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Ann. I. H. Poincaré – AN 31 (2014) 103–128

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Hybrid mountain pass homoclinic solutions of a class of semilinear elliptic PDEs

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Received 8 February 2012; received in revised form 15 February 2013; accepted 15 February 2013

Available online 26 February 2013

Abstract

Variational gluing arguments are employed to construct new families of solutions for a class of semilinear elliptic PDEs. The main tools are the use of invariant regions for an associated heat flow and variational arguments. The latter provide a characterization of critical values of an associated functional. Among the novelties of the paper are the construction of "hybrid" solutions by gluing minima and mountain pass solutions and an analysis of the asymptotics of the gluing process.

1. Introduction

During the past 20 years, direct variational methods have been developed to treat functionals defined on unbounded temporal or spatial domains. These methods lead to the existence of heteroclinic and homoclinic solutions of dynamical systems, see e.g. [6,10,26,27,22,18,11] and so-called multibump and multitransition solutions of partial differential equations [12,1,23,24,4,16]. The solutions are generally obtained as minima or mountain pass critical points of corresponding functionals. Taking advantage of further properties of these problems, variational "gluing" arguments have been developed to find more complex solutions of the equations which shadow (i.e. are near) formal concatenations of the solutions mentioned above. In a sense this work goes back to the results of Poincaré and Birkhoff on homoclinic orbits of Hamiltonian systems. Indeed in his work on the 3 body problem, Poincaré showed that if a time periodic Hamiltonian system with 1 degree of freedom has an isolated homoclinic orbit to a hyperbolic periodic orbit, then there exists an infinite number of homoclinic orbits. In volume 3 of New Methods of Celestial Mechanics, Poincaré also classified homoclinic orbits with respect to their "Morse index".

Poincaré's method was geometrical. He obtained his orbits by looking at the tangle of homoclinic intersections of the stable and unstable invariant curve, and the index of the orbits was related to the intersection index. In this paper we will use gluing arguments to find homoclinic and heteroclinic solutions for a family of semilinear elliptic

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1 Supported by the Programme "Dynamical Systems and Control Theory" of Russian Academy of Science and by RFBR grant #12-01-00441.

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.02.003

PDEs. Our approach is also geometrical, but instead of working in the phase space as did Poincaré, we employ the configuration space and use variational methods.

Aside from the one dimensional case, the work we know of using variational gluing arguments only treats solutions of the same type, i.e. minima are "glued" to minima and mountain pass solutions to mountain pass solutions. One of the novelties of the problems studied in this paper is that "hybrid" solutions will be created by gluing minima and mountain pass solutions. See also [8]. Another is that we can give a simple variational characterization of the solutions we glue. The methods we use also provide geometrical information on the location of new solutions, namely we construct invariant regions for the heat flow associated with our equation. This enables us to carry out the variational arguments in these invariant regions. Moreover the shape of the region determines the form of the associated solution. Such an approach has been used before in other settings by many authors, see e.g. [2,4,16].

The equation studied here is

$$-\Delta u + F_u(x,u) = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where Δ is the Laplace operator. We assume that F is periodic, i.e. $F \in C^2(\mathbb{T}^{n+1})$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus. Eq. (1.1) is the Euler–Lagrange equation for the functional

$$\mathcal{F}(u) = \int_{\mathbb{T}^n} L(x,u,\nabla u)\,dx, \quad u \in W^{1,2}(\mathbb{T}^n),$$

where

$$L(x,u,\nabla u) = \frac{1}{2}|\nabla u|^2 + F(x,u). \tag{1.2}$$

Standard results from the calculus of variations and elliptic partial differential equations imply that $\mathcal{F}$ attains its minimum on $W^{1,2}(\mathbb{T}^n)$ and the minimizer is a classical solution of (1.1). Any weak solution of (1.1) is a classical solution, so when we refer to a solution of (1.1), we always mean a classical solution.

A standard example of (1.1) for $n = 1$ is a pendulum with an oscillating suspension point:

$$L(t,u,\dot{u}) = \frac{1}{2}\dot{u}^2 + f(t)\bigl(1 - \cos(2\pi u)\bigr), \quad f > 0. \tag{1.3}$$

Then the set of minimizers of $\mathcal{F}$ is $\mathbb{Z}$. For this example our results are strongly related to well known results in the theory of dynamical systems, in particular in dynamics of area preserving maps (see e.g. [15]).

Another well known example is the Allen–Cahn Lagrangian:

$$F(x,u) = f(x)\bigl(u^2 - 1\bigr)^2, \quad f > 0. \tag{1.4}$$

This can be modified to put it into the above framework by redefining F outside of $-1 \leqslant u \leqslant 1$ making it 2-periodic in u and solutions of the resulting equation are solutions of the original one if $-1 \leqslant u \leqslant 1$.

Returning to (1.2), note that if u is a minimizer, so is $u + \mathbb{Z}$. By a result of Moser [20], the set of minimizers of $\mathcal{F}$ on $W^{1,2}(\mathbb{T}^n)$ is ordered: if u, v are distinct minimizers, then $u > v$ or $v < u$. Suppose there are minimizers $u_- < u_+$ such that there are no other minimizers between them. Then we refer to u_- and u_+ as a gap pair of periodic solutions of (1.1).

Remark 1.5. In fact we do not need F to be periodic in u. What is needed, as for (1.4), is that $\mathcal{F}$ has a minimum and the minimum is nonunique: there are at least two minimizers $u_- < u_+$. Then F can be modified outside the strip S between u_- and u_+ to make it periodic in u. All solutions we study lie in S, so this modification does not change anything.

We will study solutions of (1.1) which are periodic in all variables except x_1 and lie in the gap between u_- and u_+. Let $\mathcal{N} = \mathbb{R} \times \mathbb{T}^{n-1}$ and

$$\mathcal{W} = \bigl\{u \in W^{1,2}_{\mathrm{loc}}(\mathcal{N}) \colon u_- \leqslant u \leqslant u_+\bigr\}.$$

Here and in the sequel, inequalities between $W^{1,2}_{\mathrm{loc}}$ functions are understood in an a.e. sense. Let $\tau : \mathcal{N} \to \mathcal{N}$ be the right translation $\tau(x) = (x_1 + 1, x_2, \dots, x_n)$. For a function u on $\mathcal{N}$, let $\tau u = u \circ \tau^{-1}$. Thus $\tau : \mathcal{W} \to \mathcal{W}$ moves the graph of u to the right. We say that a solution $u \in \mathcal{W}$ of (1.1) is *heteroclinic* from u_- to u_+ if $\tau^{\pm k}u \to u_\mp$ in the $W^{1,2}_{loc}$ topology as $k \to \infty$.

Remark 1.6. By standard elliptic regularity results, the topology we use in the definition of a heteroclinic solution is unimportant: if a solution u satisfies $\tau^{-k}u \to u_\pm$ in the L^2_{loc} topology, then $\tau^{-k}u \to u_\pm$ in the C^2_{loc} topology. Thus the definition of a heteroclinic solution is equivalent to

$$\lim_{i\to\pm\infty} \|u - u_\pm\|_{L^2(T_i)} = 0 \quad \text{or} \quad \lim_{i\to\pm\infty} \|u - u_\pm\|_{C^2(T_i)} = 0, \tag{1.7}$$

where $T_i = [i, i+1] \times \mathbb{T}^{n-1}$.

Let $\mathcal{H}(u_-, u_+)$ be the set of heteroclinic solutions from u_- to u_+ and $\mathcal{H}(u_+, u_-)$ the set of heteroclinic solutions from u_+ to u_-. Similarly, let $\mathcal{H}(u_\pm, u_\pm)$ be the sets of homoclinic solutions to u_+ and u_- respectively.

As was shown by Bangert [5], we have:

Theorem 1.8. *There exists a heteroclinic solution $u \in \mathcal{H}(u_-, u_+)$ of (1.1) which is minimal, i.e.*

$$\int_{\mathcal{N}} \big(L\big(x, u, \nabla(u+\phi)\big) - L(x, u, \nabla u)\big)\, dx \geqslant 0.$$

for all $\phi \in W^{1,2}(\mathcal{N})$ with compact support. This solution satisfies $\tau u < u$. The set $\mathcal{M}(u_-, u_+)$ of minimal heteroclinic solutions is an ordered set.

Similarly, there exist minimal heteroclinics from u_+ to u_- which form an ordered set $\mathcal{M}(u_+, u_-) \subset \mathcal{H}(u_+, u_-)$. Solutions which satisfy $\tau^{\pm 1}u \geqslant u$ are called 1-*monotone* (in x_1) and if the inequality is strict, are called strictly 1-*monotone*.

Theorem 1.8 is a PDE version of old results of Morse and Hedlund [19,14] on minimal heteroclinic geodesics. To prove Theorem 1.8, Bangert used a limit argument based on Moser's results on the existence of periodic and quasiperiodic minimal solutions [20] of (1.1).

Remark 1.9. Bangert considered the more general types of heteroclinics and more general class of Lagrangians studied by Moser [20]. In fact most of our results hold for more general Lagrangians $L(x, u, \nabla u)$ on $\mathbb{T}^n \times \mathbb{R}^{n+1}$ provided standard convexity assumptions are satisfied (see [20]). Moreover $\mathbb{T}^n$ can be replaced by any manifold with a $\mathbb{Z}$ group action satisfying certain compactness conditions. However, to avoid technicalities, we consider only the Lagrangians (1.2) on $\mathbb{T}^n$.

To state our main results for (1.1) precisely requires a lengthy set of preliminaries. Therefore for now we will just give an informal description. Suppose $u_- < u_+$ are a gap pair of periodic solutions of (1.1). By Bangert's Theorem 1.8, there is an ordered family of solutions lying between u_- and u_+ and heteroclinic from u_- to u_+. Likewise there is a family of solutions heteroclinic from u_+ to u_-. If there is a gap pair $v_+ < w_+$ in $\mathcal{M}(u_-, u_+)$ and a gap pair $v_- < w_-$ in $\mathcal{M}(u_+, u_-)$, then as shown in [25], there exist an infinite number of homoclinic and heteroclinic locally minimizing multitransition solutions between u_- and u_+. In [7], mountain pass heteroclinic solutions, U_-, between v_- and w_-, and U_+, between v_+ and w_+, were found. The question we study in the present paper is the existence of homoclinic and heteroclinic solutions of (1.1) that are obtained by gluing together τ^k-translations of all these heteroclinic solutions.

In Section 2, some results of [25] and [7] will be reformulated in a form convenient for our goals. We will also present slight improvements of these results which will be used in the sequel. In Section 3, we prove the existence of hybrid solutions obtained by gluing a mountain pass heteroclinic, U_+, and a translation, $\tau^k w_-$, of a minimal heteroclinic. In Section 4, the limit behavior of these hybrid solutions as $k \to \infty$ is studied. In Section 5, the more complex question of the existence of homoclinic solutions obtained by gluing of two mountain pass solutions U_+ and $\tau^k U_-$ will be treated. The existence of k-transition homoclinics and heteroclinics will also be discussed briefly. Lastly, some of the technical preliminaries of Section 2 will be proved in Appendix A.

2. Preliminaries

For future use, a direct variational characterization of heteroclinic solutions given by Theorem 1.8 will be needed. This characterization was obtained in [25]. Without loss of generality, assume

$$\min_{W^{1,2}(\mathbb{T}^n)} \mathcal{F} = 0. \tag{2.1}$$

For any $u \in \mathcal{W}$ and a measurable set $A \subset \mathcal{N}$, set

$$J_A(u) = \int_A L(x, u, \nabla u)\, dx. \tag{2.2}$$

Then $J_{T_i}(u_\pm) = 0$, where $T_i = [i, i+1] \times \mathbb{T}^{n-1}$. Define the functional J on $\mathcal{W}$ by

$$J(u) = \sum_{i=-\infty}^{\infty} J_{T_i}(u) = \lim_{i\to\infty, j\to-\infty} J_{N_j^i}(u), \quad N_j^i = [j, i] \times \mathbb{T}^{n-1}. \tag{2.3}$$

It was proved in [25] that for any $u \in \mathcal{W}$, the series (2.3) either converges or diverges to $+\infty$, and J is bounded from below on $\mathcal{W}$.

Let

$$\Gamma(u_+, u_+) = \Big\{ u \in \mathcal{W}\colon \lim_{i\to\pm\infty} \|u - u_+\|_{L^2(T_i)} = 0 \Big\}.$$

and define $\Gamma(u_-, u_-)$ similarly. Then

$$\inf_{\Gamma(u_\pm, u_\pm)} J = 0. \tag{2.4}$$

Moreover from [25], we have

Lemma 2.5. *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $J(u) < \delta$ for some $u \in \Gamma(u_\pm, u_\pm)$, then*

$$\|u - u_\pm\|_{W^{1,2}(T_j)} < \varepsilon \quad \textit{for all } j \in \mathbb{Z}.$$

Next define

$$\Gamma_+ = \Gamma(u_-, u_+) = \Big\{ u \in \mathcal{W}\colon \lim_{i\to\pm\infty} \|u - u_\pm\|_{L^2(T_i)} = 0 \Big\},$$
$$\Gamma_- = \Gamma(u_+, u_-) = \Big\{ u \in \mathcal{W}\colon \lim_{i\to\pm\infty} \|u - u_\mp\|_{L^2(T_i)} = 0 \Big\}.$$

Then from [25], we have:

Proposition 2.6.

1. *The functional J attains its minimum, $c_\pm$, on $\Gamma_\pm$ and*
$$c = c_+ + c_- > 0, \quad c_\pm = \inf_{u\in\Gamma_\pm} J(u). \tag{2.7}$$
2. *Let $\mathcal{M}_\pm = \{u \in \Gamma_\pm\colon J(u) = c_\pm\}$. Any minimizer $u \in \mathcal{M}_\pm$ is a solution of (1.1) heteroclinic from $u_\mp$ to $u_\pm$ and $\tau^{\pm 1} u_\pm < u_\pm$.*
3. *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $J(v) < c_\pm + \delta$ for some $v \in \Gamma_\pm$, then there is a $u \in \mathcal{M}_\pm$ such that*
$$\|u - v\|_{W^{1,2}(T_j)} < \varepsilon \quad \textit{for all } j \in \mathbb{Z}.$$

It was further proved in [25] that the sets $\mathcal{M}_\pm = \mathcal{M}(u_\mp, u_\pm)$ of minimizing heteroclinics are the same as given by Bangert in Theorem 1.8. These sets are ordered and invariant under the translation group $\{\tau^k\}_{k\in\mathbb{Z}}$, and compact modulo translations. The graphs of minimizers $u \in \mathcal{M}_\pm$ form laminations of the strip $S = \{(x, u) \in \mathbb{T}^n \times \mathbb{R}\colon u_-(x) \leqslant u \leqslant u_+(x)\}$. We impose the following condition:

$(*)$ **No foliation assumption.** The lamination of S by minimal heteroclinics in $\mathcal{M}_\pm$ is not a foliation.

As was shown in [25], assumption $(*)$ is generic. If it holds, there are gaps in the sets of minimal heteroclinics. In particular, there is a point in S through which no graphs of minimal heteroclinics pass. Every gap in $\mathcal{M}_\pm$ is bounded by a pair of minimal heteroclinics $v_\pm < w_\pm$ which again we call a gap pair.

Remark 2.8. Condition $(*)$ never holds if L is independent of x. Indeed, then for any minimizer $u \in \mathcal{M}_\pm$ and any $k \in \mathbb{R}$, the translation $\tau^k u$ is a minimizer, and the graphs of $\{\tau^k u\}_{k\in\mathbb{R}}$ form a foliation of S.

Assuming condition $(*)$, in Section 3, it will be proved that there exists an infinite number of homoclinic and heteroclinic solutions of mountain pass and other types in the strip S.

For the rest of this paper, it will be assumed that $(*)$ holds. Then there exists a gap pair, $v_\pm < w_\pm$, in $\mathcal{M}_\pm$. In [25], it was proved that $w_\pm - v_\pm \in W^{1,2}(\mathcal{N})$. Set $E = W^{1,2}(\mathcal{N})$ equipped with the norm

$$\|\psi\| = \|\psi\|_{W^{1,2}(\mathcal{N})}.$$

Let $E_\pm = v_\pm + E$ be the affine space through $v_\pm$ and let

$$\Lambda_\pm = \{u \in E_\pm\colon v_\pm \leqslant u \leqslant w_\pm\}.$$

The sets $\Lambda_\pm$ can be identified with subsets of the Banach space E via the map $u \to u - v_\pm$. The $W^{1,2}$ topology in $\Lambda_\pm$ inherited from $E_\pm$ will be used.

The following result was proved in [7].

Proposition 2.9. *The functional J is C^1 on $E_\pm$ and it satisfies the Palais–Smale condition (PS) in $\Lambda_\pm$: if $(u_k) \subset \Lambda_\pm$ is a sequence such that $J(u_k)$ is bounded and $\|J'(u_k)\| \to 0$ as $k \to \infty$, then (u_k) has a subsequence which is convergent in the $W^{1,2}$ norm to some $u \in \Lambda_\pm$.*

Let $I = [0, 1]$ and let

$$b_\pm = \inf_h \max_{h(I)} J, \tag{2.10}$$

where the infimum is taken over all continuous paths $h : I \to \Lambda_\pm$ connecting $v_\pm$ with $w_\pm$. By item 3 of Proposition 2.6, $b_\pm > c_\pm$. Hence $b_\pm$ is a so-called mountain pass critical level. Equivalently, $b_\pm$ is the supremum of all a such that $v_\pm$ and $w_\pm$ are in different path connected components of $\Lambda_\pm^a = \{u \in \Lambda_\pm\colon J(u) \leqslant a\}$.

It is convenient to introduce the following notation. For points v, w in a topological space Λ, we write $v \sim w$ if v and w lie in the same path connected component of Λ and $v \nsim w$ if they lie in different components. Then (2.10) yields:

Lemma 2.11. *For any $\delta \in (0, b_\pm - c_\pm)$, $v_\pm \nsim w_\pm$ in $\Lambda_\pm^{b_\pm - \delta}$, but $v_\pm \sim w_\pm$ in $\Lambda_\pm^{b_\pm + \delta}$.*

Furthermore it was shown in [7] that:

Proposition 2.12. *There exists a critical point $u \in \Lambda_\pm$ of J with $J(u) = b_\pm$.*

Any u given by Proposition 2.12 will be called a mountain pass critical point since it lies in the mountain pass critical level $J^{-1}(b_\pm)$.

Proposition 2.12 was proved in [7] by a variant of the usual so-called Deformation Theorem [21]. However we will give a proof here based on a heat flow argument since the same method will be used repeatedly throughout this paper. First some preliminaries are required. Let Φ^t, $t \geqslant 0$, be the semiflow defined by the parabolic PDE

$$u_t = \Delta u - F_u(x, u). \tag{2.13}$$

Thus $u(t) = \Phi^t(u_0)$ is the solution of the initial value problem with $u(0) = u_0$ for (2.13). Several facts about Φ^t will be stated next. The details can be found in [7]. In particular:

Proposition 2.14. *For each $u_0 \in \mathcal{W}$, there is a unique solution*

$$u(t) = u(t, \cdot) = \Phi^t(u_0) \in \mathcal{W}, \quad t \geqslant 0,$$

of (2.13). *For $t > 0$, $u(t) \in C^2(\mathcal{N})$ and the map $t \to u(t)$ is in $C^1((0,\infty), C^2(\mathcal{N}))$. If $u_0 \in E_\pm$, then $u(t) \in E_\pm$ for $t \geqslant 0$ and the map $(t, u_0) \to u(t)$ is in $C^0((0,\infty) \times E_\pm, E_\pm)$.*

The parabolic flow is a standard tool for finding solutions of nonlinear elliptic PDEs (see e.g. [9]), but usually the domain of definition is compact.

By a comparison principle for (2.13) (see e.g. [7]), if $v_\pm \leqslant u_0 \leqslant w_\pm$, then $v_\pm \leqslant u(t) \leqslant w_\pm$ for all $t \geqslant 0$. Thus $\Phi^t(\Lambda_\pm) \subset \Lambda_\pm$, $t \geqslant 0$. The semiflow $\Phi^t : \Lambda_\pm \to \Lambda_\pm$ is continuous in the topology of $\Lambda_\pm$.

An important property of Φ^t is that J is a Lyapunov function, i.e. if $u(t) = \Phi^t(u_0)$,

$$\frac{d}{dt} J\big(u(t)\big) = -\int_{\mathcal{N}} \big|\Delta u(t) - F_u\big(x, u(t)\big)\big|^2 \, dx \leqslant 0. \tag{2.15}$$

There is equality in (2.15) iff u_0 is an equilibrium point of the flow, i.e. a solution of (1.1). Since $J(u(t)) \geqslant 0$, (2.15) implies there is a sequence $t_k \to \infty$ such that

$$\int_{\mathcal{N}} \big|\Delta u(t_k) - F_u\big(x, u(t_k)\big)\big|^2 \, dx \to 0.$$

Since for $u \in \Lambda_\pm$ (see e.g. [7]),

$$\big\|J'(u)\big\| \leqslant \int_{\mathcal{N}} \big|\Delta u - F_u(x,u)\big|^2 \, dx,$$

it follows that $\|J'(u(t_k))\| \to 0$ as $k \to \infty$. Then by the (PS) condition (Proposition 2.9), we obtain:

Lemma 2.16. *For any $u_0 \in \Lambda_\pm$ and any sequence $t_k \to \infty$, there exists a subsequence such that $\Phi^{t_k}(u_0)$ converges in $W^{1,2}$ to a critical point $u \in \Lambda_\pm$ of J with $J(u) = \lim_{t\to\infty} J(\Phi^t(u_0))$.*

Remark 2.17. A similar statement holds for any Φ^t-invariant set $\Lambda \subset \mathcal{W}$ such that J is finite and differentiable at every point in Λ and satisfies the (PS) condition in Λ. Such generalizations will be used several times in what follows.

Now the proof of Proposition 2.12 can be given.

Proof of Proposition 2.12. For any $\varepsilon > 0$, let $h : I \to \Lambda_\pm^{b_\pm+\varepsilon}$ be a continuous path joining $v_\pm$ and $w_\pm$ in $\Lambda_\pm^{b_\pm+\varepsilon}$. Let $h_t = \Phi^t \circ h : I \to \Lambda_\pm^{b_\pm+\varepsilon}$. Then there exists $\theta_\infty \in I$ such that $J(h_t(\theta_\infty)) \geqslant b_\pm$ for all $t \geqslant 0$. Indeed, for any $t \geqslant 0$ there is $\theta_t \in I$ such that $J(h_t(\theta_t)) \geqslant b_\pm$. Take a sequence $t_k \to \infty$ such that $\theta_{t_k} \to \theta_\infty \in I$. Suppose that $J(h_\tau(\theta_\infty)) < b_\pm$ for some $\tau > 0$. By the continuity of h_τ, $J(h_\tau(\theta_{t_k})) < b_\pm$ and $t_k > \tau$ for large k. Then $J(h_{t_k}(\theta_{t_k})) < b_\pm$, a contradiction.

By Lemma 2.16 with $u_0 = h(\theta_\infty)$, there is a sequence $s_k \to \infty$ such that $u_k = h_{s_k}(\theta_\infty)$ converges in $W^{1,2}$ to a critical point $v_\varepsilon \in \Lambda_\pm$ such that

$$b_\pm + \varepsilon \geqslant J(v_\varepsilon) = \lim_{t\to\infty} J\big(h_t(\theta_\infty)\big) \geqslant b_\pm.$$

Since $\varepsilon > 0$ is arbitrary, (PS) implies there is a sequence $\varepsilon_j \to 0$ such that v_{ε_j} converges to a critical point $u \in J^{-1}(b_\pm)$. $\square$

Next we give two preliminaries that concern the existence of locally minimal 2-transition homoclinic solutions of (1.1). These solutions are close to concatenations of two minimizing heteroclinics.

Let $c = c_- + c_+$ and $w_k = \min(w_+, \tau^k w_-)$. Then $J(w_k) < c$ and $J(w_k) \to c$ as $k \to \infty$. Indeed, let $w_k^* = \max(w_+, \tau^k w_-)$. Then $J(w_k) = J(w_+) + J(w_-) - J(w_k^*)$, where $J(w_k^*) \geqslant 0$ by Lemma 2.5. In addition $J(w_k^*) \to 0$ as $k \to \infty$.

Set

$$N^a \equiv (-\infty, a] \times \mathbb{T}^{n-1}, \qquad N_a \equiv [a, \infty) \times \mathbb{T}^{n-1}, \qquad N_a^b \equiv [a, b] \times \mathbb{T}^{n-1}.$$

Proposition 2.18. *There is a $K > 0$ such that for any $k \in \mathbb{N}$ with $k \geqslant K$, there exists a homoclinic $u_k \in \mathcal{H}(u_-, u_-)$ such that*

1. $u_k < w_k$ *and for large* $a > 0$, $u_k > \tau w_+$ *in* N^{-a} *and* $u_k > \tau^{-1} w_-$ *in* N_a.
2. $J(u_k) < J(w_k) < c$ *and* $J(v) \geqslant J(u_k)$ *for any* $v \in \mathcal{W}$ *such that* $u_k \leqslant v \leqslant w_k$.
3. $J(u_k) \to c$ *as* $k \to \infty$.
4. $\|u_k - w_k\|_{L^\infty(\mathcal{N})} \to 0$ *as* $k \to \infty$.
5. *For any* $a \in \mathbb{R}$, *as* $k \to \infty$,
$$\|u_k - w_+\|_{W^{1,2}(N^a)} \to 0, \qquad \left\|\tau^{-k} u_k - w_-\right\|_{W^{1,2}(N_a)} \to 0.$$

Proposition 2.18 is a variant of a result of [25]. It follows from a more precise Theorem A.12 which will be proved in Appendix A.

A slight modification of Proposition 2.18 is also required to define a region invariant under the heat flow of (2.13). Let $\underline{v}_\pm \leqslant v_\pm$ be the smallest minimizer in $\mathcal{M}_\pm$ such that there are no gaps between $\underline{v}_\pm$ and $v_\pm$. Then the region between $\underline{v}_\pm$ and $v_\pm$ is foliated into minimal heteroclinics from $\mathcal{M}_\pm$, and $\underline{v}_\pm$ is the upper boundary of a gap or the limit of gaps below $\underline{v}_\pm$. Generically $\underline{v}_\pm = v_\pm = \tau^{\pm 1} w_\pm$, but the case $\underline{v}_\pm < v_\pm$ cannot be ruled out. Set $\underline{w}_k = \min(\underline{v}_+, \tau^k w_-) \leqslant w_k$. Then we have a version of Proposition 2.18, with w_+ replaced by $\underline{v}_+$.

Proposition 2.19. *There is a $K > 0$ such that for all $k \in \mathbb{N}$ with $k > K$, there exists a homoclinic $v_k \in \mathcal{H}(u_-, u_-)$ such that*

1. $v_k < \underline{w}_k$.
2. $J(v_k) < J(\underline{w}_k) < c$ *and* $J(v) \geqslant J(v_k)$ *for any* $v \in \mathcal{W}$ *such that* $v_k \leqslant v \leqslant \underline{w}_k$.
3. $J(v_k) \to c$ *as* $k \to \infty$.
4. $\|v_k - \underline{w}_k\|_{L^\infty(\mathcal{N})} \to 0$ *as* $k \to \infty$.
5. *For any* $a \in \mathbb{R}$, $\|v_k - \underline{v}_+\|_{W^{1,2}(N^a)} \to 0$ *and* $\|\tau^{-k} v_k - w_-\|_{W^{1,2}(N_a)} \to 0$ *as* $k \to \infty$.

Proof. Suppose first that $\underline{v}_+$ is an upper boundary of a gap. Then Proposition 2.18 applies with w_+ replaced by $\underline{v}_+$. If $\underline{v}_+$ is not the upper boundary of a gap, then there is a sequence of gap pairs

$$\tau w_+ < V_j < W_j < \underline{v}_+$$

in $\mathcal{M}_+$ such that $V_j \to \underline{v}_+$ pointwise. Lemma A.6 shows $\|V_j - \underline{v}_+\|_{W^{1,2}(\mathcal{N})} \to 0$ and $\|W_j - \underline{v}_+\|_{W^{1,2}(\mathcal{N})} \to 0$ as $j \to \infty$. Proposition 2.18 can be applied with the gap pair $v_+ < w_+$ replaced by the gap pair $V_j < W_j$. For each j, there exists a K_j such that for $k > K_j$, there is a homoclinic $u_{jk} \leqslant U_{jk} \equiv \min(W_j, \tau^k w_-)$ satisfying the assertion of Proposition 2.18 with $\|u_{jk} - U_{jk}\|_{L^\infty(\mathcal{N})} \to 0$ as $k \to \infty$. Thus for large j and $k > K_j$, u_{jk} satisfies all assertions of Proposition 2.19 except possibly item 2. Consider the functional J, on the set $X = \{u \in \mathcal{W} \mid u_{jk} \leqslant u \leqslant \underline{w}_k\}$. It has a minimizer, $v_k \in X$. Since X is invariant under the flow of (2.13), v_k is either u_{jk} or v_k does not touch the boundary of X. In particular $J(v_k) < J(\underline{w}_k)$. Thus v_k is a homoclinic solution of (1.1) satisfying all of the assertions of Proposition 2.19. □

3. Hybrid 2-transition homoclinics

In this and the following two sections, a heat flow method will be used to prove the existence of many minimax homoclinic solutions of (1.1) in the strip S. In particular, we will prove that one can glue together minimal and mountain pass heteroclinic solutions of (1.1) to form a multitransition homoclinic solution. In a future paper we will show that whenever it makes sense geometrically, one can glue together an arbitrary number of minimal and mountain pass heteroclinic solutions.

Gluing a minimal heteroclinic in $\mathcal{H}(u_-, u_+)$ as given by Theorem 1.8 and Proposition 2.6 corresponding to c_+ to one in $\mathcal{H}(u_+, u_-)$ corresponding to c_- was already carried out in [25]. The hybrid 2-transition cases of gluing a mountain pass heteroclinic corresponding to b_+ as given by Proposition 2.12 to a minimizer in $\mathcal{H}(u_+, u_-)$ from Proposition 2.6 corresponding to c_- (or gluing a minimizer to a mountain pass heteroclinic) will be treated in this section. Gluing a pair of mountain pass heteroclinics corresponding to b_+ and b_- will be treated in Section 4.

Let v_k, w_k be as in Section 2 and let

$$\Sigma_k = \{u \in \mathcal{W}: v_k \leqslant u \leqslant w_k,\ u - w_k \in E\},$$

be the set of functions between v_k and w_k. Set

$$\Sigma_k^a = \{u \in \Sigma_k: J(u) \leqslant a\}, \qquad \Sigma_k^{a-} = \{u \in \Sigma_k: J(u) < a\}.$$

Let $q_k = \min(v_+, \tau^k w_-) \in \Sigma_k$. Define

$$d_k = \inf\{a \in \mathbb{R}: q_k \sim w_k \text{ in } \Sigma_k^a\}.$$

Our main goal in this section is to prove that d_k is a mountain pass critical value:

Theorem 3.1. *There exists a $K > 0$ such that for any $k > K$, the functional J has a critical point $U_k \in \Sigma_k$ with $J(U_k) = d_k$.*

To prove Theorem 3.1, a technical result is required. Set $d = b_+ + c_-$.

Proposition 3.2. *For any $\delta \in (0, b_+ - c_-)$, there exists a $K > 0$ such that for $k > K$, q_k and w_k are in different path connected components of $\Sigma_k^{d-\delta}$ but in the same path connected component of $\Sigma_k^{d+\delta}$. Thus $q_k \sim w_k$ in $\Sigma_k^{d+\delta}$ and $q_k \nsim w_k$ in $\Sigma_k^{d-\delta}$.*

Proposition 3.2 will be assumed for the moment. It immediately implies:

Corollary 3.3. *There exists a $K > 0$ such that for any $\varepsilon > 0$ and any $k > K$, there is a path $h : I \to \Sigma_k^{d_k+\varepsilon}$ with $h(\partial I) \subset \Sigma_k^{d_k-}$ and h is not homotopic to a path g satisfying*

$$g(I) \subset \Sigma_k^{d_k-} = \{u \in \Sigma_k: J(u) < d_k\}$$

in the class of paths $g : I \to \Sigma_k$ with $g(\partial I) \subset \Sigma_k^{d_k-}$.

Proof of Theorem 3.1. Fix K as given by Corollary 3.3 and let $k > K$. Take a sequence $\varepsilon_j \to 0$. Let $h_j : I \to \Sigma_k^{d_k+\varepsilon_j}$ be the path given by Corollary 3.3. As was the case for $\Lambda_\pm$, the functional J satisfies the (PS) condition in Σ_k and $\Phi^t : \Sigma_k \to \Sigma_k$ for $t \geqslant 0$. Hence an analogue of Lemma 2.16 holds in Σ_k. Let $h_j^t = \Phi^t \circ h_j$. Then $h_j^t(\partial I) \subset \Sigma_k^{d_k-}$ for $t \geqslant 0$. By Corollary 3.3, for any $t \geqslant 0$ there exists $\theta_j^t \in [0, 1]$ such that $d_k + \varepsilon_j \geqslant J(h_j^t(\theta_j^t)) \geqslant d_k$. Hence, arguing as in the proof of Proposition 2.12, (a) there is $\theta_j \in I$ such that $J(h^t(\theta_j)) \geqslant d_k$ for all $t \geqslant 0$, (b) there exists $t_p \to \infty$ as $p \to \infty$ such that $h^{t_p}(\theta_j)$ converges in the $W^{1,2}$ norm as $p \to \infty$ to a critical point $U_{\varepsilon_j} \in \Sigma_k$ with $J(U_{\varepsilon_j}) \in [d_k, d_k + \varepsilon_j]$, and finally (c) as $j \to \infty$, $U_{\varepsilon_j} \to U_k \in \Sigma_k$ with $J(U_k) = d_k$. □

It remains to prove Proposition 3.2. This will be done with the aid of some auxiliary maps which will be introduced and studied next. Define the maps $\phi_k : \Lambda_+ \to \Sigma_k$ and $\psi : \Sigma_k \to \Lambda_+$ by

$$\phi_k(u) = \min(u, \tau^k w_-), \qquad \psi(v) = \max(v, v_+).$$

Then $\phi_k(v_+) = q_k$ and $\psi(q_k) = v_+$. It is known (see e.g. [12]) that the maps ϕ_k, ψ are continuous with respect to $W^{1,2}$ norm.

Lemma 3.4. *For $u \in \Lambda_+$,*

$$J(\phi_k(u)) \leqslant J(u) + c_-. \tag{3.5}$$

For any $\varepsilon > 0$, there exists a $K > 0$ such that for $k > K$ and any $u \in \Sigma_k$,

$$J(\psi(u)) \leqslant J(u) - c_- + \varepsilon. \tag{3.6}$$

Proof. To prove (3.5), set

$$A=\{x\in\mathcal{N}:\ u(x)\leqslant\tau^k w_-(x)\},\qquad B=\{x\in\mathcal{N}:\ u(x)>\tau^k w_-(x)\}$$

and let $v=\max(u,\tau^k w_-)$. Then

$$\begin{aligned}J(\phi_k(u))-J(u)&=\sum_{i\in\mathbb{Z}}\left[J_{T_i\cap A}(u)+J_{T_i\cap B}(\tau^k w_-)-J_{T_i}(u)\right]\\&=\sum_{i\in\mathbb{Z}}\left[J_{T_i\cap B}(\tau^k w_-)-J_{T_i\cap B}(u)\right]\\&=\sum_{i\in\mathbb{Z}}\left[J_{T_i}(\tau^k w_-)-J_{T_i\cap A}(\tau^k w_-)-J_{T_i\cap B}(u)\right]\\&=c_--J(v).\end{aligned}$$

By (2.4), $J(v)\geqslant 0$ and (3.5) is proved.

To prove (3.6), now set

$$A=\{x\in\mathcal{N}:\ u(x)<v_+(x)\},\qquad B=\{x\in\mathcal{N}:\ u(x)\geqslant v_+(x)\},$$

and let $v=\min(u,v_+)$. We claim that $v\in\Sigma_k$. Then arguing as above yields

$$J(\psi(u))-J(u)=J(v_+)-J(v)=c_+-J(v)\leqslant c_+-\inf_{\Sigma_k}J.$$

and the result follows by item 3 of Proposition 2.19. To see that $v\in\Sigma_k$, it suffices to show that $v_k\leqslant v\leqslant w_k$. By its definition, this is certainly true if $v(x)=u(x)$ since $u\in\Sigma_k$. If $v(x)=v_+(x)$, then by item 1 of Proposition 2.19,

$$v_k(x)\leqslant \underline{w}_k(x)\leqslant \underline{v}_+(x)\leqslant v_+(x)=v(x)\leqslant u(x)\leqslant w_k(x).\qquad\square$$

Lemma 3.7. *Let $\chi_k=\psi\circ\phi_k:\Lambda_+\to\Lambda_+$. Then $\|w_+-\chi_k(w_+)\|\to 0$ as $k\to\infty$.*

Proof. Set

$$\begin{aligned}&A=\{x\in\mathcal{N}:\ w_+(x)\leqslant\tau^k w_-(x)\},\\&B=\{x\in\mathcal{N}:\ v_+(x)<\tau^k w_-(x)<w_+(x)\},\\&C=\{x\in\mathcal{N}:\ \tau^k w_-(x)\leqslant v_+(x)\}.\end{aligned}$$

Then $\mathcal{N}=A\cup B\cup C$ and $\chi_k(w_+)=w_+$ on A. Hence

$$\|\chi_k(w_+)-w_+\|\leqslant\|\tau^k w_--w_+\|_{W^{1,2}(B)}+\|v_+-w_+\|_{W^{1,2}(C)}.\tag{3.8}$$

We have $v_+-w_+\in E$. Since $B\cup C\subset N_{\gamma_k}$ with $\gamma_k\to\infty$ as $k\to\infty$, $\|v_+-w_+\|_{W^{1,2}(B\cup C)}$ represents the tail of a convergent integral and therefore $\|v_+-w_+\|_{W^{1,2}(C)}\to 0$ as $k\to\infty$.

To estimate the B term, first we show that the measure of B, $|B|\leqslant 1$. It suffices to prove that $\tau B\cap B=\emptyset$, where τB denotes the translation of B by τ. For $x\in\tau B$, we have

$$v_+(\tau^{-1}x)<\tau^k w_-(\tau^{-1}x)<w_+(\tau^{-1}x).$$

Since $w_+(x)<v_+(\tau^{-1}x)$ and $w_-(\tau^{-1}x)<w_-(x)$, we obtain $w_+(x)<\tau^k w_-(x)$. Thus $x\notin B$ and $\tau B\cap B=\emptyset$.

Now estimating the B term crudely,

$$\|\tau^k w_--w_+\|_{W^{1,2}(B)}\leqslant(|B|+1)\|\tau^k w_--w_+\|_{C^1(B)}.\tag{3.9}$$

Let $\delta>0$. Since $w_\pm$ are heteroclinic solutions, by (1.7) there is an $a=a(\delta)$ such that $u_+(x)-\delta<w_+(x)<u_+(x)$ for $x\in N_a$ and $u_+(x)-\delta<w_-(x)<u_+(x)$ for $x\in N^{-a}$. Thus $u_+(x)-\delta<\tau^k w_+(x)<u_+(x)$ for $x\in N^{k-a}$. It can be further assumed that $B\subset N_{a+1}^{k-a-1}$. Therefore

$$\|\tau^k w_--w_+\|_{L^\infty(B)}\leqslant\|u_+-w_+\|_{L^\infty(N_a^{k-a})}<\delta.\tag{3.10}$$

Since u_+, $\tau^k w_-$, and w_+ are solutions of (1.1), by Lemma A.1 in Appendix A,

$$\|u_+ - w_+\|_{C^2(N_{a+1}^{k-a-1})} \leqslant M\delta, \qquad \|u_+ - \tau^k w_-\|_{C^2(N_{a+1}^{k-a-1})} \leqslant M\delta. \tag{3.11}$$

Combining (3.9)–(3.11) yields

$$\limsup_{k\to\infty} \|\chi_k(w_+) - w_+\|_{W^{1,2}(B)} \leqslant (|B|+1)M\delta$$

from which Lemma 3.7 follows. □

Corollary 3.12. *For any $\varepsilon > 0$ and large enough k, $\chi_k(w_+) \sim w_+$ in $\Lambda_+^{c_++\varepsilon}$.*

Proof. This follows from Lemma 3.7 and the continuity of J: for large k and all $t \in [0, 1]$,

$$J\big((1-t)w_+ + t\chi_k(w_+)\big) = J\big(w_+ - t\big(w_+ - \chi_k(w_+)\big)\big) \leqslant c_+ + \varepsilon. \qquad \square$$

Finally we are ready for:

Proof of Proposition 3.2. By Lemma 3.4, for any $\delta > 0$ and large k,

$$\phi_k\big(\Lambda_+^{b_++\delta/2}\big) \subset \Sigma_k^{d+\delta/2}, \qquad \psi\big(\Sigma_k^{d-\delta}\big) \subset \Lambda_+^{b_+-\delta/2}. \tag{3.13}$$

Since $v_+ \sim w_+$ in $\Lambda_+^{b_++\delta/2}$ and ϕ_k is continuous, $q_k = \phi_k(v_+) \sim \phi_k(w_+) = w_k$ in $\Sigma_k^{d+\delta/2}$. To show that $\phi_k(v_+) \nsim \phi_k(w_+)$ in $\Sigma_k^{d-\delta}$, suppose to the contrary that $\phi_k(v_+) \sim \phi_k(w_+)$ in $\Sigma_k^{d-\delta}$. By Lemma 3.4 with $\varepsilon = \delta/2$, $\chi_k = \psi \circ \phi_k : \Lambda_+^{c_++\delta/2} \to \Lambda_+^{c_++\delta}$ for large k. Since it can be assumed that $c_+ + \delta < b_+ - \delta/2$, we have $\chi_k(v_+) \sim \chi_k(w_+)$ in $\Lambda_+^{b_+-\delta/2}$. By Lemma 3.7, $\chi_k(w_+) \sim w_+$ in $\Lambda_+^{c_++\delta} \subset \Lambda_+^{b_+-\delta/2}$ if k is large enough. Note that $\chi_k(v_+) = v_+$. Thus $v_+ \sim w_+$ in $\Lambda_+^{b_+-\delta/2}$, contrary to Lemma 2.11. □

4. Limit behavior

Next the behavior of the critical points, U_k, and critical values, d_k, in Theorem 3.1 as $k \to \infty$ will be studied. Let

$$\Omega_+ = \{u \in E_+ : \underline{v}_+ \leqslant u \leqslant w_+\}. \tag{4.1}$$

Theorem 4.2. *For the critical point U_k of Theorem 3.1, as $k \to \infty$, we have*:

- $\lim_{k\to\infty} d_k = d$.
- *$\tau^{-k}U_k \to w_-$ in C^2_{loc}.*
- *There exists a heteroclinic $V \in \Omega_+$ and a subsequence $k \to \infty$ such that $U_k \to V$ in C^2_{loc}.*

Proof. The first item follows from Proposition 3.2. For the second, by Corollary A.5, the set of solutions of (1.1) in $\mathcal{W}$ is compact in the C^2_{loc} topology. Hence, along a subsequence, the functions $\tau^{-k}U_k$ and U_k converge in C^2_{loc} to solutions, W and V, of (1.1) as $k \to \infty$. Since for any a and large k, $\tau^{-k}w_k = w_-$ in N_a, it follows that $\tau^{-k}U_k \to w_-$ in $L^\infty(N_a)$ as $k \to \infty$ and $W = w_-$. Both $\tau^{-k}U_k$ and w_- are solutions of (1.1). By Lemma A.1, $\tau^{-k}U_k \to w_-$ in $C^2(N_a)$ as $k \to \infty$. To get the last item, since $v_k \leqslant U_k \leqslant w_k$, by item 5 of Proposition 2.19, $\underline{v}_+ \leqslant V \leqslant w_+$. Thus $V \in \Omega_+$ and is a heteroclinic solution of (1.1). □

It seems probable that V is a mountain pass heteroclinic, with $J(V) = b_+$, but we are unable to prove this without a further nondegeneracy assumption which will be stated next.

($ND^\pm$) The minimizer, $u_\pm$, of the functional, $\mathcal{F}$, on $W^{1,2}(\mathbb{T}^n)$ is nondegenerate, i.e. the second variation quadratic form

$$Q(\phi) = \mathcal{F}''(u_\pm)(\phi, \phi), \quad \phi \in W^{1,2}(\mathbb{T}^n),$$

is positive for $\phi \neq 0$.

The nondegeneracy assumption will be used to improve Theorem 4.3 as follows. Again, U_k denotes the solution of (1.1) given by Theorem 3.1.

Theorem 4.3. *Suppose* (ND^+) *holds. Then the solution* $V \in \Omega_+$ *in Theorem* 4.3 *is of mountain pass type with* $J(V) = b_+$ *and, along a subsequence* $k \to \infty$,

$$\|U_k - V_k\| \to 0, \quad V_k = \min\bigl(V, \tau^k w_-\bigr). \tag{4.4}$$

A similar but simpler result holds for the solutions u_k given by Proposition 2.18.

Theorem 4.5. *Suppose condition* (ND^+) *holds. Then* $\|u_k - w_k\|_{W^{1,2}(\mathcal{N})} \to 0$ *as* $k \to \infty$.

To prove Theorem 4.3, the following consequence of (ND^+) is required.

Proposition 4.6. *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $a < b$ *are integers and* u *is a solution of* (1.1) *on* N_a^b *satisfying* $u_+ - \delta \leqslant u \leqslant u_+$, *then*:

- u *minimizes* $J_{N_a^b}$ *in the class of functions which equal* u *on* ∂N_a^b,
- *if* $a < b - 2$, *then*

$$\|u - u_+\|_{W^{1,2}(N_{a+1}^{b-1})} \leqslant \varepsilon.$$

The proof of Proposition 4.6 is given at the end of this section.

Proof of Theorem 4.3. Once we obtain (4.4), it follows that

$$d = b_+ + c_- = \lim_{k\to\infty} J(U_k) = \lim_{k\to\infty} J(V_k) = J(V) + c_-$$

so $J(V) = b_+$ and Theorem 4.3 follows from Theorem 4.3. Hence to prove Theorem 4.3, it suffices to show that for any $\nu > 0$,

$$\limsup_{k\to\infty} \|U_k - V_k\|_{W^{1,2}(\mathcal{N})} \leqslant \nu. \tag{4.7}$$

By Lemma A.6 in Appendix A, for any $a > 0$,

$$\lim_{k\to\infty} \|U_k - V_k\|_{W^{1,2}(N^a \cup N_{k-a})} = 0.$$

Thus to prove (4.7), it suffices to prove that for any $\varepsilon > 0$, there exist an $a \in \mathbb{Z}$ such that for all large k,

$$\|U_k - V_k\|_{W^{1,2}(N_a^{k-a})} \leqslant 3\varepsilon.$$

Observe that

$$\begin{aligned}\|U_k - V_k\|_{W^{1,2}(N_a^{k-a})} \leqslant {} & \|U_k - u_+\|_{W^{1,2}(N_a^{k-a})} \\ & + \|V - u_+\|_{W^{1,2}(N_a^{k-a})} \\ & + \bigl\|\tau^k w_- - u_+\bigr\|_{W^{1,2}(N_a^{k-a})}.\end{aligned}$$

Each of the terms on the right can be estimated in a similar way so only the first one will be treated. To begin, note that

$$\|U_k - u_+\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \|v_k - u_+\|_{L^\infty(N_{a-1}^{k-a+1})}. \tag{4.8}$$

Let $\delta > 0$. By item 4 of Proposition 2.19, for k large,

$$\|v_k - \underline{w}_k\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta/2 \tag{4.9}$$

and if a is large enough, then increasing k further if needed,

$$\|\underline{w}_k - u_+\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta/2. \tag{4.10}$$

Then

$$\|U_k - u_+\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta.$$

If $\delta > 0$ is sufficiently small, applying Proposition 4.6 gives

$$\|U_k - u_+\|_{W^{1,2}(N_a^{k-a})} \leqslant \varepsilon.$$

and the proof is complete. □

Proof of Theorem 4.5. Let $\varepsilon > 0$ and $a > 0$. By item 4 of Proposition 2.18, for k sufficiently large, $\|u_k - w_k\|_{W^{1,2}(N^a \cup N_{k-a})} \leqslant \varepsilon$. Thus, it suffices to find a such that

$$\|u_k - w_k\|_{W^{1,2}(N_a^{k-a})} \leqslant 2\varepsilon \tag{4.11}$$

for large k.

Let $\delta > 0$ be as in Proposition 4.6. We can choose $a \in \mathbb{N}$ so that $\|u_+ - w_+\|_{L^\infty(N_{a-1}^a)} \leqslant \delta/2$ and similarly $\|u_+ - w_-\|_{L^\infty(N_{-a}^{1-a})} \leqslant \delta/2$. By the 1-monotonicity of $w_\pm$, $\|u_+ - w_+\|_{L^\infty(N_{a-1})} \leqslant \delta/2$ and $\|u_+ - \tau^k w_-\|_{L^\infty(N^{k-a+1})} \leqslant \delta/2$. Therefore $\|u_+ - w_k\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta/2$. Again by Proposition 2.18, for large k, we have $\|u_k - w_k\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta/2$. Consequently, $\|u_+ - u_k\|_{L^\infty(N_{a-1}^{k-a+1})} \leqslant \delta$. Lastly applying Proposition 4.6 again for $u = w_+$ and $u = \tau^k w_-$ and combining gives

$$\|u_k - u_+\|_{W^{1,2}(N_a^{k-a})} \leqslant \varepsilon.$$

from which (4.11) follows. □

It remains to prove Proposition 4.6. Towards that end, we deduce some consequences of the nondegeneracy assumptions $(ND^\pm)$. For simplicity we work with (ND^+). Recall it means that the second variation bilinear form:

$$Q(\phi) = \mathcal{F}''(u_+)(\phi,\phi) = \int_{\mathbb{T}^n} \left(|\nabla\phi|^2 + a(x)\phi^2\right) dx, \quad a(x) = F_{uu}\big(x, u_+(x)\big)$$

is positive definite on $W^{1,2}(\mathbb{T}^n)$. Let λ be the smallest eigenvalue of the operator $-\Delta + a(x)$ in $W^{1,2}(\mathbb{T}^n)$. By (ND^+), $\lambda > 0$. Then for all $\phi \in W^{1,2}(\mathbb{T}^n)$,

$$Q(\phi) \geqslant \lambda \int_{\mathbb{T}^n} \phi^2\, dx.$$

Next we will prove an iterated inequality. Set $\mathbb{T}_m^n = (\mathbb{R}/m\mathbb{Z}) \times \mathbb{T}^{n-1}$. Thus if ϕ is a function on $\mathbb{T}_m^n$, then $\tau^m\phi = \phi$, so ϕ is m-periodic in x_1.

Proposition 4.12. *For any $\phi \in W^{1,2}(\mathbb{T}_m^n)$, we have*

$$Q_m(\phi) = \int_{\mathbb{T}_m^n} \left(|\nabla\phi|^2 + a(x)\phi^2\right) dx \geqslant \lambda \int_{\mathbb{T}_m^n} \phi^2\, dx. \tag{4.13}$$

Consequently the smallest eigenvalue, λ_m, of $-\Delta + a$ on $\mathbb{T}_m^n$ equals the smallest eigenvalue, λ, of $-\Delta + a$ on $\mathbb{T}^n$.

Proof. Set

$$\hat{Q}(\phi) = Q_m(\phi) - \lambda_m \int_{\mathbb{T}_m^n} \phi^2\, dx.$$

Thus $\hat{Q}(\phi) \geqslant 0$ for any $\phi \in W^{1,2}(\mathbb{T}^n_m)$ and the minimal value of $\hat{Q}$ is 0. It is straightforward to prove that nonzero minimizers exist, any such minimizer is an eigenfunction of $-\Delta + a$ on $\mathbb{T}^n_m$ corresponding to λ_m, and choosing ϕ to be an eigenfunction corresponding to λ shows $\lambda \geqslant \lambda_m$. By results and arguments of Moser [20] – see also [25] – the set of minimizers of $\hat{Q}$ is ordered. If ϕ is a minimizer, so is $\tau\phi$. If e.g. $\tau\phi \geqslant \phi$, then

$$\phi = \tau^m\phi \geqslant \cdots \geqslant \tau\phi \geqslant \phi.$$

Thus $\tau\phi = \phi$ and so $\phi \in W^{1,2}(\mathbb{T}^n)$. Similarly ϕ is 1-periodic if $\tau\phi \leqslant \phi$. Hence $\phi \in W^{1,2}(\mathbb{T}^n)$ and $\hat{Q}(\phi) \geqslant 0$ implies $\lambda_m \geqslant \lambda$, so we must have equality. □

Remark 4.14. This is a scalar phenomenon; a similar result is not true for vector valued ϕ. For $n = 1$, an analogue of Proposition 4.12 was known to Poincaré and repeatedly used by Morse and Hedlund [19,14]. It is a basis for Aubry–Mather theory [3,17]. In the PDE setting, a similar result was used by Moser [20].

The next result is standard.

Corollary 4.15. *There is a constant, $\mu > 0$, such that for all $\phi \in W^{1,2}(\mathbb{T}^n_m)$,*

$$Q_m(\phi) \geqslant 2\mu \|\phi\|^2_{W^{1,2}(\mathbb{T}^n_m)} \tag{4.16}$$

Proof. Let $\theta \in (0, 1)$. Then

$$\begin{aligned} Q_m(\phi) &= \theta \int_{\mathbb{T}^n_m} \left(|\nabla\phi|^2 + a(x)\phi^2\right) dx + (1-\theta) \int_{\mathbb{T}^n_m} \left(|\nabla\phi|^2 + a(x)\phi^2\right) dx \\ &\geqslant \theta \int_{\mathbb{T}^n_m} \left(|\nabla\phi|^2 + a(x)\phi^2\right) dx + (1-\theta)\lambda \int_{\mathbb{T}^n_m} \phi^2\, dx \\ &\geqslant \theta \int_{\mathbb{T}^n_m} |\nabla\phi|^2\, dx + \left((1-\theta)\lambda - \theta\|a\|_{L^\infty(\mathbb{T}^n)}\right) \int_{\mathbb{T}^n_m} \phi^2\, dx \\ &= \theta\|\phi\|^2_{W^{1,2}(\mathbb{T}^n_m)} \end{aligned}$$

provided

$$\theta = (1-\theta)\lambda - \theta\|a\|_{L^\infty(\mathbb{T}^n)}.$$

Thus (4.16) holds with

$$2\mu = \theta = \lambda\left(1 + \lambda + \|a\|_{L^\infty(\mathbb{T}^n)}\right)^{-1}. \qquad \square$$

Corollary 4.17. *Let $b \in L^\infty(N^m_0)$ with*

$$\|b - a\|_{L^\infty(N^m_0)} \leqslant \mu. \tag{4.18}$$

Then for any $\phi \in W^{1,2}(\mathbb{T}^n_m)$,

$$\int_{N^m_0} \left(|\nabla\phi|^2 + b(x)\phi^2\right) dx \geqslant \mu\|\phi\|^2_{W^{1,2}(\mathbb{T}^n_m)}. \tag{4.19}$$

Proof. This is immediate from Corollary 4.15. □

Proposition 4.20. *Let $\delta > 0$ be sufficiently small. Then for any $m \in \mathbb{N}$ and any solutions u, v of (PDE) such that $u_+ - \delta \leqslant u, v \leqslant u_+$ on N^m_0 and $u = v$ on ∂N^m_0, we have $u = v$ on N^m_0.*

Remark 4.21. Note that δ is independent of m. This will be important below.

Proof. Set $\phi = v - u$ on N_0^m and extend ϕ periodically so that $\phi \in W^{1,2}(\mathbb{T}_m^n)$. Subtracting $\Delta u = F_u(x,u)$ from $\Delta v = F_u(x,v)$ shows

$$\int_{N_0^m} \phi \Delta \phi \, dx = \int_{N_0^m} \int_0^1 F_{uu}(x, u + s\phi)\phi^2 \, ds \, dx.$$

Since $\phi|_{\partial N_0^m} = 0$, integrating by parts gives

$$0 = \int_{N_0^m} \left(|\nabla \phi|^2 + \phi^2 b(x)\right) dx, \quad b(x) = \int_0^1 F_{uu}(x, u + s\phi)\, ds. \tag{4.22}$$

For δ small (independently of m due to the periodicity of F), b satisfies (4.18). Thus by Corollary 4.17 and (4.22), $\phi \equiv 0$. $\square$

Next, we consider the nonquadratic functional $J_{N_0^m}$ on $W^{1,2}(N_0^m)$. Observe that if e.g. $\psi \in C^\alpha(\partial N_0^m)$, then $J_{N_0^m}$ has a minimizer, v, in the class of $W^{1,2}(J_{N_0^m})$ functions which equal ψ on ∂N_0^m. If v satisfies $u_+ - \delta \leqslant v \leqslant u_+$, then any solution u such that $u_+ - \delta \leqslant u \leqslant u_+$ on N_0^m is a minimizer for the boundary conditions $u|_{\partial N_0^m} = \psi$. Namely by Proposition 4.20, $u = v$. Therefore we obtain

Corollary 4.23. *Any solution, u, of (PDE) satisfying $u_+ - \delta \leqslant u \leqslant u_+$ on N_0^m is a minimizer of the functional, $J_{N_0^m}$, in the class of functions $\phi \in W^{1,2}(N_0^m)$ with $u_+ - \delta \leqslant \phi \leqslant u_+$ and $\phi|_{\partial N_0^m} = u|_{\partial N_0^m}$.*

Corollary 4.24. *There exists a $\delta > 0$ such that for any $m \in \mathbb{N}$ and any $u \in W^{1,2}(\mathbb{T}_m^n)$ satisfying $u_+ - \delta \leqslant u \leqslant u_+$, we have*

$$J_{N_0^m}(u) \geqslant \mu \|u - u_+\|^2_{W^{1,2}(\mathbb{T}_m^n)}.$$

Note that δ is independent of m.

Proof. Let $\phi = u - u_+$. Since $J_{N_0^m}(u_+) = 0$ and $J'_{N_0^m}(u_+) = 0$, expanding J_m about u_+ shows

$$J_{N_0^m}(u) = \int_0^1 J''_{N_0^m}(u_+ + s\phi)(\phi,\phi)(1-s)\, ds$$

$$= \int_{\mathbb{T}_m^n} \left(|\nabla \phi|^2 + \phi^2 b(x)\right) dx, \quad b(x) = \int_0^1 F_{uu}(x, u_+ + s\phi)(1-s)\, ds.$$

If $\delta > 0$ is sufficiently small (independently of m), then b satisfies (4.18). Thus by Corollary 4.17,

$$J_{N_0^m}(u) \geqslant \mu \|\phi\|^2_{W^{1,2}(\mathbb{T}_m^n)}. \qquad \square$$

Next we prove an analogue of Corollary 4.24 for non-periodic functions on N_0^m.

Proposition 4.25. *Let $\delta > 0$ be as in Corollary 4.24. For any $\varepsilon > 0$, there exists a $\rho > 0$ such that for any $m \in \mathbb{N}$ and any $u \in C^1(N_0^m)$ with $\|u_+ - u\|_{C^1(N_0^m)} \leqslant \rho$, we have*

$$J_{N_0^m}(u) \geqslant \mu \|u - u_+\|^2_{W^{1,2}(N_0^m)} - \varepsilon. \tag{4.26}$$

Proof. Define a periodic function $\psi \in W^{1,2}(\mathbb{T}^n_m)$ with $\psi = u$ on N^m_1 by linear interpolation on T_0:

$$\psi(x) = u_+(x) - u_+(0, x_2, \dots, x_n) + (1 - x_1)u(m, x_2, \dots, x_n) + x_1 u(1, x_2, \dots, x_n).$$

For $x \in N^1_0$, we obtain $|\psi - u| \leqslant \rho$ and $|\nabla\psi - \nabla u| \leqslant \rho$. Thus

$$\|u_+ - \psi\|_{C^1(N^m_0)} = \|\psi - u_+\|_{L^\infty(N^m_0)} + \|\nabla\psi - \nabla u_+\|_{L^\infty(N^m_0)} \leqslant 2\rho.$$

For $\rho < \delta/2$, by Corollary 4.24,

$$J_m(\psi) \geqslant \mu\|\psi - u_+\|^2_{W^{1,2}(\mathbb{T}^m_0)}.$$

Now

$$\|\psi - u_+\|^2_{W^{1,2}(\mathbb{T}^n_m)} = \|u - u_+\|^2_{W^{1,2}(N^m_0)} + \|\psi - u_+\|^2_{W^{1,2}(T_0)} - \|u - u_+\|^2_{W^{1,2}(T_0)}.$$

Therefore

$$\begin{aligned} J_{N^m_0}(\psi) &= J_{N^m_0}(u) + J_{T_0}(\psi) - J_{T_0}(u) \\ &\geqslant \mu\big(\|u - u_+\|^2_{W^{1,2}(N^m_0)} + \|\psi - u_+\|^2_{W^{1,2}(T_0)} - \|u - u_+\|^2_{W^{1,2}(T_0)}\big). \end{aligned}$$

For ρ small enough,

$$\begin{aligned} &\big|J_{T_0}(\psi)\big| \leqslant \varepsilon/4, \qquad \big|J_{T_0}(u)\big| \leqslant \varepsilon/4, \\ &\|\psi - u_+\|^2_{W^{1,2}(T_0)} \leqslant \varepsilon/4\mu, \qquad \|u - u_+\|^2_{W^{1,2}(T_0)} \leqslant \varepsilon/4\mu. \end{aligned}$$

Consequently (4.26) follows. □

Corollary 4.27. *For any $\sigma > 0$, there exists a $\delta > 0$ such that for any integers $a < b$, and any solution, u, on N^{b+1}_{a-1} with $u_+ - \delta \leqslant u \leqslant u_+$, we have*

$$J_{N^b_a}(u) \geqslant \mu\|u - u_+\|^2_{W^{1,2}(N^b_a)} - \sigma. \tag{4.28}$$

Proof. By Proposition 4.25, it suffices to show that if δ is sufficiently small and u is as in the hypothesis, then $\|u_+ - u\|_{C^1(N^b_a)} \leqslant \rho$. Since $\|u - u_+\|_{L^\infty(N^{b+1}_{a-1})} \leqslant \delta$, this follows from Lemma A.1. □

Proof of Proposition 4.6. The first item follows from Corollary 4.23. The second follows from Corollary 4.27. Indeed, the minimization property of u in N^{b+1}_{a-1} yields the estimate:

$$J_{N^b_a}(u) \leqslant M\delta,$$

where $M > 0$ is independent of δ. By (4.28),

$$\|u - u_+\|_{W^{1,2}(N^b_a)} \leqslant \sqrt{(\sigma + M\delta)/\mu} \leqslant \varepsilon.$$

provided that δ and σ are sufficiently small. □

5. Gluing two mountain pass heteroclinics

Next a minimax heteroclinic, $\hat{u}_k$, with $J(\hat{u}_k)$ close to $b = b_+ + b_-$ will be obtained. Since $\underline{v}_\pm$ are both upper boundaries of gaps or limits of upper boundaries for a sequence of gaps, using Proposition 2.18 as in the proof of Proposition 2.19, we find:

Proposition 5.1. *There exists a constant, $K > 0$, such that for all $k > K$, there is a homoclinic $v^*_k \in \mathcal{H}(u_-, u_-)$ such that*

- $v^*_k \leqslant \underline{u}_k = \min(\underline{v}_+, \tau^k \underline{v}_-)$.
- $J(v^*_k) < J(\underline{u}_k) < c$.

- *$J(v) \geqslant J(v_k^*)$ for any v such that $v_k^* \leqslant v \leqslant \underline{u}_k$.*
- *As $k \to \infty$, $J(v_k^*) \to c$,*
- *$\|v_k^* - \underline{u}_k\|_{C^0(\mathcal{N})} \to 0$.*
- *For any $a \in \mathbb{R}$, as $k \to \infty$,*

$$\left\|v_k^* - \underline{v}_+\right\|_{W^{1,2}(N^a)} \to 0, \qquad \left\|\tau^{-k} v_k^* - \underline{v}_-\right\|_{W^{1,2}(N_a)} \to 0.$$

- *If (ND^+) holds,*

$$\|v_k - \underline{v}_+\|_{W^{1,2}(\mathcal{N})} \to 0, \quad k \to \infty.$$

With $w_k = \min(w_+, \tau^k w_-)$ as earlier, let

$$\Omega_k = \left\{u \in \mathcal{W}\colon v_k^* \leqslant u \leqslant w_k,\ u - v_k^* \in E\right\}.$$

Then $\Sigma_k \subset \Omega_k$ and

$$\lim_{k\to\infty} \inf_{\Omega_k} J = c = c_+ + c_-. \tag{5.2}$$

Indeed, if not, there is $\delta > 0$ such that for arbitrary large k there is $w \in \Omega_k$ with $J(w) \leqslant c - \delta$. Since Ω_k is invariant under the heat flow, we may assume that w is a solution of (PDE). For arbitrary small $\varepsilon > 0$ and large k, there is $a \in \mathbb{Z}$ such that $u_+ - \varepsilon \leqslant v_k^* \leqslant w \leqslant u_+$ in N_{a-1}^{a+2}. Then by Lemma A.1, $\|w - u_+\|_{W^{1,2}(N_a^{a+1})} \leqslant C\varepsilon$. If ε is small enough, we can glue w to u_+ in N_a^{a+1} to obtain functions, $q_\pm \in \Gamma_\pm$, with

$$c \leqslant J(q_+) + J(q_-) < J(w) + \delta,$$

a contradiction. □

Following standard notation, we write $g : (A, B) \to (X, Y)$ if $B \subset A$, $Y \subset X$, and $g : A \to X$ is a continuous map such that $g(B) \subset Y$. Let $I^2 = [0, 1] \times [0, 1]$. Then we claim:

Proposition 5.3. *Let $\delta \in (0, b - c)$. For any $\varepsilon > 0$, there is a constant, $K > 0$, such that for any $k > K$, there exists a continuous map $g : (I^2, \partial I^2) \to (\Omega_k^{b+\varepsilon}, \Omega_k^{b-\delta})$ which is not homotopic to a map $(I^2, \partial I^2) \to (\Omega_k^{b-}, \Omega_k^{b-\delta})$ in the class of maps $(I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$.*

Again, let $\Omega_k^{b-} = \{u \in \Omega_k\colon J(u) < b\}$. The proof of Proposition 5.3 will be postponed until later.
Take $\delta \in (0, b - c)$ and define

$$a_k = \inf_h \max_{h(I^2)} J, \tag{5.4}$$

where the infimum is taken over all maps $h : (I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$ homotopic to g in the class of maps $(I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$.
By Proposition 5.3, $a_k \to b$ as $k \to \infty$. The above preliminaries yield:

Proposition 5.5. *There exists a constant, $K > 0$, such that for any $k > K$, J has a critical point $\hat{u}_k \in \Omega_k$ such that $J(\hat{u}_k) = a_k$.*

Proof. By Proposition 5.3, for $k > K$ and for any $\varepsilon \in (0, \delta)$, there exists a map $g : (I^2, \partial I^2) \to (\Omega_k^{a_k+\varepsilon}, \Omega_k^{b-\delta})$ which is not contractible to a map $(I^2, \partial I^2) \to (\Omega_k^{a_k-}, \Omega_k^{b-\delta})$. The set Ω_k is invariant under the parabolic semiflow $\Phi^t : \Omega_k \to \Omega_k$, $t \geqslant 0$. Let $g_t = \Phi^t \circ g$. Then $g_t(\partial I^2) \subset \Omega_k^{b-\delta}$ for all $t \geqslant 0$, and hence $g_t(I^2) \not\subset \Omega_k^{a_k-\varepsilon}$. Then the heat flow argument of e.g. Theorem 3.1 gives a critical point in Ω_k with $a_k \leqslant J \leqslant a_k + \varepsilon$. Letting $\varepsilon \to 0$, we obtain a critical point in $J^{-1}(a_k)$ as in earlier proofs. □

Remark 5.6. If there is just one critical point in $J^{-1}(a_k)$ and it is nondegenerate, then its Morse index will be 2. Abusing terminology a bit, we say that a_k is a critical level of index 2. With this terminology, mountain pass critical levels have index 1. More generally, we say that a is a critical level of index i if for arbitrary small $\delta > 0$, we have $H_i(\Omega^{a+\delta}, \Omega^{a-\delta}) \neq 0$. Thus there is an i-cycle C in $\Omega^{a+\delta}$ such that $\partial C \subset \Omega^{a-\delta}$ and C is not homologically equivalent to a cycle in $\Omega^{a-\delta}$. This implies the existence of a critical point u with $|J(u) - a| < \delta$ for all small $\delta > 0$, and hence with $J(u) = a$. If all critical points in $J^{-1}(a)$ are nondegenerate, then at least one will have Morse index i. Thus the name.

Remark 5.7. For each large k we have constructed 7 homoclinics in Ω_k. Four of them are given by local minimizers of J with J close to $c_+ + c_-$, two are of mountain pass type with J close to $c_\pm + b_\mp$, and one has J close to $b_+ + b_-$. Generically, 4 have Morse index 0, 2 have Morse index 1, and one has Morse index 2.

Next the proof of Proposition 5.3 will be given. It follows from the next proposition. In it all homology groups are taken with coefficients in $\mathbb{Z}_2$ and 1 is the generator of $H_2(I^2, \partial I^2) = \mathbb{Z}_2$.

Proposition 5.8. *For any $\varepsilon \in (0, \delta)$ and sufficiently large k, there exists a continuous map $g : (I^2, \partial I^2) \to (\Omega_k^{b+\varepsilon}, \Omega_k^{b-\delta})$ which defines a nonzero element $[g] = g_*(1) \in H_2(\Omega_k, \Omega_k^{b-\nu})$ for any $\nu > 0$.*

Using Proposition 5.8, we have the:

Proof of Proposition 5.3. If a homotopy of g into a map $(I^2, \partial I^2) \to (\Omega_k^{b-\nu}, \Omega_k^{b-\delta})$, $\nu > 0$, in the class of maps $(I^2, \partial I^2) \to (\Omega_k, \Omega_k^{b-\delta})$ were to exist, then $[g] = 0$ in $H_2(\Omega_k, \Omega_k^{b-\varepsilon})$, contrary to Proposition 5.8. □

It remains to prove Proposition 5.8. Let $X = \Lambda_+ \times \Lambda_-$ with the product norm. Define continuous maps $\phi_k : X \to \Omega_k$ and $\psi_\pm = \psi_\pm^k : \Omega_k \to \Lambda_\pm$ by

$$\phi_k(u, v) = \min\bigl(u, \tau^k v\bigr), \qquad \psi_+(u) = \max(u, v_+), \qquad \psi_-(u) = \tau^{-k} \max\bigl(u, \tau^k v_-\bigr).$$

Let $\psi_k = (\psi_+, \psi_-) : \Omega_k \to X$. Arguing somewhat as in Section 3, we will show the maps ψ_k and ϕ_k are almost inverses in the following sense. Let $\chi_k = \psi_k \circ \phi_k : X \to X$. Then we have:

Lemma 5.9. *For any compact set $A \subset X$,*

$$\sup_{(u,v)\in A} \bigl\| \chi_k(u, v) - (u, v) \bigr\| \to 0 \quad \text{as } k \to \infty.$$

Proof. Let $\chi_\pm^k = \psi_\pm \circ \phi_k$. Then $\chi_k = (\chi_+^k, \chi_-^k)$ and

$$\chi_+^k(u, v) = \max\bigl(\min\bigl(u, \tau^k v\bigr), v_+\bigr),$$

and similarly for χ_-^k. We will show that

$$\sup_{(u,v)\in A} \bigl\| \chi_+^k(u, v) - u \bigr\| \to 0 \quad \text{as } k \to \infty.$$

Let

$$B_k(u) = \bigl\{x \in \mathcal{N}\colon u(x) > \tau^k v_-(x)\bigr\} \subset C_k = \bigl\{v_+ > \tau^k v_-\bigr\}.$$

Then it is easy to see that $\chi_+^k(u, v) = u$ in $\mathcal{N} \setminus B_k(u)$. In $B_k(u)$ we have $\chi_+^k(u, v) = v_+$ or $\chi_k^+(u, v) = \tau^k v$. But

$$\|u - v_+\|_{W^{1,2}(C_k)} \to 0 \quad \text{as } k \to \infty$$

uniformly on any compact set of $u \in \Lambda_+$. So only the set where $\chi_k^+(u, v) = \tau^k v$ needs some care. This set is contained in $D_k = \{v_+ < \tau^k v < w_+\}$ which has measure less than one: $D_k \cap \tau D_k = \emptyset$. □

Next define a functional F on $X = \Lambda_+ \times \Lambda_-$ by $F(u, v) = J(u) + J(v)$.

Lemma 5.10. *For any $u \in \Lambda_+$ and any $v \in \Lambda_-$,*

$$J\big(\phi_k(u,v)\big) \leqslant F(u,v).$$

In addition, for any $\varepsilon > 0$, there exists a constant, $K > 0$, such that for $k \geqslant K$,

$$F\big(\psi_k(u)\big) \leqslant J(u) + \varepsilon, \quad u \in \Omega_k.$$

More briefly,

$$J \circ \phi_k \leqslant F_k \quad \text{and} \quad F \circ \psi_k \leqslant J + \varepsilon \quad \text{for large } k.$$

Thus F is a good approximation for J.

Proof of Lemma 5.10. The proof is similar to that of Lemma 3.4. The first inequality is easy:

$$J\big(\phi_k(u,v)\big) = J(u) + J(v) - J(w), \quad \text{where } w = \max\big(u, \tau^k v\big),$$

and $J(w) \geqslant 0$ for all $w \in \mathcal{W}$.

To prove the second inequality, set $U_+ = \{u < v_+\}$, $U_- = \{u < \tau^k v_-\}$ and define

$$\overline{w}_k = \max\big(v_+, \tau^k v_-\big), \qquad \underline{w}_k = \min\big(v_+, \tau^k v_-\big).$$

Then formally

$$\begin{aligned} F\big(\psi(u)\big) - J(u) &= J\big(\psi_+(u)\big) + J\big(\psi_-(u)\big) - J(u) \\ &= J_{U_+}(v_+) + J_{U_-}\big(\tau^k v_-\big) - J_{U_+\cap U_-}(u) + J_{N\setminus(U_+\cup U_-)}(u) \\ &= J_{U_+\cup U_-}(\overline{w}_k) + J_{U_+\cap U_-}(\underline{w}_k) - J_{U_+\cap U_-}(u) + J_{N\setminus(U_+\cup U_-)}(u). \end{aligned}$$

For a more precise proof, argue as in the proof of Lemma 3.4. Next define

$$w = \max(u, \bar{u}_k), \qquad z = \min(u, \underline{u}_k).$$

Then

$$F\big(\psi_k(u)\big) - J(u) = J(w) + J(\underline{w}_k) - J(z),$$

where again $J(w) \geqslant 0$. Since $z \in \Omega_k$, (5.2) implies that for any $\varepsilon > 0$ and large k (independent of u),

$$J(z) \geqslant c - \varepsilon/2, \quad J(\underline{u}_k) \leqslant c + \varepsilon/2.$$

Thus $F(\psi_k(u)) - J(u) \leqslant \varepsilon$. $\square$

Now the proof of Proposition 5.8 can be given.

Proof of Proposition 5.8. The idea goes back to Séré [26,27]. For any $\varepsilon > 0$ and large k, let $h_\pm : (I, \partial I) \to (\Lambda_\pm^{b_\pm+\varepsilon/2}, \Lambda_\pm^{c_\pm})$ be mountain pass paths joining $v_\pm$ and $w_\pm$. Define a map $h : I^2 \to X = \Lambda_+ \times \Lambda_-$ by $h(t,s) = (h_+(t), h_-(s))$. For sufficiently small $\varepsilon > 0$ and sufficiently large k,

$$F|_{h(I^2)} \leqslant b + \varepsilon \quad \text{and} \quad F|_{h(\partial I^2)} \leqslant \max\{c_+ + b_-, c_- + b_+\} + \varepsilon \leqslant b - 3\delta.$$

Thus $h : (I^2, \partial I^2) \to (X^{b+\varepsilon}, X^{b-3\delta})$, where $X^a = \{u \in X\colon F(u) \leqslant a\}$. For any $\nu \in (0, 3\delta)$, h defines an element $[h]$ in $H_2(X, X^{b-\nu})$.

We claim that

$$[h] \neq 0 \quad \text{in } H_2\big(X, X^{b-\nu}\big). \tag{5.11}$$

Indeed by the Kunneth formula, nonzero elements $[h_\pm] \in H_1(\Lambda_\pm, \Lambda_\pm^{b_\pm-\nu/2})$ define a nonzero element $[h_+] \otimes [h_-]$ in

$$H_2\big(X, \big(\Lambda_+^{b_+-\nu/2} \times \Lambda_-\big) \cup \big(\Lambda_+ \times \Lambda_-^{b_--\nu/2}\big)\big) \cong H_1\big(\Lambda_+, \Lambda_+^{b_+-\nu/2}\big) \otimes H_1\big(\Lambda_-, \Lambda_-^{b_--\nu/2}\big).$$

Consider the inclusion

$$j:\left(X,X^{b-\nu}\right)\to\left(X,\left(\Lambda_+^{b_+-\nu/2}\times\Lambda_-\right)\cup\left(\Lambda_+\times\Lambda_-^{b_--\nu/2}\right)\right)$$

and let

$$j_*:H_2\left(X,X^{b-\nu}\right)\to H_2\left(X,\left(\Lambda_+^{b_+-\nu/2}\times\Lambda_-\right)\cup\left(\Lambda_+\times\Lambda_-^{b_--\nu/2}\right)\right)$$

be the corresponding homomorphism of homology groups. Then $[h_+]\otimes[h_-]=j_*([h])\neq 0$. Hence $[h]\neq 0$ in $H_2(X,X^{b-\nu})$. $\square$

By Lemma 5.10, $\phi_k:(X^{b+\varepsilon},X^{b-3\delta})\to(\Omega_k^{b+\varepsilon},\Omega_k^{b-3\delta})$. Set $g=\phi_k\circ h$. Then $g:(I^2,\partial I^2)\to(\Omega_k^{b+\varepsilon},\Omega_k^{b-3\delta})$. We will show that for any $\varepsilon>0$ and large k the map g satisfies the condition of Proposition 5.8, i.e. $\phi_*[h]=[g]\neq 0$ in $H_2(\Omega_k,\Omega_k^{b-\varepsilon})$. For large k, we have $\psi_k:(\Omega_k,\Omega_k^{b-3\delta})\to(X,X^{b-2\delta})$ and $\psi_k(\Omega_k^{b-\varepsilon})\subset X^{b-\varepsilon/2}$. If $[g]=0$ in $H_2(\Omega_k,\Omega_k^{b-\varepsilon})$, then $(\psi_k)_*[g]=(\chi_k)_*[h]=[\chi_k\circ h]=0$ in $H_2(X,X^{b-\varepsilon/2})$.

Since $h(I^2)\subset X$ is compact, Lemma 5.9 implies that for any $\sigma>0$ and large enough k, we have $\|\chi(u)-u\|<\sigma$ for all $u\in h(I^2)$. Thus for large k there is a homotopy joining $\chi\circ h$ and h in the class of maps $(I^2,\partial I^2)\to(X,X^{b-\varepsilon/2})$. Then $[h]=0$ in $H_2(X,X^{b-\varepsilon/2})$, a contradiction to (5.11) for large k. $\square$

Next we consider the limit of a critical point $\hat u_k\in\Omega_k$ given by Proposition 5.3 as $k\to\infty$. Let

$$\Omega_\pm=\{u\in E_+:\ \underline{v}_\pm\leqslant u\leqslant w_\pm\}.$$

Proposition 5.12. *As $k\to\infty$,*

- $J(\hat u_k)=a_k\to b=b_-+b_+$.
- *There is a subsequence, $k\to\infty$, and heteroclinics $z_\pm\in\Omega_\pm$ such that $\hat u_k$ and $\tau^{-k}\hat u_k$ converge in C^2_{loc} to $z_\pm$ respectively.*

Proof. By Corollary A.5 the sequence $\hat u_k$ contains a subsequence convergent in C^2_{loc} to a solution $z_+\in\Omega_+$. It is evident that z_+ is a homoclinic solution. The same argument works for $\tau^{-k}\hat u_k$. $\square$

It is natural to suspect that $J(z_\pm)=b_\pm$. We pose this as:

Conjecture. *Suppose condition (ND_+) holds. Then*

- $J(z_\pm)=b_\pm$.
- *Along a subsequence $k\to\infty$, we have*

$$\|\zeta_k-\hat u_k\|\to 0,\qquad \zeta_k=\min\left(z_+,\tau^k z_-\right).$$

Remark 5.13. As we will show in a future paper, the same ideas used in the study of the 2-transition cases in Sections 3–5 can be employed to find multitransition homoclinic or heteroclinic solutions of (1.1). E.g. to get multi-transition homoclinics, choose $k\in\mathbb{N}$, let A_k be the set of $\mathbf{a}=(a_1^+,a_1^-,\dots,a_k^+,a_k^-)$ such that $a_i^\pm=c_\pm$ or $b_\pm$ for $i=1,\dots,k$ and for $P>0$, let $M_{P,k}$ be the set of $\mathbf{m}=(m_1^+,m_1^-,\dots,m_k^+,m_k^-)\in\mathbb{Z}^{2k}$ such that $m_i^--m_i^+\geqslant P$ for $i=1,\dots,k$ and $m_{i+1}^+-m_i^-\geqslant P$ for $i=1,\dots,k-1$. Then we can show that if P is large enough, there exists a homoclinic solution, $u=u_{\mathbf{am}}$, of (1.1) with $J(u)$ near $\sum_{i=1}^p(a_i^++a_i^-)$. The proof again involves the construction of an invariant region for the heat flow and a variational argument.

Appendix A

This appendix consists of 3 parts. First in Appendix A.1 some technical results which are used several times in the paper will be presented. Then in Appendix A.2, we state and prove a result, Theorem A.12, that contains Proposition 2.18. Lastly in Appendix A.3, the proof of one of the technical tools required in Appendix A.2 will be given.

A.1. Technical results

Let $\alpha \in (0,1)$.

Lemma A.1. *Let $a < b-2$ be integers and u, v any solutions of (1.1) in N_{a-1}^{b+1}. Then there is a constant, $C > 0$, which is independent of a, b, u, v such that $\phi = u - v$ satisfies*

$$\|\phi\|_{W^{2,2}(N_a^b)} \leqslant C\|\phi\|_{L^2(N_{a-1}^{b+1})}, \tag{A.2}$$

$$\|\phi\|_{C^{2,\alpha}(N_a^b)} \leqslant C\|\phi\|_{L^2(N_{a-1}^{b+1})}, \tag{A.3}$$

$$\|\phi\|_{C^{2,\alpha}(N_a^b)} \leqslant C\|\phi\|_{L^\infty(N_{a-1}^{b+1})}. \tag{A.4}$$

Proof. Standard linear elliptic estimates will be used. Since F is 1-periodic it suffices to prove (A.2) for $a = 0$ and $b = 1$. Subtracting the equations for u, v, we obtain an expression of the form $\Delta\phi = f$, where $|f| \leqslant M_1|\phi|$ and $M_1 = \|F_{uu}\|_{L^\infty}$. Using the L^p_{loc} linear elliptic estimates [13] yields for some constant M_2,

$$\|\phi\|_{W^{2,p}(N_0^1)} \leqslant M_2\big(\|f\|_{L^p(N_{-1}^2)} + \|\phi\|_{L^p(N_{-1}^2)}\big) \leqslant M_2(M_1+1)\|\phi\|_{L^2(N_{-1}^2)}.$$

For $p = 2$ this gives (A.2).

To prove (A.3), take $p > n$. The Sobolev inequality provides a $C^{1,\alpha}(N_{-1}^2)$ bound for ϕ in terms of the $W^{2,p}(N_{-1}^2)$ norm of ϕ. Then the local linear Schauder estimate gives a $C^{2,\alpha}(N_0^1)$ estimate for ϕ.

In fact (A.3) implies (A.2) and (A.4). □

Combining Lemma A.1 and the Arzela–Ascoli Theorem, we immediately find:

Corollary A.5. *Any sequence $(u_k) \subset \mathcal{W}$ of solutions to (PDE) contains a subsequence converging in $C^2_{\text{loc}}(\mathcal{N})$ to a solution $u \in \mathcal{W}$.*

To get $W^{1,2}$ convergence, additional conditions are needed.

Lemma A.6. *Let (u_k) be a sequence of solutions of (PDE) such that $\tau w_+ \leqslant u_k \leqslant w_+$ in N^a. Then (u_k) contains a subsequence convergent in $W^{1,2}(N^{a-1})$ and $C^2(N^{a-1})$.*

Proof. By Corollary A.5, there is a solution, U, of (1.1) such that along a subsequence, $u_k \to U$ in $C^2_{\text{loc}}(N^a)$. Because of this C^2_{loc} convergence and (A.3), it suffices to show that for any $\varepsilon > 0$, $\|u_k - U\|_{L^2(N^b)} \leqslant \varepsilon$ for some b near $-\infty$ and large k.

We have

$$\|u_k - U\|_{L^2(N^{b+1})} = \int\limits_{N^{b+1}} (u_k - U)^2\,dx \leqslant \max(u_+ - u_-) \int\limits_{N^{b+1}} |U - u_k|\,dx.$$

Now since $\tau w_+ \leqslant U \leqslant w_+$,

$$\int\limits_{N^{b+1}} |U - u_k|\,dx \leqslant \int\limits_{N^{b+1}} (w_+ - \tau w_+)\,dx = \int\limits_{N_b^{b+1}} w_+\,dx - \int\limits_{T_0} u_-\,dx \to 0$$

as $b \to -\infty$ and the result follows. □

Remark. A similar statement holds for solutions in N_a.

A.2. *Proof of Proposition 2.18*

Our main goal in this section is to give the proof of Proposition 2.18. A result related to parts of Proposition 2.18 was already proved in [25]. However unlike [25] where the goal was simply to construct a 2-transition homoclinic solution of (1.1), here we seek one that shadows two particular heteroclinics. This requires a somewhat different construction that can also be used to simplify the argument of [25]. Thus below, we present Theorem A.12 which implies both Theorem 6.8 of [25] and Proposition 2.18. First some preparation is required.

Let $c = c_- + c_+$ and $w_k = \min(w_+, \tau^k w_-)$. The solution u_k in Proposition 2.18 will be found by minimizing the functional J on a set $Y_k \subset \mathcal{W}$ which will be defined next. For simplicity, let

$$\rho(u) = \|u - u_+\|_{L^2(T_0)}.$$

Set $m = (m_+, m_-) \in \mathbb{Z}^2$ and $r = (r_+, r_-)$ with

$$0 < r_\pm < \rho(u_-)$$

For $k \in \mathbb{N}$, define

$$Y_k = Y_{k,m,r} = \{u \in \mathcal{W} \mid u \leqslant w_k,\ \rho(\tau^{m_+} u) \leqslant r_+,\ \rho(\tau^{m_- + k} u) \leqslant r_-\}.$$

and set

$$c_k = \inf_{u \in Y_k} J(u). \tag{A.7}$$

The parameters m, r will be selected so that J attains its minimum c_k in Y_k, and any minimizer u_k lies in the interior of Y_k. Hence $u_k \in \mathcal{H}(u_-, u_-)$ is a homoclinic solution of (1.1).

To choose m and r, note first that the sets, $\mathcal{M}_\pm$, of minimal heteroclinics are ordered and $\rho(u)$ is a strictly monotone function on $\mathcal{M}_\pm$. Therefore $\rho(\tau^\nu v_\pm) < \rho(\tau^\nu w_\pm)$ for all $\nu \in \mathbb{Z}$ and as $\nu \to \pm\infty$, $\rho(\tau^\nu v_\pm) \to 0$. For any $m_\pm$, a corresponding $r_\pm$ can be chosen so that

$$\rho(\tau^{m_\pm} v_\pm) < r_\pm < \rho(\tau^{m_\pm} w_\pm). \tag{A.8}$$

If $m_+ > 0$ and $-m_- > 0$ are sufficiently large, then $r_\pm$ will be as close to 0 as we please.

Let

$$\Gamma_\pm^* = \{u \in \Gamma_\pm \mid \rho(u) = r_\pm\}.$$

By item 3 of Proposition 2.6, we have:

Proposition A.9.

$$c_\pm^* = \inf_{\Gamma_\pm^*} J(u) > c_\pm.$$

One further smallness condition will be imposed on $r_\pm$ and then any pair m, r satisfying (A.8) is suitable for our purposes. Before imposing the condition, the following proposition is needed.

Set

$$A_r = \{u \in \Gamma(u_-, u_-) \mid \rho(u) \leqslant r\}$$

and define

$$\beta(r) = \inf_{A_r} J.$$

Then $0 < \beta(r) < c$. Indeed, for appropriate j and large k, $\tau^{-j} w_k \in A_r$, so $\beta(r) \leqslant J(\tau^{-j} w_k) = J(w_k)$ and as was shown preceding Proposition 2.18, $J(w_k) < c$. Now we have:

Proposition A.10. $\lim_{r\to 0} \beta(r) = c$.

So as not to delay the exposition, we postpone the proof of Proposition A.10 until Section 7.3.

Now we are ready to state Theorem A.12. Choose $r_\pm$ so that $\beta(r_\pm) > c/2$. According to (A.8), this also means that $|m_\pm|$ are sufficiently large. The functions w_+ and $\tau^k w_-$ satisfy, respectively, the r_+, r_- inequalities in the definition of Y_k. Therefore for any large k, w_k belongs to Y_k, so

$$0 \leqslant c_k = \inf_{u \in Y_k} J(u) \leqslant J(w_k) < c. \tag{A.11}$$

Theorem A.12. *There is a constant, $K > 0$, such that for any $k \in \mathbb{N}$ with $k \geqslant K$,*

- *J attains its minimum c_k on Y_k.*
- *Any minimizer, u_k, lies in the interior of Y_k and is a classical solution of* (1.1).
- *As $k \to \infty$, $\|u_k - w_k\|_{L^\infty(\mathcal{N})} \to 0$.*
- *For any $a \in \mathbb{R}$,*

$$\|u_k - w_+\|_{W^{1,2}(N^a)} \to 0, \qquad \left\|\tau^{-k} u_k - w_-\right\|_{W^{1,2}(N_a)} \to 0 \quad \textit{as } k \to \infty.$$

Theorem A.12 implies Proposition 2.18. Indeed, item 1 follows since u_k is in the interior of Y_k. If $u_k \leqslant v \leqslant w_k$, then $v \in Y_k$, so $J(v) \geqslant c_k$ giving the second item. The third was shown prior to the statement of Proposition 2.18 and the remaining item is copied verbatim.

For the proof of Theorem A.12, Proposition 6.27 from [25] which is useful for cutting and pasting arguments will be needed.

Proposition A.13. *Let $\sigma > 0$ and $M > 0$. There exists an $\ell_0 = \ell_0(\sigma, M) > 0$ with the property that whenever $u \in \mathcal{W}$ and $J(u) \leqslant M$, then any interval of length larger than ℓ_0 contains an integer, i, such that*

$$\|u - u_-\|_{L^2(N_{i-2}^{i+3})} \leqslant \sigma \quad \textit{or} \quad \|u - u_+\|_{L^2(N_{i-2}^{i+3})} \leqslant \sigma. \tag{A.14}$$

Thus if u is also a solution of (1.1), by Lemma A.1,

$$\|u - u_-\|_{C^{2,\alpha}(N_{i-1}^{i+2})} \leqslant C\sigma \quad \text{or} \quad \|u - u_+\|_{C^{2,\alpha}(N_{i-1}^{i+2})} \leqslant C\sigma. \tag{A.15}$$

Proof of Theorem A.12. Arguing as in the proof of Theorem 6.8 of [25], (a) there is a $u_k \in Y_k$ such that $J(u_k) = c_k$, (b) u_k is a solution of (1.1) except possibly in the integral constraint regions, and (c) $u_- < u_k < u_+$. To prove item 2, we will show that there is strict inequality in the two integral constraints for large k:

$$\rho\left(\tau^{m_+} u_k\right) < r_+, \qquad \rho\left(\tau^{m_- + k}\right) < r_- \tag{A.16}$$

and therefore by a standard elliptic regularity argument, u_k is a solution of (1.1) in the corresponding constraint regions and therefore in all of $\mathcal{N}$.

The arguments being the same, the first inequality (A.16) will be proved. Note that u_k is a solution of (1.1) in $N_{m_+ + 3}^{m_- + k - 3}$. Since $J(u_k)$ is bounded independently of k, by Proposition A.13, for any $\sigma > 0$ and sufficiently large k, there is $i \in \mathbb{Z}$ with

$$m_+ + 3 \leqslant i \leqslant m_- + k - 3$$

such that one of the following inequalities hold:

$$\|u_k - u_-\|_{C^{2,\alpha}(N_{i-1}^{i+2})} \leqslant C\sigma, \tag{A.17}$$

$$\|u_k - u_+\|_{C^{2,\alpha}(N_{i-1}^{i+2})} \leqslant C\sigma. \tag{A.18}$$

We claim that (A.18) holds. Indeed, suppose that (A.17) is satisfied. By the choice of $r_\pm$, there is a $\delta > 0$ such that $\beta(r_+) + \beta(r_-) > c + \delta$. The function, u_k, can be modified in N_i^{i+1} to obtain two functions $\phi_\pm \in \mathcal{W}$ such that

$$\phi_+|_{N^i} = u_k, \qquad \phi_+|_{N_{i+1}} = u_-, \qquad \phi_-|_{N_{i+1}} = u_k, \qquad \phi_-|_{N^i} = u_-,$$

and $\phi_\pm$ are linear in x_1 in N_i^{i+1}. Then for large k,

$$\left|J(u_k) - J(\phi_+) - J(\phi_-)\right| \leqslant \delta. \tag{A.19}$$

Since $\tau^{-m_+}\phi_+ \in A_r$, $\tau^{-(m_-+k)}\phi_- \in A_r$, by Proposition A.10 and (A.19), we have

$$J(u_k) \geqslant J(\phi_-) + J(\phi_+) - \delta \geqslant \beta(r_+) + \beta(r_-) - \delta > c. \tag{A.20}$$

But for large k, this is contrary to $\lim_{k\to\infty} J(u_k) = c$. Thus (A.18) holds.

Next we similarly modify u_k in N_i^{i+1} to obtain $\psi_\pm \in \Gamma_\pm$ such that:

$$\psi_+|_{N^i} = u_k, \qquad \psi_+|_{N_{i+1}} = u_+, \qquad \psi_-|_{N_{i+1}} = u_k, \qquad \psi_-|_{N^i} = u_+, \tag{A.21}$$

and for any $\delta > 0$ and large k,

$$\left|J(u_k) - J(\psi_+) - J(\psi_-)\right| \leqslant \delta. \tag{A.22}$$

Since $J(\psi_\pm) \geqslant c_\pm$ and $J(u_k) \to c$ as $k \to \infty$, (A.22) shows

$$\left|J(\psi_\pm) - c_\pm\right| \leqslant \delta \tag{A.23}$$

for large k. But if $\rho(\tau^{-m_+}u_k) = r_+$, then $\tau^{-m_+}\psi_+ \in \Gamma_+^*$. Hence, by Proposition A.9,

$$J(\psi_+) \geqslant c_+^* > c_+. \tag{A.24}$$

If δ is small enough, (A.23) and (A.24) are contradictory. Thus item 2 of Theorem A.12 is proved.

The limit results remain. To get item 3, let $\sigma > 0$ and let k and i be such that (A.18) holds. Define $\psi_\pm \in \Gamma_\pm$ as in (A.21). Choose $\delta > 0$. If k is sufficiently large, (A.23) holds. Let $\varepsilon \in (0, \min(r_+, r_-))$. Possibly making δ still smaller, by Proposition 2.6, there are functions $U_\pm \in \mathcal{M}_\pm$ such that $\|U_\pm - \psi_\pm\|_{W^{1,2}(T_j)} \leqslant \varepsilon$ for all $j \in \mathbb{Z}$. For $j \leqslant i-1$, $\psi_+ = u_k$ on T_j. Therefore

$$\|u_k - U_+\|_{L^2(T_j)} \leqslant \varepsilon \tag{A.25}$$

for all $j \leqslant i-1$. Since v_+ does not satisfy the integral constraint at m_+, it can be assumed that $U_+ \geqslant w_+$. But $\psi_+ \leqslant w_+$, so if U_+ satisfies (A.25), so does w_+. Thus

$$\|u_k - w_+\|_{L^2(T_j)} \leqslant \varepsilon, \quad j < i. \tag{A.26}$$

Then by Lemma A.1,

$$\|u_k - w_+\|_{C^{2,\alpha}(T_j)} \leqslant C\varepsilon, \quad j < i-1. \tag{A.27}$$

Now either $w_k = w_+$ or w_k lies between w_+ and u_k. Therefore in either event, by (A.27),

$$\|u_k - w_k\|_{L^\infty(N^i)} \leqslant \|u_k - w_+\|_{L^\infty(N^i)} \leqslant C\varepsilon. \tag{A.28}$$

A similar argument beginning with ψ_- also yields (A.28) in N_{i+2}. Finally by (A.18), we have

$$\|u_k - w_k\|_{L^\infty(N_{i-1}^{i+2})} \leqslant \|u_k - u_+\|_{L^\infty(N_{i-1}^{i+2})} \leqslant C\sigma. \tag{A.29}$$

Combining these estimates for $u_k - w_k$ yields item 3.

It remains to prove item 4. By Corollary A.5, there is a solution, U, of (1.1) such that along a subsequence, $u_k \to U$ in C^2_{loc}. By (A.26), $U = w_+$. Moreover the uniqueness of the limit function, w_+, implies the entire sequence, u_k, converges to w_+ in C^2_{loc}. Choose b so that $b+1 \leqslant \min(a, m_+)$. Then as $k \to \infty$, $u_k \to w_+$ in $C^2(N_b^{a+1})$. Hence for large k,

$$\tau w_+ \leqslant v_+ < u_k < w_+ \quad \text{in } N_b^{a+1}. \tag{A.30}$$

Suppose that (A.30) holds in N^{a+1}. Then Lemma A.6 shows $u_k \to w_+$ in $W^{1,2}(N^a)$ and the first part of item 4 is proved. The second follows in a similar fashion.

To verify (A.30) for N^{a+1}, suppose $u_k \leqslant v_+$ somewhere in N^{a+1}. Then $u_k \not\equiv \max(u_k, v_+) \equiv \Psi_k$ on N^{a+1}. Extend Ψ_k to $\mathcal{N}$ via $\Psi_k = u_k$ on N_{a+1}. Then $\Psi_k \in Y_k$ so $J(\Psi_k) \geqslant J(u_k)$. If $J(\Psi_k) = J(u_k)$, Ψ_k is a solution of (1.1) with $\Psi_k \geqslant u_k$. But $\Psi_k = u_k$ in N_b^{a+1} so by the Maximum Principle, $\Psi_k \equiv u_k$, a contradiction. Hence $J(\Psi_k) > J(u_k)$. We will show that this last inequality is impossible.

Set $\mathcal{O}_k = \{x \in N^b \mid v_+(x) > u_k(x)\}$. By the minimality property of v_+, $J_{\mathcal{O}_k}(u_k) \geqslant J_{\mathcal{O}_k}(v_+)$. On the other hand, u_k also has a minimality property in N^b so $J_{\mathcal{O}_k}(u_k) \leqslant J_{\mathcal{O}_k}(v_+)$. Consequently $J_{\mathcal{O}_k}(u_k) = J_{\mathcal{O}_k}(v_+)$. But then,

$$J(\Psi_k) = J_{\mathcal{O}_k}(v_+) + J_{\mathcal{N}\setminus\mathcal{O}_k}(u_k) = J(u_k),$$

a contradiction and (A.30) is proved. □

A.3. *Proof of Proposition A.10*

Let $\delta > 0$ be small. Choose any $w \in A_r$ such that $J(w) \leqslant \beta + \delta$. Since $w \in \mathcal{W}$ and $J(w) < \infty$, we have $\|w - u_-\|_{W^{1,2}(T_i)} \to 0$ as $|i| \to \infty$. Hence for large i we can glue w to u_- in $T_{\pm i}$ to obtain $v \in A_r$ such that $v = u_-$ in $N^{-i} \cup N_i$ and $J(v) \leqslant \beta + 2\delta$. Observe that $J_{N^i_{-i}}$ has a minimizer, u, in

$$\bigl\{z \in A_r \bigm| z|_{N^{-i}\cup N_i} = u_-\bigr\}.$$

Then $J(u) \leqslant \beta + 2\delta$ and u is also a minimizer of J_{T_0} on

$$S(u) = \bigl\{\psi \in A_r \bigm| \psi|_{N^0\cup N_1} = u\bigr\}.$$

We claim there exists a constant, $K > 0$, independent of r, and points $a, b \in (0,1)$, depending on r and u, such that $b - a > 1/2$ and

$$\|u - u_+\|_{W^{1,2}(N^b_a)} \leqslant Kr. \tag{A.31}$$

This inequality implies Proposition A.10. Indeed, let $\zeta = (a+b)/2$ and let $0 \leqslant \phi(x_1) \leqslant 1$ be a smooth function such that $\phi(x_1) = 0$ for $x_1 \notin [a,b]$, $\phi(\zeta) = 1$, and $|\phi'| \leqslant 4$. Set $u^* = u + \phi(u_+ - u)$. Then $u^* \in A_r$ satisfies $u^* = u_+$ when $x_1 = \zeta$ and by (A.31), for small r,

$$J\bigl(u^*\bigr) \leqslant J(u) + \delta \leqslant \beta + 3\delta. \tag{A.32}$$

Let $q_- = u^*$ for $x_1 \leqslant \zeta$ and $= u_+$ for $x_1 \geqslant \zeta$. Similarly let $q_+ = u_+$ for $x_1 \leqslant \zeta$ and $= u^*$ for $x_1 \geqslant \zeta$. Then $q_\pm \in \Gamma_\pm$ and so $J(u^*) = J(q_-) + J(q_+) \geqslant c$. Thus $\beta \geqslant c - 3\delta$. Since δ is arbitrary, Proposition A.10 is proved.

Next we prove (A.31). If $\rho(u) < r$, then u is a solution of (1.1) in N^1_0 and (A.31) follows from the argument of Lemma A.1. Thus assume that $\rho(u) = r$. By Lemma 2.22 of [25], there is a constant $K_1 > 0$ such that for all $z \in \mathcal{W}$,

$$J_{T_0}(z) \leqslant J(z) + K_1. \tag{A.33}$$

For $z = u$, by (A.32), $J(u) \leqslant c$. The form of J_{T_0} gives a constant, $M_3 > 0$ and independent of r such that $\|\nabla u\|_{L^2(T_0)} \leqslant M_3$. Hence there exists a constant $M_4 > 0$ such that for any $r > 0$,

$$\|\nabla u - \nabla u_+\|_{L^2(T_0)} \leqslant M_4.$$

Let $S_a = \{a\} \times \mathbb{T}^{n-1}$. Since

$$\int\limits_{[0,1]\times\mathbb{T}^{n-1}} |u - u_+|^2\,dx = r^2 \quad \text{and} \quad \int\limits_{[0,1]\times\mathbb{T}^{n-1}} |\nabla u - \nabla u_+|^2\,dx \leqslant M_4^2,$$

the measure of the set

$$B = \Biggl\{a \in [0,1]\colon \int\limits_{S_a} |u - u_+|^2\,dS > 4r^2 \text{ or } \int\limits_{S_a} |\nabla u - \nabla u_+|^2\,dS > 4M_4^2\Biggr\}$$

is less than $1/2$. We write $dS = dx_2 \cdots dx_n$. Hence we can find points a, b in

$$A = [0,1] \setminus B = \Biggl\{a \in [0,1]\colon \int\limits_{S_a} |u - u_+|^2\,dS \leqslant 4r^2,\ \int\limits_{S_a} |\nabla u - \nabla u_+|^2\,dS \leqslant 4M_4^2\Biggr\}$$

such that $b - a \geqslant 1/2$.

Since u is a minimizer of J_{T_0} on the hypersurface $\rho(u) = r$ in $\{w \in W^{1,2}(T_0)\colon w|_{\partial T_0} = u\}$, it readily follows that there is a Lagrange multiplier, $\lambda \in \mathbb{R}$, such that

$$\nabla J_{T_0}(u) = -\lambda \nabla \rho^2(u)$$

i.e. for all χ in $W_0^{1,2}(T_0)$,

$$\int_{T_0} \big(\nabla u \cdot \nabla \chi + F_u(x,u)\chi\big)\, dx = -2\lambda \int_{T_0} (u - u_+)\chi\, dx. \tag{A.34}$$

We claim that $\lambda \geqslant 0$. To see this, let $\zeta = (u_+ - u)\phi$ where $0 \leqslant \phi \leqslant 1$ is smooth with support in T_0. Then $u + \varepsilon\zeta \in A_r$ for $0 \leqslant \varepsilon \leqslant 1$, so

$$J(u) \leqslant J(u + \varepsilon\zeta) = J(u) + 2\lambda\varepsilon \int_{T_0} (u - u_+)^2 \phi\, dx + o(\varepsilon).$$

as $\varepsilon \to 0$. Hence $\lambda \geqslant 0$.

Now (A.34) and elliptic regularity arguments imply $u \in C^2(T_0)$ and satisfies

$$-\Delta u + F_u(x,u) = -\lambda(u - u_+)$$

in T_0. Let $\psi = u - u_+$. Then integrating

$$-\psi\Delta\psi + \psi\big(F_u(x, u_+ + \psi) - F_u(x, u_+)\big) = -2\lambda\psi^2 \leqslant 0$$

over N_a^b yields

$$-\int_{\partial N_a^b} \psi\nabla\psi \cdot \nu\, dS + \int_{N_a^b} |\nabla\psi|^2\, dx \leqslant M_1 \int_{N_a^b} \psi^2\, dx \leqslant M_1 r^2,$$

where $\nu = \pm e_1$ denotes the unit outward normal vector to ∂N_a^b. Since $a, b \in A$,

$$\left| \int_{\partial N_a^b} \psi\nabla\psi \cdot \nu\, dS \right| \leqslant \int_{S_a} |\psi\nabla\psi|\, dS + \int_{S_b} |\psi\nabla\psi|\, dS \leqslant 2\sqrt{16r^2M_4^2} = 8rM_4.$$

from which (A.31) follows with $K = 8M_4 + M_1 + 1$. $\square$

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