). The solution of the geodesic equation (Theorem 4.1) gives a precise sense to how geodesics in $(\mathcal{M}, g_{\mathrm{N}})$ generalize those discovered by Calabi [4,5] for the subspace of Kähler metrics, which in turn bear several similarities with constrained geodesics of the Wasserstein metric in optimal transportation [6] (cf. [12]). We also compute the curvature of g_{N} and compare it to that of the metrics g_p (Section 3.2).

Finally, it should be noted that "weighted" L^2 type metrics were also studied by several authors on the space of simple closed curves in $\mathbb{R}^2$ (see [20,22,24] and references therein), and this can also be seen as another motivation for our study. Moreover, very recently, while the present article was being prepared, Bauer, Harms and Michor [2] have written down the geodesic equation for metrics conformal to g_{E} on $\mathcal{M}$, as well as much more general Sobolev-type metrics and metrics weighted by the scalar curvature function. Their main result is that for some of these metrics the exponential mapping is a local diffeomorphism. In this article, we go into greater depth for a smaller class of metrics by solving the geodesic equation, computing the curvature, estimating the distance function, and determining the metric completion.

The article is organized as follows. In Section 2, we briefly review the relevant preliminaries about $\mathcal{M}$. In Section 3, we discuss general conformal changes, mostly focusing on those involving functions of the volume. For the model metrics g_p we compute the curvature as well as find an invariant (or 'constant of motion') of the geodesic flow. Section 4 contains the solution of the initial value problem for g_{N}-geodesics, making use of the invariant of the geodesic flow. In Section 5, we study the distance functions of g_p and determine their metric completions. Some of the technical facts needed in this analysis are proven in Appendix A. Section 6 concludes with some further remarks and a few open questions.

2. Preliminaries

Since the preliminaries relevant to our results are covered in detail in [14,17,18], we will simply briefly summarize what we need in this section.

The manifold of metrics, $\mathcal{M}$, is easily seen to be an open cone in the Fréchet space $\Gamma(S^2T^*M)$ of smooth, symmetric $(0,2)$-tensors on the finite-dimensional, compact manifold M. As such, it is endowed with the structure of a Fréchet manifold (cf. [19] for background on Fréchet manifolds), and its tangent space at $g \in \mathcal{M}$ is canonically identified with $\Gamma(S^2T^*M)$.

The Ebin metric, defined in (1), is a smooth Riemannian metric. It is, however, a weak metric, meaning that the tangent spaces of $\mathcal{M}$ are incomplete with respect to the scalar product induced on them by the Ebin metric. For weak Riemannian metrics, the existence of the Levi-Civita connection is not guaranteed by general results. Nevertheless, the Ebin metric has a Levi-Civita connection which can be directly computed. Geodesics and curvature may also be directly computed. The Riemannian curvature of $(\mathcal{M}, g_{\mathrm{E}})$ is nonpositive, and the exponential mapping at any point $g \in \mathcal{M}$ is a real-analytic diffeomorphism from an open neighborhood of zero in $T_g\mathcal{M}$ to an open neighborhood of g in $\mathcal{M}$. (Both of these neighborhoods are taken in the C^∞ topology.)

With respect to g_{E}, we may orthogonally decompose the tangent space $T_g\mathcal{M}$ into the subspaces of traceless (satisfying $\operatorname{tr}(g^{-1}h) = 0$) and pure-trace (satisfying $h = \rho g$ for some $\rho \in C^\infty(M)$) tensor fields. Corresponding to this decomposition is a product manifold structure for $\mathcal{M}$. Denote by $\mathcal{V}$ the space of all smooth, positive volume forms on M; it is an open cone in $\Omega^n(M)$, the space of smooth top-degree forms. For any $g \in \mathcal{M}$ we denote by dV_g its induced volume form. Then for any $\mu \in \mathcal{V}$, with $\mathcal{M}_\mu := \{g \in \mathcal{M}\colon dV_g = \mu\} \subset \mathcal{M}$, there is a diffeomorphism

$$i_\mu : \mathcal{V} \times \mathcal{M}_\mu \to \mathcal{M}, \quad i_\mu(\nu, h) = (\nu/\mu)^{2/n} h. \tag{2}$$

That is, i_μ maps (ν, h) to the unique metric conformal to h with volume form ν. Thus $\mathcal{M} \cong \mathcal{V} \times \mathcal{M}_\mu$, and one sees that $(i_\mu)_*(T\mathcal{V})$ is the subbundle of $T\mathcal{M}$ consisting of pure-trace tensor fields, while a tangent space to the submanifold $\mathcal{M}_\mu$ is identified with the subspace of traceless tensor fields.

An identity that will be repeatedly used below is that the differential of the map $g \mapsto dV_g$ is $h \mapsto \frac{1}{2}\operatorname{tr}(g^{-1}h)\, dV_g$. Therefore, if we denote by $V = V_g := \int_M dV_g$ the volume function on $\mathcal{M}$, then the differential of $g \mapsto V_g$ is $h \mapsto \frac{1}{2}(g, h)_{\mathrm{E}}$.

For the remainder of the paper, all differentiability properties we refer to are implicitly meant to hold in the category of Fréchet manifolds; we again refer to [19] for background.

3. Conformal deformations of the Ebin metric

Let $f : \mathcal{M} \to \mathbb{R}$ be a twice continuously differentiable function, and consider the metric on $\mathcal{M}$,

$$g_f(h,k)|_g := e^{2f(g)} g_{\mathrm{E}}(h,k)|_g = e^{2f(g)} \int_M \operatorname{tr}\bigl(g^{-1}hg^{-1}k\bigr)\, dV_g, \quad h, k \in T_g\mathcal{M}, \tag{3}$$

conformal to the Ebin metric. The purpose of this section is to characterize two metrics in the conformal class of g_{E}. One metric, the generalized Calabi metric $g_{\mathrm{N}} = g_{\mathrm{E}}/V$, is characterized by its Levi-Civita connection (Proposition 3.1), and the other, the second Ebin metric $g_2 = g_{\mathrm{E}}/V^2$, by its curvature tensor (Proposition 3.6). We then restrict to the model metrics

$$g_p := \frac{1}{V_g^p} g_{\mathrm{E}}, \quad g \in \mathcal{M}, \tag{4}$$

for some integer p. We find invariants for their geodesic flows that will be important later in integrating the geodesic equation (Corollary 3.2 and Lemma 3.4), compute their curvature (Section 3.2), and describe a natural duality map on $\mathcal{M}$ that is a conformal isometry between g_p and g_{2-p} (Proposition 3.8) and that also conformally relates their curvature tensors.

3.1. Conformal deformations and the Levi-Civita connection

Our first observation is a characterization of the *generalized Calabi metric*

$$g_{\mathrm{N}} := g_1 = g_{\mathrm{E}}/V_g, \quad g \in \mathcal{M}.$$

Proposition 3.1. *Let $f : \mathcal{M} \to \mathbb{R}$ be a differentiable function. Let ∇^f denote the Levi-Civita connection of $e^{2f} g_{\mathrm{E}}$, and suppose that $\nabla^f g = 0$, where g denotes the tautological vector field $g \mapsto g$ on $\mathcal{M}$. Then $f(g) = -\frac{1}{2} \log V_g + C$ for some constant C.*

Proof. First, note that $\nabla^{g_{\mathrm{E}}} g = \frac{n}{4}\delta$, where δ is the Kronecker tensor. This follows from the formula for the Levi-Civita connection of g_{E} [14, (4.1)],

$$\nabla_h^{g_{\mathrm{E}}} k|_g = D_h k - \frac{1}{2}\big(hg^{-1}k + kg^{-1}h\big) + \frac{1}{4}\big(\operatorname{tr}\big(g^{-1}k\big)h + \operatorname{tr}\big(g^{-1}h\big)k - \operatorname{tr}\big(g^{-1}hg^{-1}k\big)g\big),$$

where h and k are any vector fields on $\mathcal{M}$ and $D_h k|_g = \frac{d}{dt}|_{t=0} k(g + th)$.

Next, recall that [1, p. 58],

$$\nabla_h^f k = \nabla_h^{g_{\mathrm{E}}} k + (\nabla_h f)k + (\nabla_k f)h - (h,k)_{\mathrm{E}} \nabla^{g_{\mathrm{E}}} f, \tag{5}$$

so $\nabla g = 0$ is equivalent to

$$0 = \frac{n}{4} h + df(h)g + df(g)h - (h,g)_{\mathrm{E}} \nabla^{g_{\mathrm{E}}} f, \quad \text{for all } h.$$

Plugging in $h = g$ shows that $\nabla^{g_{\mathrm{E}}} f$ is proportional to g; and by inspecting the equation again then necessarily $-\frac{n}{4} = df(g) = D_g f = \nabla_g^{g_{\mathrm{E}}} f$ and $dF(h)g = (h,g)_{\mathrm{E}} \nabla^{g_{\mathrm{E}}} f$. Combining these two equations yields $\nabla^{g_{\mathrm{E}}} f = -\frac{1}{4V} g$ and substituting this back into the second equation yields $df(h) = -\frac{1}{4}(g,h)_{\mathrm{N}}$. Now consider a path $\{g(t)\}$. Then

$$\frac{d}{dt} f\big(g(t)\big) = -\frac{1}{4V} \int_M \operatorname{tr}\big(g(t)^{-1} g_t\big)\, dV_{g(t)} = -\frac{1}{2} \frac{d}{dt} \log V_{g(t)},$$

hence $f(g) = -\frac{1}{2} \log V_g + C$ (as $\mathcal{M}$ is path connected), as desired. □

Since, by the proof above, g is the gradient vector field of $2 \log V$ with respect to g_{N}, we have the following corollary, which will prove crucial in integrating the geodesic equation for g_{N} in Section 4.

Corollary 3.2. *The Hessian of* $\log V$ *satisfies*

$$\nabla^{g_{\mathrm{N}}} d \log V = 0.$$

In particular, $\log V$ *is linear and* V *is either strictly monotone or constant along* g_{N}*-geodesics.*

By the above corollary, if $g(t)$ is a g_{N}-geodesic and $(V_{g(t)})_t(0) = 0$, then $V_{g(t)}$ is constant. This gives the following fact.

Corollary 3.3. *For any $v \in \mathbb{R}_+$, the submanifold $\mathcal{M}_v := \{g \in \mathcal{M}\colon\ V_g = v\}$ is totally geodesic in $(\mathcal{M}, g_{\mathrm{N}})$.*

As Corollary 3.5 below will imply, the above statement is true for g_p only when $p = 1$. In particular, it is false for the Ebin metric.

By using the Koszul formula (or else by using (5) and the known expression for $\nabla^{g_{\mathrm{E}}}$), one can directly compute the Levi-Civita connection of g_{N} for constant vector fields h, k to be

$$\nabla_h^{g_{\mathrm{N}}} k|_g = \frac{1}{4} g_{\mathrm{N}}(k,h) g - \frac{1}{4}\operatorname{tr}\bigl(g^{-1}hg^{-1}k\bigr) g - \frac{1}{2} hg^{-1}k - \frac{1}{2} kg^{-1}h + \frac{1}{4}\operatorname{tr}\bigl(g^{-1}h\bigr)k + \frac{1}{4}\operatorname{tr}\bigl(g^{-1}k\bigr)h - \frac{1}{4} g_{\mathrm{N}}(h,g)k - \frac{1}{4} g_{\mathrm{N}}(k,g)h. \tag{6}$$

It is torsion free (symmetric in h and k), and one checks directly that it is metric compatible, hence it is the Levi-Civita connection.

It is well known that along g_{E}-geodesics the volume is quadratic [17]. This is explained by the following lemma and corollary, which are in a similar vein to Corollary 3.2.

Lemma 3.4. *We have*

$$\nabla^{g_p} dV^{1-p} = \frac{n}{8}(1-p)^2 g_p.$$

Proof. By (5) and (6) for constant vector fields h, k,

$$\nabla_h^{g_p} k|_g = \frac{p}{4}(k,h)_{\mathrm{N}} g - \frac{1}{4}\operatorname{tr}\bigl(g^{-1}hg^{-1}k\bigr) g - \frac{1}{2} hg^{-1}k - \frac{1}{2} kg^{-1}h + \frac{1}{4}\operatorname{tr}\bigl(g^{-1}h\bigr)k + \frac{1}{4}\operatorname{tr}\bigl(g^{-1}k\bigr)h - \frac{p}{4}(h,g)_{\mathrm{N}} k - \frac{p^2}{4}(k,g)_{\mathrm{N}} h.$$

Thus,

$$\begin{aligned}
\nabla^{g_p} dV^{1-p}(h,k) &= \frac{1}{2}\nabla_h^{g_p}(1-p)V^{-p}(g,k)_{\mathrm{E}} - \frac{1}{2}(1-p)V^{-p}\bigl(g, \nabla_h^{g_p} k\bigr)_{\mathrm{E}} \\
&= -\frac{p(1-p)}{4V^{p+1}}(g,h)_{\mathrm{E}}(g,k)_{\mathrm{E}} - \frac{1-p}{2V^p}(h,k)_{\mathrm{E}} + \frac{1-p}{4V^p}\bigl(g, \operatorname{tr}\bigl(g^{-1}h\bigr)k\bigr)_{\mathrm{E}} \\
&\quad - \frac{np(1-p)}{8V^p}(h,k)_{\mathrm{E}} + \frac{n(1-p)}{8V^p}(h,k)_{\mathrm{E}} + \frac{1-p}{2V^p}(h,k)_{\mathrm{E}} \\
&\quad - \frac{1-p}{4V^p}\bigl(g, \operatorname{tr}\bigl(g^{-1}h\bigr)k\bigr)_{\mathrm{E}} + \frac{p(1-p)}{4V^{p+1}}(g,h)_{\mathrm{E}}(g,k)_{\mathrm{E}} \\
&= \frac{n}{8}(1-p)^2 g_p(h,k). \qquad \square
\end{aligned}$$

Corollary 3.5. *Along unit-speed g_p-geodesics $g(t)$, V^{1-p} grows quadratically ($p \neq 1$),*

$$V^{1-p}(t) = \frac{n}{16}(1-p)^2 t^2 + \frac{1-p}{2} a_0 t + V^{1-p}(0), \tag{7}$$

where

$$a_0 := \frac{1}{V_{g_0}^p}\int_M f(0)\, dV_{g_0}, \qquad f(t) := \operatorname{tr}\bigl(g^{-1} g_t\bigr).$$

In particular, $V_{g(t)}$ converges to 0 (if $p < 1$) or to ∞ (if $p > 1$) in finite time precisely along constant conformal directions, i.e., if $g_t(0) = \lambda g(0)$ for $\lambda \in \mathbb{R}$ and λ negative (if $p < 1$) or positive (if $p > 1$). Also, along a unit-speed g_p-geodesic (for any $p \in \mathbb{R}$),

$$\frac{d}{dt}\left(\frac{1}{V^p}\int_M f\, dV_g\right) = \frac{n}{4}(1-p). \tag{8}$$

Proof. Let $g(t)$ be a unit-speed g_p-geodesic. By the previous lemma, we have that

$$\frac{d^2}{dt^2}V^{1-p}(t) = \frac{d}{dt}\,dV^{1-p}\big(g_t(t)\big) = \nabla_{g_t(t)}\,dV^{1-p}\big(g_t(t)\big) = \frac{n}{8}(1-p)^2.$$

This proves that V^{1-p} grows quadratically, and (7) follows from the observation that $\frac{d}{dt}V^{1-p}(0) = \frac{1-p}{2}a_0$.

By the quadratic formula applied to (7), the equation $V^{1-p}(t_0) = 0$ has a real solution t_0 if and only if $a_0^2 - nV^{1-p}(0) \geqslant 0$. Furthermore, by the Cauchy–Schwarz inequality for the Hilbert–Schmidt scalar product on symmetric matrices, $f(t)^2 = \operatorname{tr}(g^{-1}g_t)^2 \leqslant \operatorname{tr}((g^{-1}g_t)^2)\operatorname{tr}(I) = n\operatorname{tr}((g^{-1}g_t)^2)$, with equality if and only if $g_t = \lambda g$ for some $\lambda \in \mathbb{R}$. Therefore, again using Cauchy–Schwarz,

$$\begin{aligned} a_0^2 &= \left(\frac{1}{V_{g_0}^p}\int_M f(0)\,dV_{g_0}\right)^2 \leqslant V_{g_0}^{1-p}\left(\frac{1}{V_{g_0}^p}\int_M f(0)^2\,dV_{g_0}\right) \\ &\leqslant V_{g_0}^{1-p}\left(\frac{n}{V_{g_0}^p}\int_M \operatorname{tr}\big(\big(g(0)^{-1}g_t(0)\big)^2\big)\,dV_{g_0}\right) = nV_{g_0}^{1-p}, \end{aligned}$$

where the last equality holds because $g_p(g_t(0), g_t(0)) = 1$ by assumption. Note that equality holds in the above if and only if $g_t(0)$ is a constant scalar multiple of $g(0)$. Thus this is the only condition under which $a_0^2 - nV^{1-p}(0) \geqslant 0$, and the statement about $V_{g(t)}$ converging to 0 or ∞ in finite time follows.

Finally, to prove (8), note that $\frac{d}{dt}V^{1-p}(t) = \frac{1-p}{2}\cdot\frac{1}{V^p(t)}\int_M f(t)\,dV_{g(t)}$. The result then follows from differentiating this equation and applying the fact, shown above, that $\frac{d^2}{dt^2}V^{1-p}(t) = \frac{n}{8}(1-p)^2$. □

Eq. (8) in the corollary above provides an integral for the geodesic flow of g_p which allows solving the geodesic equation explicitly, in the spirit of the work of Freed–Groisser. This will be carried out for g_{N} using Corollary 3.2 in Section 4.

3.2. Conformal deformations and curvature

Our main purpose in this subsection is to study which conformal deformations of g_{E} still have nonpositive curvature. The next result shows that there is precisely one metric of the form g_{E}/V^p, besides g_{E} itself, whose curvature is nonpositive—it is also the unique such metric with curvature conformal to that of g_{E}—and this characterizes the *second Ebin metric*

$$g_2 = g_{\mathrm{E}}/V^2,$$

among all conformal deformations that depend on the volume.

Proposition 3.6. *The curvature of g_p is nonpositive if and only if $p = 0$ or 2. Moreover, the curvature of $g_2 = g_{\mathrm{E}}/V^2$ is conformal to the curvature of g_{E},*

$$R^{g_2} = R^{g_{\mathrm{E}}}/V^2,$$

and this property characterizes g_2, up to scaling, among all conformal deformations $e^{2f}g_{\mathrm{E}}$ with $f : \mathcal{M} \to \mathbb{R}$ a smooth function depending only on V_g.

In the proof we make use of the following computation:

Proposition 3.7. *The curvature tensor of g_p is given by*

$$R^{g_p} = \frac{1}{V^p}R^{g_{\mathrm{E}}} + \frac{2p-p^2}{16}g_p \owedge \left(V^{2p-2}g^{\flat_p}\otimes g^{\flat_p} - \frac{n}{2}V^{p-1}g_p\right), \tag{9}$$

where $g^{\flat_p}$ is the 1-form dual to the tautological vector field g with respect to g_p and $\owedge$ denotes the Kulkarni–Nomizu product.

Let h, k be tangent vectors that are orthonormal with respect to g_p. The sectional curvature of the plane $\mathbb{R}\{h,k\}$ is

$$\sec^{g_p}(h,k) = \frac{1}{V^p}\sec^{g_{\mathrm{E}}}(h,k) - \frac{2p-p^2}{16}\big(V^{2p-2}(g,k)_p^2 + V^{2p-2}(g,h)_p^2 - nV^{p-1}\big). \tag{10}$$

Proof. The formula for the curvature under the conformal change $g_{\mathrm{E}} \mapsto e^{2f} g_{\mathrm{E}}$ is [1, p. 58],

$$R = \frac{1}{V}\left(R^{g_{\mathrm{E}}} + g_{\mathrm{E}} \owedge \left(\nabla^{g_{\mathrm{E}}} df - df \otimes df + \frac{1}{2}|df|_{\mathrm{E}}^2 g_{\mathrm{E}}\right)\right), \tag{11}$$

and this applies in infinite dimensions as can be verified from its proof. (Note that our convention is $R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$, the opposite of Besse's.) Let $f(g) := -\frac{1}{2}\log V_g$ and $f_p(g) := pf(g) = -\frac{p}{2}\log V_g$. Assume first that $p = 1$. We claim that

$$\nabla^{g_{\mathrm{E}}} df = \frac{1}{8}(g,\cdot)_{\mathrm{N}}(g,\cdot)_{\mathrm{N}} - \frac{n}{16}g_{\mathrm{N}} = 2\,df \otimes df - \frac{n}{16}g_{\mathrm{N}}. \tag{12}$$

To see this, compute using the formula $\nabla_h^{g_{\mathrm{E}}} g = \frac{n}{4}h$ (cf. the proof of Proposition 3.1) and the metric property of $\nabla^{g_{\mathrm{E}}}$ to deduce

$$\begin{aligned}\nabla^{g_{\mathrm{E}}} df(h,k) &= \big(\nabla_h^{g_{\mathrm{E}}} df\big)(k) = \nabla_h^{g_{\mathrm{E}}}\big(df(k)\big) - df\big(\nabla_h^{g_{\mathrm{E}}} k\big)\\ &= -\frac{1}{4}\nabla_h^{g_{\mathrm{E}}}\left(\frac{1}{V}(g,k)_{\mathrm{E}}\right) + \frac{1}{4}\big(g, \nabla_h^{g_{\mathrm{E}}} k\big)_{\mathrm{N}}\\ &= \frac{1}{8}(g,h)_{\mathrm{N}}(g,k)_{\mathrm{N}} - \frac{n}{16}(h,k)_{\mathrm{N}} - \frac{1}{4}\big(g, \nabla_h^{g_{\mathrm{E}}} k\big)_{\mathrm{N}} + \frac{1}{4}\big(g, \nabla_h^{g_{\mathrm{E}}} k\big)_{\mathrm{N}}\\ &= \frac{1}{8}(g,h)_{\mathrm{N}}(g,k)_{\mathrm{N}} - \frac{n}{16}(h,k)_{\mathrm{N}}.\end{aligned} \tag{13}$$

Second, note that $df = -\frac{1}{4}g^{\flat_1}$, $|df|_{\mathrm{N}}^2 = |\nabla f|_{\mathrm{N}}^2 = \frac{1}{16}|g|_{\mathrm{N}}^2 = \frac{n}{16}$, and $|df|_{\mathrm{N}}^2 g_{\mathrm{N}} = |df|_{\mathrm{E}}^2 g_{\mathrm{E}}$. Thus,

$$R^{g_{\mathrm{N}}} = \frac{1}{V}R^{g_{\mathrm{E}}} + \frac{1}{16}g_{\mathrm{N}} \owedge \left(g^{\flat_1} \otimes g^{\flat_1} - \frac{n}{2}g_{\mathrm{N}}\right). \tag{14}$$

To conclude the proof, note now that $\nabla^{g_{\mathrm{E}}} df_p = p\nabla^{g_{\mathrm{E}}} df = 2p\,df \otimes df - \frac{pn}{16}g_{\mathrm{N}}$, $df_p \otimes df_p = p^2\,df \otimes df$, and $\frac{1}{2}|df_p|_{\mathrm{E}}^2 g_{\mathrm{E}} = \frac{p^2}{2}|df|_{\mathrm{E}}^2 g_{\mathrm{E}} = \frac{p^2 n}{16}g_{\mathrm{N}}$. From (11) we thus obtain

$$R^{g_p} = \frac{1}{V^p}R^{g_{\mathrm{E}}} + \big(2p - p^2\big)g_p \owedge \left(df \otimes df - \frac{n}{32}V^{p-1}g_p\right).$$

Since $df = -\frac{1}{4}(g,\cdot)_{\mathrm{N}} = -\frac{1}{4}g^{\flat_1} = -\frac{1}{4}V^{p-1}g^{\flat_p}$, (9) follows.

Next, recall that

$$G \owedge H(a,b,c,d) = G(a,c)H(b,d) + G(b,d)H(a,c) - G(a,d)H(b,c) - G(b,c)H(a,d).$$

So if h and k are g_p-orthonormal, $g_p \owedge g_p(h,k,k,h) = 2(h,k)_p^2 - 2|h|_p^2|k|_p^2 = -2$, and

$$\begin{aligned}g_p \owedge \big(g^{\flat_p} \otimes g^{\flat_p}\big)(h,k,k,h) &= 2(h,k)_p\big(g^{\flat_p} \otimes g^{\flat_p}\big)(h,k)\\ &\quad - (h,h)_p\big(g^{\flat_p} \otimes g^{\flat_p}\big)(k,k) - (k,k)_p\big(g^{\flat_p} \otimes g^{\flat_p}\big)(h,h)\\ &= -(g,k)_p^2 - (g,h)_p^2.\end{aligned}$$

By definition, $\sec^{g_p}(h,k) = g_p(R^{g_p}(h,k)k,h)$, and so (10) follows. □

Proof of Proposition 3.6. Let $f = f(V_g)$ be a smooth function on $\mathcal{M}$. Then $df = f'\,dV = \frac{1}{2}f'(g,\cdot)_{\mathrm{E}}$, or $\nabla^{g_{\mathrm{E}}} f = \frac{1}{2}f'g$ and $\frac{1}{2}|\nabla^{g_{\mathrm{E}}} f|_{\mathrm{E}}^2 g_{\mathrm{E}} = \frac{n}{8}(f')^2 V g_{\mathrm{E}}$, while $df \otimes df = \frac{1}{4}(f')^2 g^{\flat_{\mathrm{E}}} \otimes g^{\flat_{\mathrm{E}}}$. A computation similar to (13) gives $\nabla^{g_{\mathrm{E}}} df = \frac{1}{4}f'' g^{\flat_{\mathrm{E}}} \otimes g^{\flat_{\mathrm{E}}} + \frac{n}{8}f' g_{\mathrm{E}}$. So

$$R^{g_f} = e^{2f} R^{g_E} + \frac{1}{4} e^{2f} g_E \owedge \left(\left(f'' - \left(f' \right)^2 \right) g^{\flat_E} \otimes g^{\flat_E} + \frac{n}{2} \left(f' + V \left(f' \right)^2 \right) g_E \right),$$

and analogously to the proof of (10), we may compute

$$\sec^{g_f}(h,k) = e^{2f} \sec^{g_E}(h,k) - \frac{f'' - (f')^2}{4e^{4f}} \left((g,k)_f^2 + (g,h)_f^2 \right) - \frac{n}{4e^{2f}} \left(f' + V \left(f' \right)^2 \right).$$

Suppose now that $\sec^{g_f} = e^{2f} \sec^{g_E}$. Then considering directions h, k tangent to $\mathcal{M}_\mu$ gives that $f' + V(f')^2 = 0$, from which it follows that either $f = -\log V + C$, i.e., $g_f = e^{2C} g_E / V^2$, or else $f' = 0$, i.e., $g_f = e^{2C} g_E$. □

3.3. Conformal transformations and a duality map

By Proposition 3.7, g_{2-p} and g_p have the same curvature tensor, up to a conformal factor. Here we observe that there is also a conformal diffeomorphism $F : \mathcal{M} \to \mathcal{M}$ that relates these two metrics, so they are in fact isometric, and in this sense g_2 does not provide a new geometry compared to g_E.

Consider the map $F : \mathcal{M} \to \mathcal{M}$ defined by $F(g) := V^q g$. Let $h \in T\mathcal{M}$ be a constant vector field. Then

$$dF(h) = \frac{d}{dt}\bigg|_{t=0} (g + th) V_{g+th}^q = V^q h + \frac{1}{2} q (g,h)_E V^{q-1} g.$$

Hence, a careful computation shows that

$$g_p\big(dF(h), dF(k)\big)\big|_{F(g)} = g_{p+\frac{n}{2}q(p-1)}(h,k) + \left(\frac{n}{4} q^2 + q \right) V^{(1-p)(1+\frac{n}{2}q)-2} (g,h)_E (g,k)_E.$$

To summarize, we have:

Proposition 3.8. *The diffeomorphism $F(g) = V^{-\frac{4}{n}} g$ of $\mathcal{M}$ is an isometry between the spaces $(\mathcal{M}, g_{2-p})$ and $(\mathcal{M}, g_p)$, and we have $V_{F(g)} = V_g^{-1}$ and $F^{-1} = F$. In particular, $(\mathcal{M}, g_2)$ and $(\mathcal{M}, g_E)$ are isometric.*

It is interesting to note that using this result one obtains rather effortlessly the solution of the geodesic equation for g_2, building on the much simpler one for g_E ([18, Thm. 3.2], [17, Thm. 2.3]). In fact, a direct solution of the g_2-geodesic equation using the fact that the inverse of the volume is quadratic (Corollary 3.5) is substantially more involved.

Remark 3.9. If ϕ is a positive differentiable function, and $F(g) := \phi(V_g) g$, then

$$\big(dF(h), dF(k)\big)_p\big|_{F(g)} = \frac{\phi^{n/2}(V_{F(g)})}{V^p} (h,k)_E|_g + V_{F(g)} (g,h)_E (g,k)_E \frac{\phi^{n/2} \phi'}{\phi} \left(\frac{n\phi'}{4\phi} V + 1 \right).$$

Hence, the only such map F that is an isometry between g_E and any g_p is given by Proposition 3.8.

4. Geometry of the generalized Calabi metric

In this section, we study the geometry of the metric g_N in more detail. In the first subsection we solve its geodesic equation for any given initial data. In the second subsection, we compute the sectional curvature, and examine the extrinsic geometry of certain submanifolds in the spirit of [12], showing that the Riemannian analogues of the space of Kähler metrics are totally geodesic. These spaces are naturally isometric (via the Calabi–Yau Theorem) to the usual spaces of Kähler metrics. These facts, together with the explicit formula for geodesics, give a precise meaning to the statement that g_N generalizes Calabi's geometry on the space of Kähler metrics.

4.1. Geodesics

From (6) we obtain the geodesic equation for $(\mathcal{M}, g_N)$,

$$\left(g^{-1} g_t\right)_t = \frac{1}{4} \operatorname{tr}\left(g^{-1} g_t g^{-1} g_t\right) \delta - \frac{1}{2} \operatorname{tr}\left(g^{-1} g_t\right) g^{-1} g_t + \frac{1}{2} (g_t, g)_N g^{-1} g_t - \frac{1}{4} |g_t|_N^2 \delta, \tag{15}$$

where δ denotes the Kronecker tensor corresponding to the identity matrix. The last two terms are the new terms compared to the geodesic equation for g_{E}. Since they are non-local, the solution of the equation becomes substantially more involved and requires making use the 'constant of motion' of the geodesic flow found in (8).

The solution of the initial value problem for the geodesic equation is given by the following theorem.

Theorem 4.1. *Let $g(0) \in \mathcal{M}$, and let $\mu_0 := dV_{g(0)}$. Then the geodesic in $(\mathcal{M}, g_{\mathrm{N}})$ emanating from $g(0)$, with initial tangent vector $(\alpha, A) \in T_{\mu_0} \operatorname{Vol}(M) \times T_{g(0)}\mathcal{M}_{\mu_0}$, is given by the following.*

Define $\sigma := |(\alpha, A)|_{\mathrm{N}}$ and

$$a_0 := \frac{2}{V(0)} \int_M \alpha, \qquad b_0 := \sqrt{\frac{n\sigma^2 - a_0^2}{4}}, \qquad q := \frac{\alpha}{\mu_0} - \frac{a_0}{2}, \qquad r := \sqrt{\frac{n}{4} \operatorname{tr}\left(\left(g(0)^{-1}A\right)^2\right)}.$$

First, if $b_0 = 0$, then $g(t, x) = e^{t\sigma\sqrt{n}} g(0, x)$.

If $b_0 \neq 0$, then for each $x \in M$,

$$g(t, x) = \left(\frac{1}{2}\left(1 - \frac{q^2 + r^2}{b_0^2}\right)\cos(b_0 t) + \frac{q}{b_0}\sin(b_0 t) + \frac{1}{2}\left(1 + \frac{q^2 + r^2}{b_0^2}\right)\right)^{\frac{2}{n}} \cdot e^{a_0 t/n} g(0) \exp\left[\frac{2}{r}\tan^{-1}\left(\frac{r\sin(b_0 t)}{b_0 + b_0\cos(b_0 t) + q\sin(b_0 t)}\right) g(0)^{-1} A\right]. \tag{16}$$

Here, we take the exponential term to be the identity if $A(x) = 0$.

If $A(x) \neq 0$, then in (16), arctangent takes values in $[\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2}]$ whenever $t \in [\frac{2\pi(k-1)+\theta}{b_0}, \frac{2\pi k+\theta}{b_0}]$ for $k \in \mathbb{Z}$, where

$$\theta(x) := \begin{cases} 2\pi - \cos^{-1}\left(\frac{q(x)^2 - b_0^2}{q(x)^2 + b_0^2}\right) & \text{if } q(x) \geqslant 0, \\ \cos^{-1}\left(\frac{q(x)^2 - b_0^2}{q(x)^2 + b_0^2}\right) & \text{if } q(x) < 0. \end{cases}$$

(Here, arccosine takes values in $[0, \pi]$.)

The domain of definition of $g(t)$ is $[0, \infty)$ if $b_0 = 0$. If $b_0 \neq 0$, the domain of definition is $[0, t_0)$, where t_0 is the infimum of $\theta(x)$ at points where $A(x) = 0$. (We take the infimum to be ∞ if there are no such points.) In the case where the geodesic exists for only finite time, it approaches a limit point on the boundary of $\mathcal{M} \subset \Gamma(S^2 T^* M)$ as $t \to t_0$; i.e., $\mu_{g(t)}(x) \to 0$ for at least one point $x \in M$.

Remark 4.2. Theorem 4.1 gives a precise meaning to the statement that g_{N} generalizes Calabi's geometry on the space of Kähler metrics. Indeed, on the level of volume forms, Calabi's geodesics in the space of Kähler metrics (or, via the Calabi–Yau Theorem, on the space of volume forms with total volume v) are given by

$$dV_{g(t)} = dV_g\left(G\sqrt{v}\sin\left(\frac{1}{2}t/\sqrt{v}\right) + \cos\left(\frac{1}{2}t/\sqrt{v}\right)\right)^2,$$

where $(dV_{g(0)})_t = G\, dV_{g(0)}$ [12, Remark 5.7]. On the other hand, in proving (16) one shows that the volume forms along g_{N}-geodesics satisfy an equation of a similar form—see (28)—and the two equations can actually be shown to exactly coincide when $A \equiv 0$ and $a_0 = 0$ by using trigonometric formulas and carefully identifying the integration constants.

Before we give the proof of this theorem, let us point out a contrast to the Ebin metric. Like the case of g_{E} (cf. [17, §2]), geodesics in $(\mathcal{M}, g_{\mathrm{N}})$ exist for all time if $A(x) \neq 0$ for all $x \in M$. However, the converse of this statement also holds—if $A(x) = 0$ for some $x \in M$, then the geodesic only exists for finite time—unless $b_0 = 0$. (In the case of g_{E}, this happens only when there is a point where $A(x) = 0$ and $(\alpha/\mu_0)(x) < 0$.)

Note also that, as in the case of the Ebin metric, any conformal class—a submanifold of the form $\mathcal{P}g$ with $g \in \mathcal{M}$—is totally geodesic, as can be seen from Theorem 4.1 by putting $A \equiv 0$.

We will solve the geodesic equation in the following subsections, beginning with general considerations and then considering various special cases.

4.1.1. The general case

We let $C := g^{-1}g_t$ and decompose into pure trace and traceless parts: $C =: E + \frac{f}{n}I$, with $\operatorname{tr} E = 0$, i.e., $f = \operatorname{tr} C$. From (15), we obtain the pair of coupled equations

$$E_t = -\frac{1}{2} f E + \frac{E}{2V}\int_M f\,dV_g, \tag{17}$$

and

$$f_t = \frac{n}{4}\operatorname{tr}\bigl(E^2\bigr) - \frac{f^2}{4} - \frac{n\sigma^2}{4} + \frac{f}{2V}\int_M f\,dV_g, \tag{18}$$

where $\sigma = |g_t|_{\mathrm{N}}$, which is constant since g is a geodesic. The last term in the first equation and the last two terms in the second equation are new compared to the unnormalized metric.

The following relations hold between E, f, and data related to the splitting $\mathcal{M} \cong \mathcal{V} \times \mathcal{M}_{\mu_0}$, where $\mu_0 := dV_{g_0}$ (cf. Section 2). We write $g = (\frac{\mu_g}{\mu_0})^{2/n}h$, where $\mu_g = dV_g$ and $h \in \mathcal{M}_{\mu_0}$, i.e., h is the unique metric conformal to g with $dV_h = \mu_0$. Then $g^{-1}g_t = h^{-1}h_t + \frac{2}{n}\frac{(\mu_g)_t}{\mu_g}I$, implying that $E = h^{-1}h_t$ and $f = 2\frac{(\mu_g)_t}{\mu_g}$.

We define

$$\phi := f - \frac{1}{V}\int_M f\,dV_g.$$

Note that $V^{-1}\int_M f\,dV_g = 2\frac{d}{dt}(\log V_{g(t)})$. Hence, by Corollary 3.2, this quantity is constant along $g(t)$. So defining

$$a_0 := \frac{1}{V(0)}\int_M f(0)\,dV_{g(0)},$$

we have $\phi = f - a_0$ and $\phi_t = f_t$.

Now, note that

$$-\frac{f^2}{4} + \frac{f}{2V}\int_M f = -\frac{1}{4}\phi^2 + \frac{1}{4}\left(\frac{1}{V}\int_M f\right)^2. \tag{19}$$

Using this, together with the considerations of the previous paragraph, we can rewrite (17)–(18) in terms of ϕ,

$$E_t = -\frac{\phi}{2}E, \tag{20}$$

$$\phi_t = \frac{n}{4}\operatorname{tr}\bigl(E^2\bigr) - \frac{n\sigma^2}{4} - \frac{\phi^2}{4} + \frac{a_0^2}{4}. \tag{21}$$

Note that

$$\bigl(\operatorname{tr}\bigl(E^2\bigr)\bigr)_t = 2\operatorname{tr}(E_t E) = \left(V^{-1}\int_M f\,dV_g - f\right)\operatorname{tr}\bigl(E^2\bigr) = -\phi\operatorname{tr}\bigl(E^2\bigr)$$

(so $\operatorname{tr}(E^2) = \exp(\int_0^t (V^{-1}\int_M f\,dV_g - f)\,ds)\operatorname{tr}(E^2(0))$). Hence, differentiating (21) yields

$$\phi_{tt} = -\frac{n}{4}\phi\operatorname{tr}\bigl(E^2\bigr) - \frac{1}{2}\phi\phi_t,$$

and substituting for $\operatorname{tr}(E^2)$ using (21) we obtain

$$4\phi_{tt} + 6\phi\phi_t + \phi^3 = \phi\bigl(a_0^2 - n\sigma^2\bigr). \tag{22}$$

We now let

$$p := \frac{\mu_g}{\mu_0}e^{-a_0 t/2}. \tag{23}$$

It follows that $2p_t/p = 2(\mu_g)_t/\mu_0 - a_0 = \phi$. Thus, the left-hand side of (22) equals $8p_{ttt}/p$, hence p satisfies

$$4p_{ttt} - \left(a_0^2 - n\sigma^2\right)p_t = 0. \tag{24}$$

Let

$$b_0 := \sqrt{\frac{n\sigma^2 - a_0^2}{4}}.$$

Note that b_0 is real-valued and positive, since $\sigma = |g_t(0)|_{\mathrm{N}}$ and so

$$\sigma^2 = \frac{1}{V(0)} \int_M \operatorname{tr}\left(C(0)^2\right) dV_{g(0)} \geqslant \frac{1}{nV(0)} \int_M f(0)^2 \, dV_{g(0)}$$
$$\geqslant \frac{1}{n}\left(\frac{1}{V(0)} \int_M f(0) \, dV_{g(0)}\right)^2 = \frac{a_0^2}{n}, \tag{25}$$

where the second inequality is Cauchy–Schwarz. Note that the first inequality is an equality if and only if $E(0) \equiv 0$, and the second inequality is an equality if and only if f is constant. Therefore, $b_0 = 0$ if and only if $g_t(0) = \lambda g(0)$ for some $\lambda \in \mathbb{R}$.

Now, integrating (24), we have

$$p_{tt} + b_0^2 p = C, \tag{26}$$

for some $C \in \mathbb{R}$. It follows that

$$p(t) = \begin{cases} C_1 \cos(b_0 t) + C_2 \sin(b_0 t) + C_3, & \text{if } b_0 \neq 0, \\ C_1 t^2 + C_2 t + C_3, & \text{if } b_0 = 0. \end{cases} \tag{27}$$

By (23), then,

$$\frac{\mu_g}{\mu_0} = \begin{cases} (C_1 \cos(b_0 t) + C_2 \sin(b_0 t) + C_3)e^{a_0 t/2}, & \text{if } b_0 \neq 0, \\ (C_1 t^2 + C_2 t + C_3)e^{a_0 t/2}, & \text{if } b_0 = 0. \end{cases} \tag{28}$$

We now consider the initial value data needed to determine the constants of integration. Note that (23) implies that $p(0) = 1$ and $p_t(0) = \alpha/\mu_0 - a_0/2$.

To determine $p_{tt}(0)$, we first use that $\phi = 2p_t/p$ to see that on the one hand,

$$\phi_t(0) = 2\left(\frac{p_t}{p}\right)_t(0) = 2\frac{p_{tt}(0)p(0) - (p_t(0))^2}{p(0)^2} = 2p_{tt}(0) - 2\left(\frac{\alpha}{\mu_0} - \frac{a_0}{2}\right)^2.$$

On the other hand, we see by (21) and the fact that $f(0) = 2\alpha/\mu_0$ that

$$\begin{aligned} p_{tt}(0) &= \frac{n}{4}\operatorname{tr}\left(E(0)^2\right) - \frac{n\sigma^2}{4} - \frac{\phi^2}{4} + \frac{a_0^2}{4} + 2\left(\frac{\alpha}{\mu_0} - \frac{a_0}{2}\right)^2 \\ &= \frac{n}{4}\operatorname{tr}\left(\left(g(0)^{-1}A\right)^2\right) - \frac{1}{4}\left(2\frac{\alpha}{\mu_0} - a_0\right)^2 - b_0^2 + 2\left(\frac{\alpha}{\mu_0} - \frac{a_0}{2}\right)^2 \\ &= \frac{1}{2}\left(q^2 + r^2 - b_0^2\right), \end{aligned}$$

where $q := \alpha/\mu_0 - a_0/2$ and $r := \sqrt{\frac{n}{4}\operatorname{tr}((g(0)^{-1}A)^2)}$.

This gives all the information needed to solve for μ_g/μ_0 in the individual cases. To solve for h, we must use (20) and the fact that $\phi = 2p_t/p$ to see that $E_t = -(\log p)_t E$, implying $E = E(0)/p = g(0)^{-1}A/p$. Since $E = h^{-1}h_t$, this gives

$$h^{-1}h_t(t) = g(0)^{-1}A/p(t). \tag{29}$$

We now give the solution of the geodesic equation for each special case.

4.1.2. The case $b_0 = 0$

In this case, we have $a_0 = \sigma\sqrt{n}$ and $g_t(0) = \lambda g(0)$ for some $\lambda \in \mathbb{R}$, as noted after (25). Therefore, $A \equiv 0$ and $q \equiv 0 \equiv r$, implying $p_t(0) \equiv 0 \equiv p_{tt}(0)$. This gives, in light of (27), $C_1 = 0 = C_2$, and $C_3 = 1$. Thus, $\mu_g/\mu_0 = e^{\sigma\sqrt{n}t/2}$ by (28), and $h(t) = g(0)$ by (29). The solution of the geodesic equation in this case now follows.

4.1.3. The case $b_0 \neq 0$, $A(x) = 0$

Here, (27) implies that

$$C_1 = \frac{1}{2}\left(1 - \frac{q^2}{b_0^2}\right), \qquad C_2 = \frac{q}{b_0}, \qquad C_3 = \frac{1}{2}\left(1 + \frac{q^2}{b_0^2}\right),$$

and thus

$$\frac{\mu_g}{\mu_0} = \frac{1}{2}\left(\left(1 - \frac{q^2}{b_0^2}\right)\cos(b_0 t) + 2\frac{q}{b_0}\sin(b_0 t) + 1 + \frac{q^2}{b_0^2}\right)e^{a_0 t/2}.$$

As in the previous case, since $A(x) = 0$, (29) gives $h(t) = g(0)$, so the solution of the geodesic equation in this case follows.

It remains only to determine the domain of definition of $g(t)$. Eq. (16) implies $g(t)$ is a smooth Riemannian metric unless the coefficient of $g(0)$ in that equation vanishes at some point $x \in M$, which happens if and only if $p(t, x) = 0$.

To see when this occurs in the case we are considering, set $a := \cos(b_0 t)$, so that $\sin(b_0 t) = \pm\sqrt{1 - a^2}$. Setting $p(t, x)$ equal to zero then leads to the quadratic equation

$$\left(1 - 2\frac{q^2}{b_0^2} + \frac{q^4}{b_0^4}\right)a^2 + 2\left(1 - \frac{q^4}{b_0^4}\right)a + \left(1 + 2\frac{q^2}{b_0^2} + \frac{q^4}{b_0^4}\right) = 4\frac{q^2}{b_0^2}\left(1 - a^2\right),$$

or

$$\left(\left(1 + \frac{q^2}{b_0^2}\right)a + \left(1 - \frac{q^2}{b_0^2}\right)\right)^2 = 0.$$

Plugging the solution $\cos(b_0 t) = a = \frac{q^2 - b_0^2}{q^2 + b_0^2}$ back into the original equation gives that $\sin(b_0 t)$ must be negative if $q > 0$, and positive if $q < 0$. Therefore, letting arccosine take values in $[0, \pi]$, we have that $p(t, x) = 0$ if and only if

$$t = \begin{cases} \frac{1}{b_0}\left(2\pi k - \cos^{-1}\left(\frac{q^2 - b_0^2}{q^2 + b_0^2}\right)\right), \ k \in \mathbb{Z}, & \text{if } q \geqslant 0, \\ \frac{1}{b_0}\left(2\pi k + \cos^{-1}\left(\frac{q^2 - b_0^2}{q^2 + b_0^2}\right)\right), \ k \in \mathbb{Z}, & \text{if } q < 0. \end{cases} \tag{30}$$

We also note that $p(t, x)$ is periodic in t, is zero for exactly one value of t in each period, and is positive for $t = 2\pi k/b_0$. Therefore, $p(t, x)$ is nonnegative for all t.

4.1.4. The case $A(x) \neq 0$

Similarly to the last case, we can compute the constants C_1, C_2, and C_3 to find

$$\frac{\mu_g}{\mu_0} = \frac{1}{2}\left(\left(1 - \frac{q^2 + r^2}{b_0^2}\right)\cos(b_0 t) + 2\frac{q}{b_0}\sin(b_0 t) + 1 + \frac{q^2 + r^2}{b_0^2}\right)e^{a_0 t/2}.$$

Either by integrating (29) or directly verifying that the following solves that equation, one sees that in this case,

$$\begin{aligned} h(t, x) &= g(0, x)\exp\left(\left(\int_0^t p(s)^{-1}\, ds\right)g(0, x)^{-1}A(x)\right) \\ &= g(0, x)\exp\left[\frac{2}{r}\left(\tan^{-1}\left(\frac{q}{r} + \frac{q^2 + r^2}{b_0 r}\tan\left(\frac{b_0}{2}t\right)\right) - \tan^{-1}\left(\frac{q}{r}\right)\right)g(0)^{-1}A\right]. \end{aligned} \tag{31}$$

Using the sum formula for arctangent and the half-angle formula for tangent, we can write this more elegantly as

$$h(t,x)=g(0,x)\exp\left[\frac{2}{r}\tan^{-1}\left(\frac{r\sin(b_0t)}{b_0+b_0\cos(b_0t)+q\sin(b_0t)}\right)g(0,x)^{-1}A(x)\right]. \tag{32}$$

As in the last case, (16) implies that $g(t)$ is a smooth Riemannian metric unless the coefficient of $g(0)$ is nonpositive. We claim that in this case, $p(t,x)>0$ for all t, implying the coefficient is always positive at x. To see this, we write

$$p(t,x)=\frac{r^2}{2b_0}\big(1-\cos(b_0t)\big)+\frac{1}{2}\left(\left(1-\frac{q^2}{b_0^2}\right)\cos(b_0t)+2\frac{q}{b_0}\sin(b_0t)+1+\frac{q^2}{b_0^2}\right).$$

Since $r>0$ in this case, the first term (involving r) is always nonnegative, and it is zero exactly when t is an integer multiple of $2\pi/b_0$.

On the other hand, the second term (involving q) is formally exactly the same as $p(t,x)$ from the previous case. In particular, it is always nonnegative, and is zero exactly for those values of t given in (30). But this shows that when the first term is zero, the second term is positive, and vice versa. Therefore $p(t,x)>0$ for all t.

Finally, to be precise, we must specify the branch of arctangent for various ranges of t in (32). That entails determining when the argument of arctangent in (32) becomes unbounded, and so we begin by finding when the denominator is zero. Again substituting $a:=\cos(b_0t)$ and setting the denominator equal to zero leads to the quadratic equation

$$b_0^2(1+a)^2=q^2\big(1-a^2\big),$$

which has solutions $a_1=\frac{q^2-b_0^2}{q^2+b_0^2}$ and $a_2=-1$. These two solutions coincide if $q=0$, and if $q\neq 0$, then the argument of arctangent in (32) approaches r/q as $t\to\pi=\cos^{-1}(-1)$. Therefore the argument remains bounded in this case, and so we are only interested in a_1. Substituting $\cos(b_0t)=a_1$ and $\sin(b_0t)=\pm\sqrt{1-a_1^2}$ into $b_0+b_0\cos(b_0t)+q\sin(b_0t)=0$ shows that in this case, $\sin(b_0t)$ must be negative if $q>0$ and positive if $q<0$. Note also that as b_0t approaches a_1 from below, the argument of arctangent in (32) approaches $+\infty$. Thus, the branch of arctangent jumps as t approaches the values

$$t=\begin{cases}\frac{1}{b_0}(2\pi k-\cos^{-1}(\frac{q^2-b_0^2}{q^2+b_0^2})),\ k\in\mathbb{Z}, & \text{if } q\geqslant 0,\\ \frac{1}{b_0}(2\pi k+\cos^{-1}(\frac{q^2-b_0^2}{q^2+b_0^2})),\ k\in\mathbb{Z}, & \text{if } q<0.\end{cases} \tag{33}$$

Since $p(t,x)>0$ for all t, the integral $\int_0^t p(s,x)^{-1}\,ds$ is strictly increasing; therefore, the branch of arctangent in (32) "jumps upwards" at each value of t in (33).

This completes the analysis of the final case in Theorem 4.1.

4.2. Curvature and relation with Calabi's space of Kähler metrics

We next restate Proposition 3.7 in the case $p=1$.

Theorem 4.3. *The curvature tensor of $g_{\mathrm N}=g_{\mathrm E}/V$ is given by*

$$R^{g_{\mathrm N}}=\frac{1}{V}R^{g_{\mathrm E}}+\frac{1}{16}g_{\mathrm N}\owedge\left(g^\flat\otimes g^\flat-\frac{n}{2}g_{\mathrm N}\right), \tag{34}$$

where $g^\flat$ is the 1-form dual to the tautological vector field g with respect to $g_{\mathrm N}$. Let h and k be unit tangent vectors with $(h,k)_{\mathrm N}=0$ and $|h|_{\mathrm N}^2=|k|_{\mathrm N}^2=1$. The sectional curvature of the plane $\mathbb{R}\{h,k\}$ is

$$\sec^{g_{\mathrm N}}(h,k)=\frac{1}{V}\sec^{g_{\mathrm E}}(h,k)-\frac{{(g,k)_{\mathrm N}}^2}{16}-\frac{{(g,h)_{\mathrm N}}^2}{16}+\frac{n}{16}. \tag{35}$$

A conformal class $\mathcal{P}g$ is totally geodesic (put $A \equiv 0$ in (16)). However, unlike the Ebin metric, it is no longer flat, and its curvature now changes sign. Furthermore, since $\sec^{g_{\mathrm{E}}}$ is nonpositive [17, Corollary 1.17], the sectional curvature of g_{N} is bounded from above by $\frac{n}{16}$.

Let (M, J, ω) be a compact closed Kähler manifold of complex dimension $m = n/2$, where J denotes the integrable almost complex structure and ω denotes the Kähler form. Denote by $\mathcal{H}$ the space of Kähler metrics cohomologous to ω. The higher-dimensional Riemannian analogue of $\mathcal{H}$ is the space of metrics of fixed volume v within a conformal class, $\mathcal{P}g \cap \mathcal{M}_v$ (where $\mathcal{M}_v := \{g\colon V_g = v\}$); in fact, these notions coincide for Riemann surfaces, while in higher dimensions, using the Calabi–Yau Theorem [26], $\mathcal{H}$ is isometric to $\mathcal{P}g \cap \mathcal{M}_v$ [12, §4.2]. Now, $\mathcal{H}$ is not totally geodesic in $(\mathcal{M}, g_{\mathrm{E}})$ [12, §3]. Yet it has constant positive curvature in the induced metric. This geometry on $\mathcal{H}$ is called Calabi's geometry [4,5] (see also [12]). The following corollary describes another sense (in addition to Theorem 4.1) in which g_{N} generalizes Calabi's geometry on the space of Kähler metrics. It is one of our motivations in introducing the metric g_{N}.

Corollary 4.4. *The space of metrics of fixed volume within a conformal class $\mathcal{P}g \cap \mathcal{M}_v$ is totally geodesic in $(\mathcal{M}, g_{\mathrm{N}})$, and has constant curvature $\frac{n}{16}$. In particular, when M is a Riemann surface the space of Kähler metrics $\mathcal{H}$ is a totally geodesic portion of a sphere in $(\mathcal{M}, g_{\mathrm{N}})$.*

In fact, for $p = 1$ and $m = 1$, $\mathcal{H} \subset \mathcal{M}$ is the intersection of the totally geodesic submanifolds $\mathcal{M}_v$ (cf. Corollary 3.3) and $\mathcal{P}g$.

In other words, g_{N} equips $\mathcal{M}$ with a geometry for which the "Riemannian Kähler spaces" $\mathcal{P}g \cap \mathcal{M}_v$ (which are isometric to $\mathcal{H}$) are totally geodesic portions of spheres, and in this sense extends Calabi's geometry to the whole of $\mathcal{M}$.

Remark 4.5. By (10), the space $\mathcal{P}g \cap \mathcal{M}_v$ has constant curvature $\frac{np(2-p)}{16V^{1-p}}$ in $(\mathcal{M}, g_p)$. However, by adapting the proof of [12, Proposition 3.1], one may readily show that this space is no longer totally geodesic for $p \neq 1$. In a related vein, but with a little more work, one may also show that $\mathcal{H}$ is no longer totally geodesic for g_{N} when $m > 1$.

5. The distance functions and the metric completions

In this section, we analyze the distance function d_p of g_p, especially in comparison with the much better-studied distance function d_{E} of the Ebin metric. These distance functions are defined in the usual way as the infimum of lengths of piecewise differentiable curves between two points.

Our main result gives one further way that the metric g_{N} is distinguished among the family considered in this article. Namely, the (metric) completion of $(\mathcal{M}, d_{\mathrm{N}})$ is strictly smaller than that of any other d_p. In fact, we will see that for each p, the completion of $(\mathcal{M}, d_p)$ is given by a quotient of the space of symmetric $(0,2)$-tensors that are measurable (as sections of S^2T^*M) and positive semidefinite. (The quotient is given by identifying tensors that agree wherever they are positive definite; equivalent tensors may disagree over a set where they are not positive definite.) However, if $p = 1$, then the completion consists only of such tensors with finite, positive total volume. If $p < 1$, the completion contains a point representing all such tensors with zero volume, and if $p > 1$, the completion contains a "point at infinity". (For precise statements, we refer to Section 5.3.)

In the process of proving the completion result, we will show that d_{E} and d_p, for $p \neq 1$, are equivalent on subsets of metrics with fixed bounds on their total volume (Section 5.2). It turns out that d_{E} and d_{N} are also equivalent on such subsets, but only locally (i.e., on small metric balls). While we suspect d_{E} and d_{N} are inequivalent when considered on the entirety of such a subset, we have no proof of this fact as yet.

5.1. The metric completion

To state the result about the completions of $(\mathcal{M}, d_p)$ in each of the cases mentioned above, we must introduce some notation.

Definition 5.1. We denote by $\mathcal{M}_f$ the set of measurable, positive-semidefinite sections $g : M \to S^2T^*M$ with finite total volume. That is, a section $g \in \mathcal{M}_f$ if and only if its restriction to any coordinate charts is a measurable mapping between subsets of Euclidean space, $g(x)(X,Y) \geqslant 0$ for any $x \in M$ and any $X, Y \in T_xM$, and $V_g = \int_M dV_g < \infty$. Here, dV_g is as usual given locally by $\sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$ (which induces a nonnegative measure since g is measurable and positive semidefinite).

We also define $\widehat{\mathcal{M}_f} := \mathcal{M}_f/\sim$. The equivalence relation $\sim$ is defined by $g \sim h$ if and only if the following statement holds almost surely (up to a Lebesgue-nullset): $g(x) \neq h(x)$ implies $\det g(x) = \det h(x) = 0$.

Remark 5.2. We note that the concept of a Lebesgue-nullset on a manifold, used in the above definition, is well defined independently of a volume form as a set whose image under any coordinate chart is a Lebesgue-nullset in $\mathbb{R}^n$.

We can now state the result, which we will prove in the remainder of this section.

Theorem 5.3. *The metric completion $\overline{(\mathcal{M}, d_p)}$ of $(\mathcal{M}, d_p)$ can be identified with*

1. $\widehat{\mathcal{M}_{f+}} := \mathcal{M}_{f+}/\sim$ *if $p = 1$, where $\mathcal{M}_{f+} \subset \mathcal{M}_f$ consists of those elements with positive total volume*;
2. $\widehat{\mathcal{M}_f}$ *if $p < 1$*;
3. $\widehat{\mathcal{M}_{f+}} \cup \{g_\infty\}$ *if $p > 1$, where g_∞ is a "point at infinity" represented by the single equivalence class of Cauchy sequences $\{h_k\}$ with $\lim_{k\to\infty} V_{h_k} = \infty$.*

In particular, $\overline{(\mathcal{M}, g_{\mathrm{N}})}$ is strictly contained in $\overline{(\mathcal{M}, g_p)}$ for all $p \neq 1$.

For $p \neq 1$, one can very heuristically view these completions as cones, where for $p < 1$ (resp. $p > 1$), metrics with zero (resp. infinite) volume are identified to a point. (Of course, there are other identifications occurring, so this picture is not very rigorous.) In the special, scale-invariant case $p = 1$, this cone is opened to a cylinder.

We begin proving the above theorem by showing the equivalence result mentioned at the beginning of the section.

5.2. The (local) equivalence of d_p and d_{E}

In this subsection, we show that d_p and d_{E} are equivalent metrics—as long as $p \neq 1$—on any subset of $\mathcal{M}$ satisfying an upper and lower bound on the total volume of any element in the subset. Furthermore, we will show that for any p, they are equivalent on small balls (of some uniformly positive radius) in such subsets. To do so, we first show that the function sending a metric to its total volume is continuous on $(\mathcal{M}, d_p)$ for any p. This allows us to prove the uniform local equivalence for any p mentioned above. Following that, we state a result that, in particular, implies that subsets of metrics with certain bounds on their total volumes have bounded diameter with respect to both d_p and d_{E}, for $p \neq 1$. (It is at this point that the proof fails for $p = 1$; however, we do not yet know whether $p \neq 1$ is an essential assumption.) A simple metric space argument then gives the global equivalence on the subsets we are considering.

We begin this process with the following lemma, which was inspired by [21, §3.3] and generalizes [8, Lemma 12].

Lemma 5.4. *Let $g, h \subseteq \mathcal{M}$. Then*

$$d_p(g,h) \geqslant \begin{cases} \frac{4}{(1-p)\sqrt{n}} |V_h^{\frac{1-p}{2}} - V_g^{\frac{1-p}{2}}|, & p \neq 1, \\ \frac{2}{\sqrt{n}} |\log(\frac{V_h}{V_g})|, & p = 1. \end{cases}$$

In particular, the function $g \mapsto V_g$ is continuous on $(\mathcal{M}, d_p)$.

Proof. Let $\gamma(t)$, $t \in [0,1]$, be any path from g to h, and define $k(t) := \gamma_t(t)$. We compute

$$\partial_t V_{\gamma(t)} = \frac{1}{2} \int_M \mathrm{tr}\big(\gamma^{-1}k\big)\, dV_\gamma \leqslant \frac{1}{2}\sqrt{V_\gamma} \left(\int_M \big(\mathrm{tr}\big(\gamma^{-1}k\big)\big)^2\, dV_\gamma \right)^{1/2}, \tag{36}$$

where we have used Hölder's inequality in the second line.

Let $k_0(t)$ denote the trace-free part of $k(t)$. By the orthogonality of traceless and trace-free matrices in the Hilbert–Schmidt product $\langle A, B\rangle = \operatorname{tr}(AB^T)$, and since $k = k_0 + \frac{1}{n}\operatorname{tr}(g^{-1}k)\gamma$, we have

$$\big(\operatorname{tr}(\gamma^{-1}k)\big)^2 = n\big(\operatorname{tr}\big((\gamma^{-1}k)^2\big) - \operatorname{tr}\big((\gamma^{-1}k_0)^2\big)\big) \leqslant n\operatorname{tr}\big((\gamma^{-1}k)^2\big).$$

Applying this to (36) gives

$$\partial_t V_{\gamma(t)} \leqslant \frac{1}{2}\sqrt{V_\gamma}\left(n\int_M \operatorname{tr}\big((\gamma^{-1}k)^2\big)\,dV_\gamma\right)^{1/2} \leqslant \frac{\sqrt{n}}{2}\sqrt{V_\gamma}|k|_E = \frac{\sqrt{n}}{2}V_{\gamma(t)}^{\frac{1+p}{2}}|k|_p.$$

Now, let $p \neq 1$. We estimate

$$\begin{aligned} V_h^{\frac{1-p}{2}} - V_g^{\frac{1-p}{2}} &= \int_0^1 \partial_t V_{\gamma(t)}^{\frac{1-p}{2}}\,dt = \frac{1-p}{2}\int_0^1 \partial_t V_{\gamma(t)} V_{\gamma(t)}^{\frac{-1-p}{2}}\,dt \\ &\leqslant \frac{(1-p)\sqrt{n}}{4}\int_0^1 \big|k(t)\big|_p\,dt = \frac{(1-p)\sqrt{n}}{4}L_p(\gamma). \end{aligned} \tag{37}$$

Since this inequality holds for all paths from g to h, and we can repeat the computation with g and h interchanged, it implies the result for $p \neq 1$. The case $p = 1$ follows analogously to (37) if one begins with the quantity $\log(V_h) - \log(V_g)$ on the left-hand side. □

The following is an immediate corollary, and hints at the completions described in the introduction to this section.

Corollary 5.5. *If $\{h_k\} \subset \mathcal{M}$ is a d_p-Cauchy sequence, then $\{V_{h_k}\}$ converges in $\mathbb{R}_+ \cup \{0\}$ (for $p < 1$), $\mathbb{R}_+$ (for $p = 1$), or $\mathbb{R} \cup \{+\infty\}$ (for $p > 1$).*

Lemma 5.4 also yields the following comparison between d_p and d_{E}.

Corollary 5.6. *Let $v' > v > 0$ be given. Define $\mathcal{M}_{v,v'} := \{g \in \mathcal{M}:\ v < V_g < v'\}$. Then there exists $\delta = \delta(v, v') > 0$ such that if $g \in \mathcal{M}_{v,v'}$ and $h \in \mathcal{M}$, then*

1. *$d_{\mathrm{E}}(g,h) < \delta$ implies $d_p(g,h) < \max\{(2v')^{-p}, (\frac{v}{2})^{-p}\}\, d_{\mathrm{E}}(g,h)$, and*
2. *$d_p(g,h) < \delta$ implies $d_{\mathrm{E}}(g,h) < \max\{(2v')^{p}, (\frac{v}{2})^{p}\}\, d_p(g,h)$.*

Proof. By Lemma 5.4, the function $g \mapsto V_g$ is uniformly continuous with respect to both d_p and d_{E} on $\mathcal{M}_{v,v'}$. So we can choose δ small enough that if $g \in \mathcal{M}_{v,v'}$, $h \in \mathcal{M}$, and either $d_p(g,h) < 2\delta$ or $d_{\mathrm{E}}(g,h) < 2\delta$, then $\frac{v}{2} < V_h < 2v'$.

Let g, h, and δ be as above, let $0 < \epsilon < \delta$ be arbitrary, and let $\{\gamma(t)\}_{t\in[0,1]}$ be any piecewise differentiable path connecting g and h that satisfies $L_{\mathrm{E}}(\gamma) < d_{\mathrm{E}}(g,h) + \epsilon$, where we denote by L_{E} and L_p the length with respect to g_{E} and g_p, respectively. Since $d_{\mathrm{E}}(g,\gamma(t)) < 2\delta$ for any $t \in [0,1]$, $\frac{v}{2} < V_{\gamma(t)} < 2v'$ for all t. Thus we may estimate

$$L_p\big(\gamma(t)\big) = \int_0^1 \big|\gamma_t(t)\big|_p\,dt = \int_0^1 V^{-p}\big|\gamma_t(t)\big|_E\,dt \leqslant \max\left\{\big(2v'\big)^{-p}, \left(\frac{v}{2}\right)^{-p}\right\} L_{\mathrm{E}}\big(\gamma(t)\big).$$

Since $L_{\mathrm{E}}(\gamma(t)) < d_{\mathrm{E}}(g,h) + \epsilon$ and ϵ was arbitrarily small, this proves statement (1). Statement (2) is then proved completely analogously. □

Since g_p is, as discussed in Section 2, a weak Riemannian metric, the distance function d_p does not a priori induce a metric space structure on $\mathcal{M}$ as it is not a priori positive definite. (Note that the other metric space axioms follow, as in the finite-dimensional case, directly from the definition of the distance function as the infimum of the lengths of piecewise differentiable curves between two given points; e.g., the triangle inequality follows from an argument involving the concatenation of two curves.) In fact, there are examples (e.g., due to Michor and Mumford [20,21]) of weak Riemannian manifolds with induced distance between any two points zero. However, it has been shown [8, Theorem 18] that d_{E} does induce a metric space structure on $\mathcal{M}$, and so Corollary 5.6 gives:

Corollary 5.7. *$(\mathcal{M}, d_p)$ is a metric space.*

We now give a proposition that estimates d_p from above in a way that is, at least in spirit, converse to Lemma 5.4. This proposition allows us to bound the distance between two metrics based only on their total volumes and the intrinsic volumes of the set on which they differ. A direct consequence is a diameter bound for subsets of metrics satisfying a bound on their total volumes.

Proposition 5.8. *Suppose that $g, h \in \mathcal{M}$, and let $E := \operatorname{carr}(h - g) = \{x \in M \mid g(x) \neq h(x)\}$. If $p \neq 1$, then there exists a constant $C(p, n)$, depending only on p and $n = \dim M$, such that*

$$d_p(g, h) \leqslant C(p, n) \cdot \bigl(V_g^{-p/2}\sqrt{\operatorname{Vol}(E, g)} + V_h^{-p/2}\sqrt{\operatorname{Vol}(E, h)}\bigr).$$

In particular, let $0 < v < \infty$. Then if $p < 1$, we have

$$\operatorname{diam}_{d_p}\bigl(\bigl\{\tilde{g} \in \mathcal{M} \bigm| \operatorname{Vol}(M, \tilde{g}) \leqslant v\bigr\}\bigr) \leqslant 2C(p, n) v^{\frac{1-p}{2}}.$$

If $p > 1$, then we have

$$\operatorname{diam}\bigl(\bigl\{\tilde{g} \in \mathcal{M} \bigm| \operatorname{Vol}(M, \tilde{g}) \geqslant v\bigr\}\bigr) \leqslant 2C(p, n) v^{\frac{1-p}{2}}.$$

Since the proof of this proposition is rather lengthy, we postpone it to Appendix A.

Corollary 5.6 and Proposition 5.8 imply, with just a little extra work, that d_p ($p \neq 1$) and d_{E} are equivalent on the sets $\mathcal{M}_{v,v'}$ defined in Corollary 5.6.

Corollary 5.9. *Let $p \neq 1$ and $0 < v, v' < \infty$. Then d_p and d_{E} are equivalent on $\mathcal{M}_{v,v'}$ (where by d_p and d_{E} we mean the extrinsic distance induced by g_p and g_{E}, respectively, on this subset).*

Proof. Let $g, h \in \mathcal{M}_{v,v'}$.

Corollary 5.6 implies that there exist $\epsilon > 0$ and $1 \leqslant \eta < \infty$ such that if either $d_p(g, h) \leqslant \epsilon$ or $d_{\mathrm{E}}(g, h) \leqslant \epsilon$, then

$$\eta^{-1} d_p(g, h) \leqslant d_{\mathrm{E}}(g, h) \leqslant \eta\, d_p(g, h). \tag{38}$$

On the other hand, let $d_{\mathrm{E}}(g, h) > \epsilon$; then the preceding paragraph gives $d_p(g, h) > \eta^{-1}\epsilon$. Furthermore, Proposition 5.8 implies that there exists $D < \infty$ such that the diameter of $\mathcal{M}_{v,v'}$ is at most D with respect to both d_p and d_{E}, so we also have $d_{\mathrm{E}}(g, h), d_p(g, h) \leqslant D$. Thus,

$$d_p(g, h) > \eta^{-1}\epsilon = \frac{\eta^{-1}\epsilon}{D} D \geqslant \frac{\eta^{-1}\epsilon}{D} d_{\mathrm{E}}(g, h),$$

and

$$d_p(g, h) \leqslant D = \frac{D}{\epsilon}\epsilon < \frac{D}{\epsilon} d_{\mathrm{E}}(g, h).$$

This completes the proof. □

5.3. The completion of $(\mathcal{M}, d_p)$

Using these results, together with the characterization of the completion of $(\mathcal{M}, d_{\mathrm{E}})$ in [9], we can prove Theorem 5.3.

First, though, we need to recall the completion of $(\mathcal{M}, d_{\mathrm{E}})$, as determined in [9]. This requires some background discussion.

Definition 5.10. Let $\mathcal{M}_x := S^2_+ T^*_x M$ denote the set of positive-definite $(0,2)$-tensors at $x \in M$; its tangent spaces are given by $T_a \mathcal{M}_x \cong S^2 T^*_x M$. Define a Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{M}_x$ by $\langle b, c \rangle_a := \operatorname{tr}(a^{-1} b a^{-1} c) \sqrt{\det(\tilde{g}(x)^{-1} a)}$, where $\tilde{g} \in \mathcal{M}$ is any fixed reference metric.

Let d_x denote the distance function of $\langle \cdot, \cdot \rangle$ on $\mathcal{M}_x$. Define a metric (in the sense of metric spaces) on $\mathcal{M}$ by

$$\Omega_2(g, h) := \left(\int_M d_x\big(g(x), h(x)\big)^2 \, dV_{\tilde{g}} \right)^{1/2}.$$

It is not hard to see that Ω_2 is indeed a metric, and one can show that it does not depend on the arbitrary choice of $\tilde{g}$ (see [11]). The completion of $(\mathcal{M}_x, d_x)$ is given by $\mathrm{cl}(\mathcal{M}_x)/\partial \mathcal{M}_x$, that is, by all positive-semidefinite $(0,2)$-tensors at x, with tensors that are not positive definite identified to a point. A sequence $\{a_k\} \subset \mathcal{M}_x$ converges in the completion to $[0]$, the equivalence class of the zero tensor, if and only if $\det(\tilde{g}(x)^{-1} a_k) \to 0$ [11, Proposition 18]. One can use this fact to show that the metric Ω_2 can also be extended in a well-defined way to $\widehat{\mathcal{M}_f}$ [11, §4.1].

In fact, we have the following theorem, which in particular says that, like curvature and geodesics, the distance between points (and in a sense the completion) of $(\mathcal{M}, g_{\mathrm{E}})$ can be computed "fiberwise".

Theorem 5.11. *(See [9, Theorem 5.17], [11, Theorem 22].) For all $g, h \in \mathcal{M}$, $d_{\mathrm{E}}(g, h) = \Omega_2(g, h)$.*

The metric completion $\overline{(\mathcal{M}, g_{\mathrm{E}})}$ of $(\mathcal{M}, g_{\mathrm{E}})$ is identified with $\widehat{\mathcal{M}_f}$. That is, for each d_{E}-Cauchy sequence $\{h_k\} \subset \mathcal{M}$, there exists a unique element $h \in \widehat{\mathcal{M}_f}$ such that $\Omega_2(h_k, h) \to 0$. Furthermore, if $\{\tilde{h}_k\} \subset \mathcal{M}$ is another d_{E}-Cauchy sequence with $\lim_{k\to\infty} d(h_k, \tilde{h}_k) = 0$, then $\Omega_2(\tilde{h}_k, h) \to 0$ as well.

Using the (local) equivalence of d_{E} and d_p, as well as the completion of $(\mathcal{M}, g_{\mathrm{E}})$ as a basis for comparison, we can now prove Theorem 5.3.

Proof of Theorem 5.3. We begin with general arguments. Following that, we treat the specifics of each of the three cases.

Let $\{h_k\}$ be a d_p-Cauchy sequence. By Corollary 5.5, $\{V_{h_k}\}$ converges either to a nonnegative real number or infinity. Let's assume that it converges to a positive number. Then there exist $0 < v \leqslant v' < \infty$ such that $\{h_k\} \subset \mathcal{M}_{v,v'}$ (with notation as in Corollary 5.6). But then Corollary 5.6 implies that $\{h_k\}$ is d_{E}-Cauchy as well. Therefore, by Theorem 5.11, $\{h_k\}$ Ω_2-converges to a unique limit point h in $\widehat{\mathcal{M}_f}$ with $V_h > 0$. This shows there exists a mapping from the set of d_p-Cauchy sequences in $\mathcal{M}$ with positive volume in the limit to $\widehat{\mathcal{M}_{f+}}$.

To see that this induces a well-defined mapping from a subset of the completion $\overline{(\mathcal{M}, d_p)}$ to $\widehat{\mathcal{M}_{f+}}$, we must show that if $\{h_k\}$ and $\{\tilde{h}_k\}$ are d_p-Cauchy sequences with positive volume in the limit and $\lim_{k\to\infty} d_p(h_k, \tilde{h}_k) = 0$, then $\{h_k\}$ and $\{\tilde{h}_k\}$ Ω_2-converge to the same element $h \in \widehat{\mathcal{M}_{f+}}$. But in this case there exist $0 < \tilde{v} \leqslant \tilde{v}' < \infty$ such that $\{h_k\}$ and $\{\tilde{h}_k\}$ both lie in $\mathcal{M}_{\tilde{v},\tilde{v}'}$, so this is implied by Corollary 5.6 and Theorem 5.11.

On the other hand, the same argument, with the roles of d_{E} and d_p reversed, shows that if $\{h_k\}$ is a d_{E}-Cauchy sequence with $\lim_{k\to\infty} V_{h_k} > 0$, then $\{h_k\}$ is d_p-Cauchy. Therefore, the mapping from this subset of $\overline{(\mathcal{M}, d_p)}$ to $\widehat{\mathcal{M}_{f+}}$ is surjective.

To see that the mapping from this subset of $\overline{(\mathcal{M}, d_p)}$ to $\widehat{\mathcal{M}_{f+}}$ is injective, we must show that if $\{h_k\}$ and $\{\tilde{h}_k\}$ are Cauchy sequences with positive volume in the limit and $\lim_{k\to\infty} d_p(h_k, \tilde{h}_k) \neq 0$, then the Ω_2-limits of $\{h_k\}$ and $\{\tilde{h}_k\}$ differ. But as in the proof that the mapping is well defined, this follows from Corollary 5.6 and Theorem 5.11.

Now, consider the case $p = 1$. Here, Corollary 5.5 implies that all Cauchy sequences have positive volume in the limit, so the preceding arguments suffice for this case.

If $p < 1$, the only remaining d_p-Cauchy sequences $\{h_k\}$ are those for which $\lim_{k\to\infty} V_{h_k} = 0$, again by Corollary 5.5. To complete the proof of the theorem, we must show that if $\{h_k\}$ and $\{\tilde{h}_k\}$ are two such sequences, then $\lim_{k\to\infty} d_p(h_k, \tilde{h}_k) = 0$. But this follows from Proposition 5.8.

The case $p > 1$ follows from the case $p < 1$ using the isometry of Proposition 3.8. □

6. Remarks and open questions

6.1. (Non-)control over geometry via d_p

In [10, Example 4.17], it was shown that the metric d_{E} is too weak to control, in any reasonable way, various geometric quantities associated to elements of $\mathcal{M}$. That is, functions mapping a metric in $\mathcal{M}$ to its curvature, distance function, diameter, or injectivity radius are discontinuous, even in some weakened sense.

In fact, the same examples constructed in [10] for d_{E} are also valid for d_p. To see this, and make it precise, we give a result analogous to Proposition 5.8, with a statement weakened in order to handle the case $p = 1$. It only gives an upper bound on the distance between metrics that agree as tensors somewhere on M. On the other hand, if two metrics differ everywhere (the generic case), this proposition gives no information.

Proposition 6.1. *Suppose that $g, h \in \mathcal{M}$, and let $E := \operatorname{carr}(h - g) = \{x \in M \mid g(x) \neq h(x)\}$. Given a measurable subset $A \subseteq M$ and $\tilde{g} \in \mathcal{M}$, let*

$$V^A_{p,\tilde{g}} := \max\bigl\{V_{\tilde{g}}^{-p/2}, \operatorname{Vol}(M \setminus A, \tilde{g})^{-p/2}\bigr\}.$$

Then there exists a constant $C(n)$, depending only on $n = \dim M$, such that

$$d_p(g, h) \leqslant C(n) \cdot \bigl(V^E_{p,g}\sqrt{\operatorname{Vol}(E, g)} + V^E_{p,h}\sqrt{\operatorname{Vol}(E, h)}\bigr).$$

The proof of this proposition is postponed to Appendix A.

In [10], taking $M = T^2$, the two-dimensional torus, several examples of sequences $\{h_k\} \subset \mathcal{M}$ with the following properties were constructed:

- $dV_{h_k} = dV_h$ for all $k \in \mathbb{N}$, where h denotes the standard flat metric on T^2 (with both radii equal to 1);
- for each $k \in \mathbb{N}$, there exists a set $U_k \subseteq M$ with $h_k = h$ outside of U_k; and
- $\operatorname{Vol}(U_k, h) \to 0$ as $k \to \infty$.

The above properties imply, by Proposition 6.1, that $d_p(h_k, h) \to 0$. Furthermore, various sequences with the above properties were constructed so that, depending on the sequence,

- no curvature quantity of (M, h_k) converges to the corresponding quantity for (M, h), even outside of some small-measure subset;
- the distance function induced on M by h_k does not converge to that of h, either in the Gromov–Hausdorff sense or some sense relevant to metric-measure spaces;
- $\operatorname{diam}(M, h_k)$ does not converge to $\operatorname{diam}(M, h)$; or
- the injectivity radius of (M, h_k) does not converge to that of (M, h), either as a function of M outside of some small-measure subset, or taking the infimum of this function.

Since these examples apply to d_p, it seems the advantage of d_p, when considered in the context of convergence of Riemannian manifolds, is that it eliminates collapse of the metrics over the entire manifold if $p = 1$.

To the best of our knowledge, it remains an open question to find a simple Riemannian metric on $\mathcal{M}$ with a distance function that offers some control over the geometry of elements of $\mathcal{M}$—for instance, one for which convergence with respect to the distance function of the Riemannian metric implies Gromov–Hausdorff convergence (or some other synthetic–geometric convergence). While this is certainly the case for Sobolev H^s metrics when $s > n/2$ (cf. [14, p. 20] or [2]), it might be the case that there are simpler Riemannian metrics with this desirable property. (Compare [21,20] for analogous examples of this in the setting of submanifold geometry.)

6.2. The exponential mapping of g_{N}

It is possible, though a bit tricky (see the next two subsections), to see that the exponential mapping of g_{N} is surjective onto any conformal class, but not onto all of $\mathcal{M}$. This is also true for the Ebin metric. It would be interesting to find a Diff(M)-invariant geodesically convex Riemannian metric on $\mathcal{M}$, that is, one for which geodesics exist between any two points. However, at this point the authors know of no such metric.

6.2.1. Conformal classes

Let us now show that for any $g \in \mathcal{M}$, $\exp_g$ is a diffeomorphism onto the conformal class $\mathcal{P}g$ of g, when restricted to an appropriate open neighborhood of 0 in $T_g(\mathcal{P}g)$. (The same is true for the Ebin metric, as is immediately apparent from the explicit formula for its exponential mapping [17, Theorem 2.3], [18, Theorem 3.2].) We show this in the remainder of this subsection.

Indeed, the completion of the set $\mathcal{V}_v$ of smooth volume forms with fixed total volume $v = \mathrm{Vol}(M, g)$ is isometric to a section of a sphere in a Hilbert space when endowed with the metric induced from the Ebin metric via the map i_μ (2) [12, §4.4]. In particular, one can deduce that the exponential mapping of $\mathcal{V}_v$ is a diffeomorphism from a subset of $T_\nu \mathcal{V}_v$ onto $\mathcal{V}_v$ for any $\nu \in \mathcal{V}_v$.

Consider now the set $\mathcal{P}g \cap \mathcal{M}_v = \{h \in \mathcal{P}g\colon \ \mathrm{Vol}(M, h) = v\}$. Since the metric induced by g_{N} on $\mathcal{P}g \cap \mathcal{M}_v$ is equal (up to a factor $1/v$) to the Ebin metric, and i_μ induces a diffeomorphism between $\mathcal{P}g \cap \mathcal{M}_v$ and $\mathcal{V}_v$, one also sees that the exponential mapping at g of $(\mathcal{P}g \cap \mathcal{M}_v, g_{\mathrm{N}})$ is a diffeomorphism when restricted to the appropriate domain. Furthermore, as noted above Remark 4.5, $\mathcal{P}g \cap \mathcal{M}_v \subset \mathcal{M}$ and $\mathcal{P}g \subset \mathcal{M}$ are totally geodesic. Therefore, the exponential mapping of $(\mathcal{M}, g_{\mathrm{N}})$, restricted to vectors tangent to $\mathcal{P}g \cap \mathcal{M}_v$, coincides with that of $(\mathcal{P}g \cap \mathcal{M}_v, g_{\mathrm{N}})$.

Now, let notation be as in Theorem 4.1, and let $\{g(t)\}_{t\in[0,1]}$ be any geodesic emanating from $g(0) = g$ with initial tangent vector $(\alpha, 0)$, where $a_0 = \frac{2}{v}\int_M \alpha = 0$; that is, $(\alpha, 0)$ is tangent to $\mathcal{P}g \cap \mathcal{M}_v$ and $\mathrm{Vol}(M, g(t)) = v$ for all t. Let us now consider the geodesic $\tilde{g}(t)$ emanating from $g(0)$ with initial tangent vector $(\alpha + \lambda\mu_0, 0)$, where $\lambda \in \mathbb{R}$. One then computes that under this change, a_0 becomes 2λ, but b_0, q, and r do not change. Examining (16), then, $\tilde{g}(t) = e^{\lambda t/n} g(t)$. Since $\mathcal{P}g = \mathbb{R}_{>0} \cdot (\mathcal{P}g \cap \mathcal{M}_v)$, one deduces that $\exp_g$ is a diffeomorphism from an appropriate domain in $T_g(\mathcal{P}g)$ onto $\mathcal{P}g$.

6.2.2. Nonsurjectivity on $\mathcal{M}$

To show that for no $g \in \mathcal{M}$ is $\exp_g$ surjective onto $\mathcal{M}$, we continue to use the notation of Theorem 4.1, and consider any geodesic $\{g(t)\}_{t\in[0,T)}$ with $g(0) = g$ and $g_t(0) = (\alpha, A)$. Let $\|A(x)\| := \sqrt{\mathrm{tr}((g(0,x)^{-1}A(x))^2)}$ denote the fiberwise norm of A, and $\overline{A}(x) := A(x)/\|A(x)\|$ the fiberwise normalization of A; then $\frac{2}{r} g(0)^{-1} A = \frac{4}{\sqrt{n}} g(0)^{-1}\overline{A}$.

Now, recall that the branch of arctangent in (16) "jumps upward" when $t \mapsto t + \frac{2\pi}{b_0}$. Furthermore, its argument has period $\frac{2\pi}{b_0}$; therefore the arctangent term increases by adding π when $t \mapsto t + \frac{2\pi}{b_0}$. In particular, using the considerations of the previous paragraph as well, we have $g(\frac{2\pi k}{b_0}) = g(0)\exp(\frac{4\pi k}{\sqrt{n}} g(0)^{-1}\overline{A})$ for any $k \in \mathbb{N}$.

To complete the proof of nonsurjectivity, note that at each $x \in M$,

$$g(t, x) = a(t, x) g(0, x) \exp\bigl(b(t, x) g(0, x)^{-1} \overline{A}(x)\bigr),$$

where a and b are real-valued functions. Furthermore, from (31) (and the nonnegativity of p in that equation), it follows that $b(\cdot, x)$ is monotonically nondecreasing for each $x \in M$. From the last paragraph, we also see that $b(\frac{2\pi k}{b_0}, x) = \frac{4\pi k}{\sqrt{n}}$ for any $x \in M$ and $k \in \mathbb{N}$. Since also $\|\overline{A}(x)\| = 1$ for all $x \in M$, we see that it is impossible for the image of $\exp_g$ to contain, for example, any metrics of the form $Rg(0)\exp(S)$, where $R : M \to \mathbb{R}_{>0}$ and S is any $(1,1)$-tensor with $\sqrt{\mathrm{tr}(S^2(x))} < \frac{4\pi k_0}{\sqrt{n}}$ and $\sqrt{\mathrm{tr}(S^2(y))} > \frac{4\pi k_0}{\sqrt{n}}$ for some points $x, y \in M$ and number $k_0 \in \mathbb{N}$.

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Appendix A

Here, we present the proofs of Propositions 6.1 and 5.8.

Proof of Proposition 6.1. This proposition is analogous to [9, Proposition 4.1], so we will follow that proof, with modifications to compensate for the conformal factor V^{-p} of g_p.

For each $k \in \mathbb{N}$ and $s \in (0, 1]$, we define three families of metrics as follows. The set E is open, and we may choose closed sets $F_k \subseteq E$ such that $\mathrm{Vol}(E, g) - \mathrm{Vol}(F_k, g) \leqslant 1/k$. (This is possible because the Lebesgue measure is regular.) Let $f_{k,s} \in C^\infty(M)$ be functions with the following properties:

1. $f_{k,s}(x) = s$ if $x \in F_k$,
2. $f_{k,s}(x) = 1$ if $x \notin E$, and
3. $s \leqslant f_{k,s}(x) \leqslant 1$ for all $x \in M$.

Now, for $t \in [0, 1]$, define

$$\hat{g}^{k,s}(t) := \big((1-t) + t f_{k,s}\big) g, \qquad \bar{g}^{k,s}(t) := f_{k,s}\big((1-t)g + th\big),$$
$$\tilde{g}^{k,s}(t) := \big((1-t) + t f_{k,s}\big) h.$$

We view these as paths in t depending on the family parameter s. Furthermore, we define a concatenated path

$$g^{k,s} := \hat{g}^{k,s} * \bar{g}^{k,s} * \big(\tilde{g}^{k,s}\big)^{-1},$$

where of course the inverse means we run through the path backwards. Then $g^{k,s}(0) = g$ and $g^{k,s}(1) = h$ for all s.

We now investigate the lengths of each piece of $g^{k,s}$ separately, starting with that of $\hat{g}^{k,s}$. We first compute

$$\begin{aligned} L\big(\hat{g}^{k,s}\big) &= \int_0^1 \bigg(V_{\hat{g}^{k,s}(t)}^{-p} \int_M \mathrm{tr}_{((1-t)+tf_{k,s})g}\Big(\big((f_{k,s}-1)g\big)^2\Big)\, dV_{\hat{g}^{k,s}(t)} \bigg)^{1/2} dt \\ &= \int_0^1 \bigg(V_{\hat{g}^{k,s}(t)}^{-p} \int_E \big((1-t) + t f_{k,s}\big)^{\frac{n}{2}-2} \mathrm{tr}_g\Big(\big((1-f_{k,s})g\big)^2\Big)\, dV_g \bigg)^{1/2} dt. \end{aligned} \tag{39}$$

Note that in the last line, we only integrate over E, since $1 - f_{k,s} \equiv 0$ on $M \setminus E$. Note also that since, additionally, $f_{k,s} \leqslant 1$, we have $\mathrm{Vol}(M \setminus E, g) \leqslant V_{\hat{g}^{k,s}(t)} \leqslant V_g$. Furthermore, since $s > 0$, we have $(1 - f_{k,s})^2 \leqslant (1-s)^2 < 1$, from which

$$L\big(\hat{g}^{k,s}\big) < V_{p,g}^E \int_0^1 \bigg(n \int_E \big((1-t) + t f_{k,s}\big)^{\frac{n}{2}-2} dV_g \bigg)^{1/2} dt.$$

Now, to estimate this, we note that for $n \geqslant 4$, $\frac{n}{2} - 2 \geqslant 0$ and therefore $f_{k,s} \leqslant 1$ implies that

$$L\big(\hat{g}_t^{k,s}\big) < V_{p,g}^E \sqrt{n\, \mathrm{Vol}(E, g)}. \tag{40}$$

For $n \leqslant 3$, $\frac{n}{2} - 2 < 0$ and therefore one can compute that $f_{k,s} \geqslant s > 0$ implies

$$\big((1-t) + t f_{k,s}\big)^{\frac{n}{2}-2} < (1-t)^{\frac{n}{2}-2}.$$

In this case, then,

$$L\big(\hat{g}_t^{k,s}\big) < V_{p,g}^E \sqrt{n\, \mathrm{Vol}(E, g)} \int_0^1 (1-t)^{\frac{n}{4}-1}\, dt = V_{p,g}^E \sqrt{\mathrm{Vol}(E, g)} \cdot \frac{4}{\sqrt{n}}. \tag{41}$$

Putting together (40) and (41) therefore gives

$$L\big(\hat{g}_t^{k,s}\big) \leqslant C(n) V_{p,g}^E \sqrt{\mathrm{Vol}(E, g)}, \tag{42}$$

where $C(n)$ is a constant depending only on n.

In exact analogy, we can show that the same estimate holds for $\tilde{g}^{k,s}$ with h in place of g.

Next, we look at the second piece of $g^{k,s}$. Here we have, using that $h-g=0$ on $M\setminus E$,

$$\left|\bar{g}_t^{k,s}\right|_s^2 = V_{f_{k,s}((1-t)g+th)}^{-p}\int_M \operatorname{tr}_{f_{k,s}((1-t)g+th)}\left(\left(f_{k,s}(h-g)\right)^2\right)dV_{f_{k,s}((1-t)g+th)}$$

$$= V_{f_{k,s}((1-t)g+th)}^{-p}\int_E f_{k,s}^{n/2}\operatorname{tr}_{(1-t)g+th}\left((h-g)^2\right)dV_{(1-t)g+th}.$$

Since $f_{k,s}(x)\leqslant 1$ for all $x\in M$ it follows that $V_{f_{k,s}((1-t)g+th)}\leqslant V_{(1-t)g+th}$. Additionally, since $f_{k,s}(x)=s>0$ for all $x\in M$ and $f_{k,s}\equiv 1$ on E, we have $V_{f_{k,s}((1-t)g+th)}>\operatorname{Vol}(M\setminus E,(1-t)g+th)$. Thus, defining (for $A\subseteq M$ measurable)

$$W_{p,g,h}^A := \max\left\{V_{(1-t)g+th}^{-p}, \operatorname{Vol}\left(M\setminus A,(1-t)g+th\right)^{-p}\colon t\in[0,1]\right\},$$

the above estimate becomes

$$\left|\bar{g}_t^{k,s}\right|_E^2 \leqslant s^{n/2}W_{p,g,h}^E\int_{F_k}\operatorname{tr}_{(1-t)g+th}\left((h-g)^2\right)dV_{(1-t)g+th}$$
$$+ W_{p,g,h}^E\int_{E\setminus F_k}\operatorname{tr}_{(1-t)g+th}\left((h-g)^2\right)dV_{(1-t)g+th}. \tag{43}$$

For each fixed t, one can see that the first term in the above goes to zero as $k\to\infty$ followed by $s\to 0$. Additionally, by our assumption on the sets F_k, the second term in (43) goes to zero as $k\to\infty$ for each fixed t (it does not depend on s at all). Since t only ranges over the compact interval $[0,1]$ and all terms in the integrals depend smoothly on t, both of these convergences are uniform in t. From this,

$$\lim_{s\to 0}\lim_{k\to\infty} L\left(\bar{g}^{k,s}\right)=0. \tag{44}$$

Combining (42), its analogue for $\tilde{g}^{k,s}$, and (44), together with $\lim_{k\to\infty} V_{p,g}^E = V_{p,g}^E$ (and similarly for $V_{p,h}^E$), gives the desired estimate. □

Proof of Proposition 5.8. The proof is divided into three cases: $p\leqslant 0$, $0<p<1$, and $p>1$.

First, let $p\leqslant 0$. In this case, the result follows from Proposition 6.1, since we have $\max\{V_g^{-p/2},\operatorname{Vol}(M\setminus E,g)^{-p/2}\}=V_g^{-p/2}$, and similarly for h.

Now, let $0<p<1$. We use the notation of the proof of Proposition 6.1, and continue from (39). Note that, since $p>0$ and $f_{k,s}\geqslant s$,

$$V_{\hat{g}^{k,s}(t)}^{-p} = \left(\int_M\left((1-t)+tf_{k,s}\right)^{n/2}dV_g\right)^{-p}\leqslant\left(1-(1-s)t\right)^{-pn/2}V_g^{-p}. \tag{45}$$

Assume $n\leqslant 3$. Then $\frac{n}{2}-2<0$, and therefore

$$\left((1-t)+tf_{k,s}\right)^{\frac{n}{2}-2}\leqslant\left(1-(1-s)t\right)^{\frac{n}{2}-2}. \tag{46}$$

Also, $(1-f_{k,s})\leqslant(1-s)$, so combining this with (45) and (46) allows us to transform (39) into the estimate (with $\tau:=(1-s)t$)

$$L\left(\hat{g}^{k,s}\right)\leqslant\int_0^1\left(V_g^{-p}\int_E\left(1-(1-s)t\right)^{\frac{(1-p)n}{2}-2}\operatorname{tr}_g\left(\left((1-s)g\right)^2\right)dV_g\right)^{1/2}dt$$

$$= V_g^{-p/2}\sqrt{n\operatorname{Vol}(E,g)}\int_0^1\left(1-(1-s)t\right)^{\frac{(1-p)n}{4}-1}(1-s)\,dt$$

$$
\begin{aligned}
&= V_g^{-p/2}\sqrt{n\,\mathrm{Vol}(E,g)}\int_0^{1-s}(1-\tau)^{\frac{(1-p)n}{4}-1}\,d\tau \\
&\leqslant V_g^{-p/2}\sqrt{n\,\mathrm{Vol}(E,g)}\int_0^{1}(1-\tau)^{\frac{(1-p)n}{4}-1}\,d\tau \\
&\leqslant C(p,n)V_g^{-p/2}\sqrt{\mathrm{Vol}(E,g)}, \qquad (47)
\end{aligned}
$$

where the last line follows since $p < 1$ and $n \leqslant 3$.

Now, assume $n \geqslant 4$. On F_k, we have $f_{k,s} \equiv s$, so we may carry out the same estimate as above (which, at least on F_k, does not depend on (46)) to obtain

$$
\begin{aligned}
L\big(\hat{g}^{k,s}\big) &\leqslant C(p,n)V_g^{-p/2}\sqrt{\mathrm{Vol}(F_k,g)} \\
&\quad + \int_0^1 \Bigg(V_{\hat{g}^{k,s}(t)}^{-p}\int_{E\setminus F_k}\big((1-t)+tf_{k,s}\big)^{\frac{n}{2}-2}\,\mathrm{tr}_g\big(\big((1-f_{k,s})g\big)^2\big)\,dV_g\Bigg)^{1/2}dt. \qquad (48)
\end{aligned}
$$

Since, in this case, $\frac{n}{2}-2 \geqslant 0$, the fact that $f_{k,s} \leqslant 1$ implies $((1-t)+tf_{k,s})^{\frac{n}{2}-2} \leqslant 1$. Also, since $f_{k,s} > 0$, we have that $1 - f_{k,s} < 1$. Using these facts, together with (45), the second term on the right-hand side of the above expression can be estimated from above by

$$
V_g^{-p/2}\sqrt{n\,\mathrm{Vol}(E\setminus F_k,g)}\int_0^1\big(1-(1-s)t\big)^{-pn/2}\,dt.
$$

The value of the integral in the above is finite for each fixed $s > 0$ and does not depend on k. Furthermore, by our assumptions on the sets E and F_k, the above expression goes to zero as $k \to \infty$. Combining this fact with (47) and (48) shows that for any n,

$$
\lim_{k\to\infty} L\big(\hat{g}^{k,s}\big) \leqslant C(p,n)V_g^{-p/2}\sqrt{\mathrm{Vol}(E,g)}.
$$

A similar estimate holds for $L(\tilde{g}^{k,s})$, and we can show exactly as in the proof of Proposition 6.1 that

$$
\lim_{s\to 0}\lim_{k\to\infty} L\big(\bar{g}^{k,s}\big) = 0.
$$

This completes the proof for $0 < p < 1$.

Finally, let $p > 1$. In this case, we use the isometry F from Proposition 3.8 and the result for $p < 1$ to see

$$
d_p(g,h) \leqslant C(p,n)\cdot\Big(V_{F(g)}^{\frac{p-2}{2}}\sqrt{\mathrm{Vol}\big(E,F(g)\big)} + V_{F(h)}^{\frac{p-2}{2}}\sqrt{\mathrm{Vol}\big(E,F(h)\big)}\Big).
$$

Recalling that $V_{F(g)} = V_g^{-1}$ and noting that $\mathrm{Vol}(E,F(g)) = V^{-2}\,\mathrm{Vol}(E,g)$ (and similarly for $F(h)$) then leads to the result. □

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