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Gradient integrability and rigidity results for two-phase conductivities in two dimensions

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Abstract

This paper deals with higher gradient integrability for σ-harmonic functions u with discontinuous coefficients σ, i.e. weak solutions of $\operatorname{div}(\sigma\nabla u) = 0$ in dimension two. When σ is assumed to be symmetric, then the optimal integrability exponent of the gradient field is known thanks to the work of Astala and Leonetti and Nesi. When only the ellipticity is fixed and σ is otherwise unconstrained, the optimal exponent is established, in the strongest possible way of the existence of so-called exact solutions, via the exhibition of optimal microgeometries.

We focus also on two-phase conductivities, i.e., conductivities assuming only two matrix values, σ_1 and σ_2, and study the higher integrability of the corresponding gradient field $|\nabla u|$ for this special but very significant class. The gradient field and its integrability clearly depend on the geometry, i.e., on the phases arrangement described by the sets $E_i = \sigma^{-1}(\sigma_i)$. We find the optimal integrability exponent of the gradient field corresponding to any pair $\{\sigma_1, \sigma_2\}$ of elliptic matrices, i.e., the worst among all possible microgeometries.

We also treat the unconstrained case when an arbitrary but finite number of phases are present.

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Keywords: Beltrami system; Quasiconformal mappings; Elliptic equations; Composites; Gradient integrability

Contents

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1. Introduction

Let Ω be a bounded, open and simply connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. We are interested in elliptic equations in divergence form with L^∞ coefficients, specifically,

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Here σ is a matrix valued coefficient, referred to as *conductivity*, and any weak solution $u \in H^1_{\mathrm{loc}}(\Omega)$ to the equation is called a σ*-harmonic* function. The case of discontinuous conductivities σ is particularly relevant in the context of non-homogeneous and composite materials. With this motivation, we only assume ellipticity. Denote by $\mathbb{M}^{2\times 2}$ the space of real 2×2 matrices and by $\mathbb{M}^{2\times 2}_{sym}$ the subspace of symmetric matrices.

Definition 1.1. Let $\lambda \in (0, 1]$. We say that $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ belongs to the class $\mathcal{M}(\lambda, \Omega)$ if it satisfies the following uniform bounds

$$\sigma\xi \cdot \xi \geqslant \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega, \tag{1.2}$$

$$\sigma^{-1}\xi \cdot \xi \geqslant \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega. \tag{1.3}$$

We denote by $\mathcal{M}_{sym}(\lambda, \Omega)$ the set of functions in $\mathcal{M}(\lambda, \Omega)$ which are a.e. symmetric.

Finally, we say that σ is elliptic if it belongs to the class $\mathcal{M}(\lambda, \Omega)$ for some positive λ.

The reader may wonder why to use the notion of ellipticity given in Definition 1.1. The reason is the interest in one class of applications related to the theory of the so-called composite materials. Physically this takes into account the possible presence of several well separated length scales. From the mathematical point of view one is forced to consider sequences of problems of type (1.1) and to study the limiting equation in a sense that has later been called homogenization. This process has been first undertaken, historically, in the case of symmetric conductivities giving rise to the notion of G-limit. Later, the study has been extended to the non-necessarily symmetric case and called H-convergence. It is exactly at this point that Murat and Tartar (see [18]) observed that only the ellipticity given in Definition 1.1 has the property to give H-stability. In other words, a class of pdes with uniform bounds of the latter type, H-converge to a pde with the same ellipticity as opposed to what happens if different notions of ellipticity are assumed. For a detailed explanation related to the relationship between composite materials and H-convergence we refer the reader to [2].

It is well known that the gradient of σ-harmonic functions locally belongs to some L^p with $p > 2$. Any σ-harmonic function u can be seen as the real part of a complex map $f : \Omega \mapsto \mathbb{C}$ which is a H^1_{loc} solution to the *Beltrami equation*

$$f_{\bar z} = \mu f_z + \nu \overline{f_z}, \quad \text{in } \Omega, \tag{1.4}$$

where the so-called complex dilatations μ and ν, both belonging to $L^\infty(\Omega, \mathbb{C})$, are given by

$$\mu = \frac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{1 + \operatorname{tr}\sigma + \det\sigma}, \qquad \nu = \frac{1 - \det\sigma + i(\sigma_{12} - \sigma_{21})}{1 + \operatorname{tr}\sigma + \det\sigma}, \tag{1.5}$$

and satisfy the ellipticity condition

$$\big\| |\mu| + |\nu| \big\|_{L^\infty} < 1. \tag{1.6}$$

Let us recall that weak solutions to (1.4) are called quasiregular mappings. They are called quasiconformal if, in addition, they are injective. The ellipticity (1.6) is often expressed in a different form. Indeed, it implies that there exists $0 \leqslant k < 1$ such that $\| |\mu| + |\nu| \|_{L^\infty} \leqslant k < 1$ or equivalently that

$$\big\| |\mu| + |\nu| \big\|_{L^\infty} \leqslant \frac{K-1}{K+1}, \tag{1.7}$$

for some $K > 1$. The corresponding solutions to (1.4) are called K-quasiregular, and K-quasiconformal if, in addition, they are injective. In 1994, K. Astala [3] proved one of the most important pending conjectures in the field at that time, namely that planar K-quasiregular mappings have Jacobian determinant in $L^{K/(K-1)}_{weak}$. Astala's work represented a benchmark for the issue of determining the optimal integrability exponent which was previously studied in the work of Bojarski [8] and Meyers [14].

Summarizing, to any given $\sigma \in \mathcal{M}(\lambda, \Omega)$ one can associate a corresponding pair of complex dilations via (1.5) and therefore, via the Beltrami equation (1.4) a quasiregular mapping. Therefore, given $\lambda \in (0, 1)$ and given $\sigma \in \mathcal{M}(\lambda, \Omega)$ one can find $K = K(\sigma)$ by using (1.5) and (1.7) in such a way that the σ-harmonic function u, solution to (1.1) is the real part of a K-quasiregular mapping. The Astala regularity result in this context reads as $|\nabla u| \in L^{p_K}_{weak}(\Omega)$, where $p_K := \frac{2K}{K-1}$. A more refined issue is to determine weighted estimates for the Jacobian determinant of a quasiconformal mapping. A first result in this direction was given in [7]. A much finer and more recent result is given in [6, formula (1.6)]. Throughout the present paper we focus on the simpler framework of the classical L^p spaces.

The first question is to determine the best possible (i.e. the minimal) constant $K(\sigma)$ such that if u is σ-harmonic with $\sigma \in \mathcal{M}(\lambda, \Omega)$, then u is the real part of a $K(\sigma)$-quasiregular mapping. Astala writes in his celebrated paper that his result implies sharp exponents of integrability for the gradient of solutions of planar elliptic pdes of the form (1.1), and he also remarks that K (and therefore the optimal integrability exponent) depends in a complicated manner on all the entries of the matrix σ rather than just on its ellipticity. Alessandrini and Nesi [2], in the process of proving the G-stability of Beltrami equations, made a progress which can be found in their Proposition 1.8. Let us rephrase it here (see also [1] for the estimate (1.9)).

Proposition 1.2. *Let $\lambda \in (0, 1]$. Then*

$$K_\lambda := \sup_{\sigma \in \mathcal{M}(\lambda, \Omega)} K(\sigma) = \frac{1 + \sqrt{1-\lambda^2}}{\lambda}, \tag{1.8}$$

$$K^{sym}_\lambda := \sup_{\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)} K(\sigma) = \frac{1}{\lambda}. \tag{1.9}$$

In Section 2.2 we give a simpler and more geometrical proof of Proposition 1.2 based on the real formulation of the Beltrami equation (see Propositions 2.2 and 2.3). As a straightforward corollary, it follows that any σ-harmonic function with $\sigma \in \mathcal{M}(\lambda, \Omega)$ satisfies the property $|\nabla u| \in L^{p_{K_\lambda}}_{weak}$, where K_λ is given by (1.8) and $p_{K_\lambda} := \frac{2K_\lambda}{K_\lambda - 1}$. This has to be compared with the version that holds true assuming a priori that $\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)$; in that case K_λ can be replaced by K^{sym}_λ defined in (1.9) (see Theorem 2.7).

A natural question is whether the bounds (1.8), (1.9) are optimal. Optimality in the symmetric case (1.9) was proved by Leonetti and Nesi [13]; optimality means that there exists $\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)$ for which the estimate $|\nabla u| \in L^{p_{K_\lambda}}_{weak}$ is sharp. The optimal microgeometry for σ constructed in [13] is given by a polycrystal: σ is symmetric, the eigenvalues are λ and λ^{-1} but the eigenvectors change from point to point.

The original question implicitly raised by Astala to prove optimality in the general case when $\sigma \in \mathcal{M}(\lambda, \Omega)$, namely, without assuming that σ is symmetric, was apparently forgotten. One of the goals of this paper is to analyze this case. In fact, we will give a complete answer to the problem of finding the best integrability exponent in this unconstrained class showing optimality of (1.8). This is our Theorem 1.5 in which we show that there exists a conductivity σ taking only two special values for which the corresponding solution has the desired critical exponent. In fact we also prove a sort of converse statement under the mild assumption that σ takes a finite number of values. This is our Theorem 1.6.

The case when σ is symmetric has been explored in depth. There has been a number of increasingly refined results showing optimality of Astala's theorem for specific classes of symmetric matrices σ. Specifically Faraco [9] treats the case of two isotropic materials, i.e. when σ takes values in the set of only two matrices of the form $\{KI, \frac{1}{K}I\}$, with I the identity matrix, which was originally conjectured to be optimal for the exponent $\frac{2K}{K-1}$ in a remarkable paper by Milton [15]. In a further advance a more refined version was given in [4], where the authors proved optimality in the stronger sense of exact solutions (see Definition 1.3 below). This is a very strong result the proof of which requires a machinery called the Baire category method.

In order to present our results we need first to explain why two-phase conductivities are representative for our problem. Recall that when σ is smooth, the corresponding σ-harmonic function is necessarily smooth and hence with bounded gradient. So the issue of higher exponent of integrability is really related to discontinuous coefficients. The simplest class of examples is when one has a conductivity taking only two values. We therefore ask the following question. Given two elliptic matrices, σ_1 and σ_2, consider the class of matrices $\sigma \in \mathcal{M}(\lambda, \Omega)$ of the special form $\sigma(x) = \sigma_1\chi_{E_1} + \sigma_2\chi_{E_2}$, where $\{E_1, E_2\}$ is a measurable partition of Ω and χ_{E_i} denotes the characteristic function of the set E_i. In the jargon of composite materials this is called a two-phase composite. What is the best possible information one can extrapolate from Astala's theorem? As already explained, to the ellipticity λ of σ there corresponds a suitable constant $K(\sigma)$ in the Beltrami equation. We are naturally led to the following related question: given $\mu, \nu \in L^\infty(\Omega; \mathbb{C})$ satisfying (1.7) with $K(\mu, \nu) > 1$, is it possible to transform μ and ν, by a suitable change of variables, specifically, by affine transformations, in order to decrease K and thus gain a better integrability for the solution of the transformed Beltrami equation? The key observation here is that the integrability of solutions of the Beltrami equation is invariant under such transformations, while $K(\mu, \nu)$ is not. It is then well defined the minimal Beltrami constant $K^{\min}$ attainable under such transformations (see Definition 2.8). This issue has been addressed by Faraco in [10] in the framework of Beltrami equation (1.4), in the canonical case when $\nu = 0$. The author observes that affine transformations correspond to Moebius transformations of μ, and expresses $K^{\min}$ in terms of the diameter (in the hyperbolic metric) of the range of μ. In terms of elliptic systems, Faraco's result corresponds to σ symmetric and with determinant constantly equal to one. Moreover, for two-phase conductivities in the latter class, his result is sharp.

In the present work, we consider the class of two-phase conductivities, without further restrictions, and we find an explicit formula for $K^{\min}$ in terms of all the entries of σ_1 and σ_2, see Proposition 4.2. Moreover, $K^{\min}$ gives a sharp measure of the integrability properties of solutions to (1.1). In order to clarify this issue, we will need the following definition.

Definition 1.3. Let $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ be elliptic. We say that $p^* = p^*(\sigma_1, \sigma_2) \in (2, +\infty)$ is critical for σ_1, σ_2 if the following two conditions hold.

1) For every boundary condition $u_0 \in H^{\frac{1}{2}}(\Omega)$, $p < p^*$ and every ball B compactly contained in Ω, there exists a constant $c > 0$ such that the solution u to

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in } \Omega, \\ u(x) = u_0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

with σ any (measurable) two-phase conductivities with values in $\{\sigma_1, \sigma_2\}$, satisfies

$$\int_B |\nabla u|^p \, dx \leqslant c.$$

2) There exist a boundary condition $u_0 \in H^{\frac{1}{2}}(\Omega)$ and a sequence σ_j of (measurable) two-phase conductivities with values in $\{\sigma_1, \sigma_2\}$ such that the solutions u_j to (1.10) (with σ replaced by σ_j) satisfy

$$\int_B |\nabla u_j|^{p^*} \, dx \to \infty \quad \text{as } j \to \infty \text{ for every ball } B \subset \Omega.$$

Finally, we say that p^* is critical in the stronger sense of exact solutions if 1) holds and 2) is replaced by the following stronger condition:

2s) There exist a boundary condition $u_0 \in H^{\frac{1}{2}}(\Omega)$ and a two-phase conductivity σ such that the corresponding solution u of (1.10) satisfies

$$\int_B |\nabla u|^{p^*}\,dx = \infty \quad \text{for every ball } B \subset \Omega.$$

The optimality result for two-phase conductivities is stated in Theorem 1.4 below.

Theorem 1.4. *Let σ_1, σ_2, be elliptic. Then $p_{K^{\min}}$ is the critical integrability exponent corresponding to σ_1, σ_2 in the sense of exact solutions as given by Definition 1.3,* 1) *and* 2s)*. Hence, we have $p^* = p_{K^{\min}}$.*

The strategy to prove 2) and 2s) is to describe the problems in terms of differential inclusions, following the program of Gromov, as developed by Kirchheim in the context of pdes and exploited in [4] and [9]. Specifically, to prove the weaker form of Theorem 1.4, namely the existence of so-called approximate solutions, we exhibit explicit laminates microgeometries, adapting the construction in [9]. To prove the stronger statement of existence of exact solutions, we rely on the methods of convex integrations and Baire category approach following [4]. In fact, we have to extend the results in [4,9] to a larger class of symmetric conductivities, specifically, to matrices of the form

$$\sigma = \chi_{E_1}\operatorname{diag}\bigl(S_1, \lambda^{-1}\bigr) + \chi_{E_2}\operatorname{diag}(S_2, \lambda), \quad \text{with } \lambda \leqslant S_1, S_2 \leqslant \lambda^{-1}, \tag{1.11}$$

thus generalizing the isotropic case $S_1 = \lambda^{-1}$, $S_2 = \lambda$, considered in [4] and [9]. The proof of the existence of approximate solutions is comparatively simpler then that of exact solutions. For the reader convenience, both are presented in Appendix A.

Our next theorem is really a corollary of Theorem 1.4, but we state it separately since it answers the general question to prove that the bound (1.8) is optimal in the unconstrained case when no symmetry assumptions are made on σ.

Theorem 1.5. *Let $\lambda \in (0, 1)$ and let σ_1, σ_2 be defined by*

$$\sigma_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a = \lambda,\ b = \pm\sqrt{1-\lambda^2}. \tag{1.12}$$

The exponent $p_{K_\lambda} := \frac{2K_\lambda}{K_\lambda - 1}$, with K_λ given by (1.8), is critical in the sense of Definition 1.3, i.e, $p^*(\sigma_1, \sigma_2) = p_{K_\lambda}$. More precisely, there exist a two-phase conductivity $\sigma : \Omega \mapsto \{\sigma_1, \sigma_2\}$ and a corresponding solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of (1.1) with affine boundary conditions such that $\nabla u \notin L^{p_{K_\lambda}}(B)$ for every disk $B \subset \Omega$.

Clearly, up to relabeling of the σ_i's, one can always choose b positive.

Finally, a natural question, both in the symmetric and in the unconstrained case, is whether there are other two-phase critical coefficients, that is to say, two-phase coefficients σ for which the bounds in Proposition 1.2 are attained and optimal in the sense of Definition (1.3). In Theorem 4.1 we give a complete answer to this question, characterizing all the critical conductivities with fixed ellipticity. In the symmetric case, the critical conductivities are given (up to rotations) exactly by those in (1.11) (for suitable partitions E_1, E_2). In the unconstrained case, the only critical conductivities are as in (1.12).

We believe that the results proved in this paper for two-phase conductivities could give some hints to treat the general case when σ takes an arbitrary number of values. Such generalization would require a specific analysis that is beyond the purposes of this paper. Nevertheless, we state here a simple generalization of our results in the unconstrained case, for conductivities taking an arbitrary finite number of values (see Section 4 for its proof).

Theorem 1.6. *Assume that $\sigma \in \mathcal{M}(\lambda, \Omega)$ takes a finite number of values. Suppose there exists a σ-harmonic function u (solution to (1.1)) such that for each ball $B \subset \Omega$,*

$$\int_B |\nabla u|^{p_{K_\lambda}} = +\infty.$$

Then, there exist two dense subsets E_1, E_2 of Ω with positive measure, such that $\sigma_i := \sigma \llcorner E_i$, $i = 1, 2$ are as in (1.12).

2. More about σ-harmonic functions and the Beltrami system

In the present section we review some well-known connections between σ-harmonic functions and the Beltrami system which we use in the rest of the paper. We refer the interested reader to [2] for a more detailed presentation of the argument, and to [5] for a general and comprehensive treatment.

2.1. Complex vs real formulation of a Beltrami system

Consider the Beltrami equation (1.4). It can be rewritten in the equivalent form

$$Df^t H Df = G \det Df, \tag{2.1}$$

where G and H are real matrix fields depending on μ and ν. Specifically,

$$\begin{aligned} G &= \frac{1}{d}\begin{pmatrix} |1+\mu|^2 - |\nu|^2 & 2\Im(\mu) \\ 2\Im(\mu) & |1-\mu|^2 - |\nu|^2 \end{pmatrix}, \\ H &= \frac{1}{d}\begin{pmatrix} |1-\nu|^2 - |\mu|^2 & -2\Im(\nu) \\ -2\Im(\nu) & |1+\nu|^2 - |\mu|^2 \end{pmatrix}, \end{aligned} \tag{2.2}$$

where

$$d = \sqrt{\big(1 - (|\nu| - |\mu|)^2\big)\big(1 - (|\nu| + |\mu|)^2\big)}.$$

We will refer to (1.4) as well as to (2.1) as the Beltrami system. Let $SL(2)$ be the subset of $\mathbb{M}^{2\times 2}$ of the invertible matrices with determinant one, and let $SL_{sym}(2) = \mathbb{M}^{2\times 2}_{sym} \cap SL(2)$. Notice that G and H belong to $SL_{sym}(2)$ and that they are positive definite. In fact injective solutions to (2.1) have a very neat geometrical interpretation. They are mapping $f : \Omega \to \Omega'$ which are conformal, i.e., they preserves angles, provided one uses the right scalar products, namely the one induced by G in Ω and H in Ω'. This interpretation has many consequences. We will get back to this point later in the paper. Inversion of the above formulas yields

$$\mu = \frac{G_{11} - G_{22} + 2iG_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}, \qquad \nu = \frac{H_{22} - H_{11} - 2iH_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}. \tag{2.3}$$

By combining (2.2) and (1.5) we obtain a formula for G and H as functions of σ,

$$G(\sigma) = \frac{1}{\sqrt{\det \sigma^S}}\begin{pmatrix} \sigma_{22} & -\frac{\sigma_{12}+\sigma_{21}}{2} \\ -\frac{\sigma_{12}+\sigma_{21}}{2} & \sigma_{11} \end{pmatrix}, \qquad H(\sigma) = \frac{1}{\sqrt{\det \sigma^S}}\begin{pmatrix} \det \sigma & \frac{-\sigma_{12}+\sigma_{21}}{2} \\ \frac{-\sigma_{12}+\sigma_{21}}{2} & 1 \end{pmatrix}, \tag{2.4}$$

where $\sigma^S = \frac{\sigma + \sigma^T}{2}$. Inversion of (2.4) gives

$$\sigma = \frac{1}{H_{22}}\big(G^{-1} + H_{12} J\big) \tag{2.5}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

Moreover, we can express σ as a function of μ, ν inverting the algebraic system (1.5),

$$\sigma = \begin{pmatrix} \frac{|1-\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} & \frac{2\Im(\nu-\mu)}{|1+\nu|^2 - |\mu|^2} \\ \frac{-2\Im(\nu+\mu)}{|1+\nu|^2 - |\mu|^2} & \frac{|1+\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} \end{pmatrix}. \tag{2.7}$$

Let us clarify the relationship between the Beltrami equation and σ-harmonic maps. Given positive definite matrices G and H in $L^\infty(\Omega; SL_{sym}(2))$, let $f = (u, v)$ be solution to (2.1). Then, the function u is σ-harmonic, with σ defined by (2.5). Conversely, given σ satisfying the ellipticity conditions (1.2)–(1.3) and given a σ-harmonic function u, the map $f := (u, v)$ solves (2.1), where G and H are defined by (2.4), v is such that

$$J^T \nabla v = \sigma \nabla u, \tag{2.8}$$

and J^T is the transpose of J defined in (2.6). The function v is called *stream function* of u, and is defined up to additive constants. Moreover, $\|\nabla f\|_{L^p}$ is finite if and only if $\|\nabla u\|_{L^p}$ is finite.

2.2. *Different formulations of ellipticity and higher gradient integrability*

Here we introduce classical notions of ellipticity for elliptic and Beltrami equations, and we recall the fundamental summability results due to Astala [3] and some of its consequences due to Leonetti and Nesi [13]. From now on, we will always assume that the values of μ, ν, G, H and σ are related according to (1.5) and (2.2).

The ellipticity corresponding to any pair $\mu, \nu \in L^\infty(\Omega; \mathbb{C})$ satisfying (1.6) is the positive constant $k(\mu, \nu)$ defined by

$$k(\mu, \nu) := \big\| |\mu| + |\nu| \big\|_{L^\infty}. \tag{2.9}$$

An alternative measure of ellipticity, that will be most convenient in our analysis, is provided by the following quantity

$$K(\mu, \nu) := \frac{1 + k(\mu, \nu)}{1 - k(\mu, \nu)}. \tag{2.10}$$

By a slight abuse of notation, we identify $k(G, H)$ and $K(G, H)$ in the natural way, i.e.,

$$k(G, H) = k(\mu, \nu), \qquad K(G, H) = K(\mu, \nu), \tag{2.11}$$

and whenever no confusion may arise, we will omit the dependence on their argument. In the sequel we will repeatedly use the following result relating the eigenvalues of the matrices G and H with the ellipticity inherited by the Beltrami equation as defined in (2.10).

Proposition 2.1. *Let $G, H \in L^\infty(\Omega; SL_{sym}(2))$ be positive definite. Denote by $g(x)$ and $h(x)$ the maximum eigenvalue of $G(x)$ and $H(x)$, respectively. Then*

$$K(G, H) = \|gh\|_{L^\infty(\Omega)}. \tag{2.12}$$

Proof. A direct computation shows that the maximum eigenvalues of G and H are given by

$$g = \frac{\sqrt{(1 - |\nu| + |\mu|)(1 + |\nu| + |\mu|)}}{\sqrt{(1 + |\nu| - |\mu|)(1 - |\nu| - |\mu|)}}, \qquad h = \frac{\sqrt{(1 + |\nu| - |\mu|)(1 + |\nu| + |\mu|)}}{\sqrt{(1 - |\nu| + |\mu|)(1 - |\nu| - |\mu|)}}.$$

Therefore $gh = \frac{1+|\mu|+|\nu|}{1-(|\mu|+|\nu|)}$, which yields

$$\|gh\|_{L^\infty} = \frac{1 + \||\mu| + |\nu|\|_\infty}{1 - \||\mu| + |\nu|\|_\infty} = \frac{1 + k}{1 - k} = K. \qquad \square$$

Next, we relate the ellipticity bounds for the second order elliptic operator (1.1) with the ellipticity of the associated Beltrami equation. Following the notation of (2.11), we set $K(\sigma) := K(G, H)$, where G, H and σ are related by (2.4)–(2.5). The following result has been proved in [13] and [2]; for the reader's convenience, we give here a proof based on Proposition 2.1.

Proposition 2.2. *Let $\lambda \in (0, 1]$. For each $\sigma \in \mathcal{M}(\lambda, \Omega)$ we have*

$$K(\sigma) \leqslant \frac{1 + \sqrt{1 - \lambda^2}}{\lambda}. \tag{2.13}$$

If in addition σ is symmetric, then

$$K(\sigma) \leqslant \frac{1}{\lambda}. \tag{2.14}$$

Proof. Let λ_1, λ_2 be the eigenvalues of σ^S, with $\lambda_1 \leqslant \lambda_2$. Then, from the assumption $\sigma \in \mathcal{M}(\lambda, \Omega)$ and the relationship

$$\left(\sigma^{-1}\right)^S = \frac{\det \sigma^S}{\det \sigma}\left(\sigma^S\right)^{-1},$$

it follows

$$\lambda_2 \geqslant \lambda_1 \geqslant \lambda, \tag{2.15}$$

$$\frac{\det \sigma^S}{\lambda_2 \det \sigma} = \frac{\lambda_1}{\det \sigma} \geqslant \lambda. \tag{2.16}$$

Next let g and h be the largest eigenvalue of G and H, respectively. By (2.5), it is readily seen that

$$\sigma^S = \frac{1}{H_{22}} G^{-1},$$

and hence

$$g = \frac{1}{H_{22}} \frac{1}{\lambda_1} = \frac{\sqrt{\det \sigma^S}}{\lambda_1}. \tag{2.17}$$

From (2.4) it follows

$$h + \frac{1}{h} = \frac{1}{\sqrt{\det \sigma^S}}(\det \sigma + 1). \tag{2.18}$$

Set $P := \frac{\det \sigma + 1}{\sqrt{\det \sigma^S}}$. Solving (2.18) and choosing the root which is bigger than one, yields

$$h = \frac{P + \sqrt{P^2 - 4}}{2}. \tag{2.19}$$

Then, using (2.17)–(2.19) and the inequalities (2.15)–(2.16), we obtain the following upper bound for gh,

$$\begin{aligned}
gh &= \frac{1}{2\lambda_1}\left[\det \sigma + 1 + \sqrt{(\det \sigma + 1)^2 - 4 \det \sigma^S}\right] \\
&\leqslant \frac{1}{2\lambda_1}\left[\frac{\lambda_1}{\lambda} + 1 + \sqrt{\left(\frac{\lambda_1}{\lambda} + 1\right)^2 - 4\lambda_1^2}\right] \\
&= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda_1} + \sqrt{\frac{1}{\lambda^2} + \frac{1}{\lambda_1^2} + \frac{2}{\lambda\lambda_1} - 4}\right) \\
&\leqslant \frac{1}{2}\left(\frac{2}{\lambda} + \sqrt{\frac{4}{\lambda^2} - 4}\right) \\
&= \frac{1 + \sqrt{1 - \lambda^2}}{\lambda}.
\end{aligned}$$

Now suppose that σ is symmetric and denote by λ_1 and λ_2 its eigenvalues, with $\lambda_1 \leqslant \lambda_2$. Since $\sigma \in \mathcal{M}(\lambda, \Omega)$, we have

$$\lambda \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \frac{1}{\lambda}. \tag{2.20}$$

Formula (2.4) reduces itself to

$$G = \sqrt{\det \sigma}\, \sigma^{-1}, \qquad H = \frac{1}{\sqrt{\det \sigma}} \begin{pmatrix} \det \sigma & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$g = \frac{1}{\lambda_1}\sqrt{\det\sigma}, \qquad h = \frac{1}{\sqrt{\det\sigma}}\max\{\lambda_1\lambda_2, 1\}. \tag{2.21}$$

In the case when $\lambda_1\lambda_2 < 1$, we find

$$K = \left\|\frac{1}{\lambda_1}\right\|_{L^\infty} \leqslant \frac{1}{\lambda}.$$

If otherwise $\lambda_1\lambda_2 \geqslant 1$, we have

$$K = \|\lambda_2\|_{L^\infty} \leqslant \frac{1}{\lambda}. \qquad \square$$

In the next proposition we look at conductivities σ attaining the bounds (2.13) and (2.14).

Proposition 2.3. *Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ for some $\lambda \in (0, 1)$. Then the bound (2.13) is attained if and only if on a set of positive measure there holds*

$$\sigma = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \text{with } a = \lambda,\ b = \pm\sqrt{1-\lambda^2}. \tag{2.22}$$

Moreover, if σ is symmetric (2.14) is attained if and only if either (1.2) or (1.3) is attained on a set of positive measure.

Proof. Keeping the notation introduced in the proof of Proposition 2.2, one can see that the bound (2.13) is attained if and only if the inequalities (2.15)–(2.16) hold as equalities, namely,

$$\lambda_2 = \lambda_1 = \lambda, \qquad \frac{\lambda_1}{\det\sigma} = \lambda.$$

It is readily seen that this is equivalent to (2.22). The symmetric case is straightforward. $\square$

We now recall the higher integrability results for gradients of solutions to (1.1) and (1.4). For $K > 1$, set $p_K := \frac{2K}{K-1}$. We start with the celebrated result in [3].

Theorem 2.4. *Let $f \in H^1_{\mathrm{loc}}(\Omega; \mathbb{C})$ be solution to (1.4) with $K(\mu, \nu) \geqslant 1$. Then*

$$\nabla f \in L^p_{\mathrm{loc}}(\Omega) \quad \forall p \in [2, p_{K(\mu,\nu)}).$$

Remark 2.5. In fact one can show that (see [3,13,9]), for each $p < p_{K^{\min}(\sigma)}$ and each ball B compactly contained in Ω, there exists a positive constant c such that

$$\|\nabla f\|_{L^p(B)} \leqslant c\|\nabla f\|_{L^2(\Omega)}.$$

Remark 2.6. In fact, one has the following striking optimal result as proved in [6, Corollary 4.1]

$$\frac{1}{|\Omega|}\int_\Omega |\nabla f|^p\,dx \leqslant \frac{2K(\mu,\nu)}{2K(\mu,\nu) - p(K(\mu,\nu)-1)} \quad \text{for } 2 \leqslant p < \frac{2K(\mu,\nu)}{K(\mu,\nu)-1},$$

for solutions f to (1.4) continuous up to the boundary, and equal to the identity on the boundary.

Recall that K_λ and K_λ^{sym} are defined by (1.8) and (1.9), respectively. A straightforward computation yields

$$p_{K_\lambda} = \frac{2 + 2\sqrt{1-\lambda^2}}{1 - \lambda + \sqrt{1-\lambda^2}}, \qquad p_{K_\lambda^{sym}} = \frac{2}{1+\lambda}. \tag{2.23}$$

As a consequence of Proposition 2.2 and Theorem 2.4, we obtain the following result which was proved in [13,2].

Theorem 2.7. *Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ for some $\lambda \in (0,1)$. Then, any solution $u \in H^1_{\mathrm{loc}}(\Omega)$ to (1.1) satisfies*

$$\nabla u \in L^p_{\mathrm{loc}}(\Omega) \quad \forall p \in [2, p_{K_\lambda}),$$

and, if $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$,

$$\nabla u \in L^p_{\mathrm{loc}}(\Omega) \quad \forall p \in [2, p_{K_\lambda^{sym}}),$$

where p_{K_λ} and $p_{K_\lambda^{sym}}$ are given in (2.23).

2.3. Affine transformations in the Beltrami equation

Affine changes of variables both in the domain and in the target space do not change the exponent of integrability of the gradient field. As explained in the introduction, this observation has been already made by Faraco in [10] in the special case $\nu = 0$, corresponding in the real formulation of Eq. (2.1) to $H = I$.

We begin with an optimization over affine transformations working with the real formulation of Eq. (2.1). Let $A, B \in SL(2)$ and set

$$\tilde f(x) := A^{-1} f(Bx), \qquad \tilde G(x) := B^t G(Bx) B, \qquad \tilde H(x) := A^t H(Bx) A. \tag{2.24}$$

A straightforward computation shows that, whenever $f : \Omega \mapsto \mathbb{R}^2$ is solution to (2.1), $\tilde f$ solves

$$D\tilde f^t \tilde H D\tilde f = \tilde G \det D\tilde f \quad \text{ in } B(\Omega). \tag{2.25}$$

Clearly $\tilde f$ enjoys the same integrability properties as f. This motivates the following definition.

Definition 2.8. Given μ, ν as in (1.7), G, H and σ as in (2.2) and (2.7) respectively, we set

$$K^{\min}(\sigma) \equiv K^{\min}(G, H) := \min_{A,B \in SL(2)} \left\| \tilde g(A,B) \tilde h(A,B) \right\|_{L^\infty}, \tag{2.26}$$

where $\tilde g(A,B) = \tilde g(B)$ and $\tilde h(A,B) = \tilde h(A)$ denote the maximum eigenvalue of $\tilde G$ and $\tilde H$, respectively.

The above definition is well posed since the minimum in (2.26) is attained by ellipticity.

From now on, to ease notation, we will drop the superscript and write $g(A,B), h(A,B)$ in place of $\tilde g(A,B)$, $\tilde h(A,B)$. Recalling (2.12), a straightforward generalization of Theorem 2.4 (see also Remark 2.5) leads to the following result.

Proposition 2.9. *Let $G, H \in SL_{sym}(2)$ and let $K^{\min}(G,H)$ be defined as in (2.26). Then any $f \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ solution to (2.1) satisfies*

$$\nabla f \in L^p_{\mathrm{loc}}(\Omega) \quad \forall p \in [2, p_{K^{\min}(G,H)}).$$

Moreover, for every ball B compactly contained in Ω and $p \in [2, p_{K^{\min}(G,H)})$, there exists $c > 0$ such that

$$\|\nabla f\|_{L^p(B)} \leqslant c \|\nabla f\|_{L^2(\Omega)}.$$

In the language of σ-harmonic functions, Proposition 2.9 has the following counterpart: any solution $u \in H^1_{\mathrm{loc}}(\Omega)$ to (1.1), satisfies

$$\|\nabla u\|_{L^p(B)} \leqslant c \|\nabla u\|_{L^2(\Omega)},$$

for every ball B compactly contained in Ω and $p \in [2, p_{K^{\min}(\sigma)})$.

3. Two-phase Beltrami coefficients

In the present section we focus on two-phase Beltrami coefficients. In this class, we find the ellipticity $K^{\min}$ defined in (2.26) and we characterize the Beltrami coefficients for which $K = K^{\min}$. From now on, to ease notation, we will omit the dependence on G and H in the ellipticity constants.

3.1. Two-phase Beltrami equation

Let E_1 be a measurable subset of Ω and let $E_2 := \Omega \setminus E_1$. Fix $\{G_1, G_2, H_1, H_2\} \subset SL_{sym}(2)$ positive definite (symmetric and with determinant one), and consider the functions

$$G := \chi_{E_1} G_1 + \chi_{E_2} G_2, \qquad H := \chi_{E_1} H_1 + \chi_{E_2} H_2, \tag{3.1}$$

where χ_{E_1} and χ_{E_2} are the characteristic functions of E_1 and E_2, respectively. From (2.12) it follows that for G and H of the form (3.1), one has

$$K = \max\{|gh| \llcorner E_1,\ |gh| \llcorner E_2\} = \max\{g_1 h_1, g_2 h_2\},$$

where g_i and h_i denote the largest eigenvalue in E_i of G and H, respectively. Set

$$\hat{K} := \sqrt{g_1 h_1 g_2 h_2}.$$

Lemma 3.1. *The following inequality holds*

$$K^{\min} \leqslant \hat{K} \leqslant K.$$

Proof. The inequality $\hat{K} \leqslant K$ is trivial. Let us prove that $K^{\min} \leqslant \hat{K}$. Without loss of generality we may assume that $g_1 h_1 \geqslant g_2 h_2$. Set

$$\lambda := \sqrt{\frac{g_2 h_2}{g_1 h_1}} \leqslant 1.$$

We can have either of the following cases: $h_1 < \max\{g_1, g_2, h_1, h_2\}$ or $h_1 = \max\{g_1, g_2, h_1, h_2\}$. Suppose we are in the first case. Up to a diagonalization, G_1 is of the form

$$G_1 = \begin{pmatrix} g_1 & 0 \\ 0 & \frac{1}{g_1} \end{pmatrix}.$$

We want to use the change of variables (2.24), and we recall that $g(A, B)$ and $h(A, B)$ denote the maximum eigenvalue of $\tilde{G}$ and $\tilde{H}$, respectively. We choose

$$B = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix},$$

and $A = I$. Then $g_1(A, B) = \lambda g_1$ and $g_2(A, B) \leqslant \frac{1}{\lambda} g_2$. Therefore

$$g_1(A, B) h_1(A, B) = \hat{K} \quad \text{and} \quad g_2(A, B) h_2(A, B) \leqslant \hat{K}.$$

We deduce

$$K^{\min} \leqslant g_1(A, B) h_1(A, B) = \hat{K}.$$

Suppose now that $h_1 = \max\{g_1, g_2, h_1, h_2\}$. Then, after diagonalization of H_1, we choose $B = I$ and $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, and we proceed as before. $\square$

Remark 3.2. A direct consequence of Lemma 3.1 is that $K^{\min} < K$ whenever $g_1 h_1 < g_2 h_2$.

Proposition 3.3. *The following formula for $K^{\min}$ holds*:

$$K^{\min}(G, H) = \sqrt{g_2\big(G_1^{-1/2}, H_1^{-1/2}\big) h_2\big(G_1^{-1/2}, H_1^{-1/2}\big)}. \tag{3.2}$$

Proof. In view of Lemma 3.1, it is enough to prove that for each $A, B \in SL(2)$ we have

$$g_2\big(G_1^{-1/2}, H_1^{-1/2}\big)h_2\big(G_1^{-1/2}, H_1^{-1/2}\big) \leqslant g_1(A,B)h_1(A,B)g_2(A,B)h_2(A,B). \tag{3.3}$$

For this purpose, we show that if $G_1 = H_1 = Id$, then for each $A, B \in SL(2)$,

$$g_2 \leqslant g_1(A,B)g_2(A,B), \tag{3.4}$$

$$h_2 \leqslant h_1(A,B)h_2(A,B). \tag{3.5}$$

Let $B \in SL(2)$ and set

$$\tilde{G}_1 := B^T B, \qquad \tilde{G}_2 := B^T G_2 B.$$

For every $v \in \mathbb{R}^2$ we have

$$\frac{1}{g_1(A,B)}\|v\|^2 \leqslant \langle \tilde{G}_1 v, v\rangle = \|Bv\|^2,$$

and hence

$$g_2(A,B) = \sup_{\|v\|\leqslant 1} \langle \tilde{G}_2 v, v\rangle = \sup_{\|v\|\leqslant 1} \langle G_2 Bv, Bv\rangle \geqslant \frac{g_2}{g_1(A,B)},$$

which proves (3.4). The proof of (3.5) is fully analogous. □

3.2. *Characterization of critical coefficients*

In the next proposition we will show that if the bound $K^{\min} \leqslant K$ is achieved, then G_i and H_i can be simultaneously diagonalized.

Proposition 3.4. *Let G and H be as in (3.1) and assume that $K^{\min} = \hat{K}$. Then, there exist rotations $A, B \in SO(2)$ such that*

$$A^T G_1 A := \begin{pmatrix} g_1 & 0 \\ 0 & \frac{1}{g_1} \end{pmatrix}, \qquad A^T G_2 A := \begin{pmatrix} \frac{1}{g_2} & 0 \\ 0 & g_2 \end{pmatrix}, \tag{3.6}$$

$$B^T H_1 B := \begin{pmatrix} h_1 & 0 \\ 0 & \frac{1}{h_1} \end{pmatrix}, \qquad B^T H_2 B := \begin{pmatrix} \frac{1}{h_2} & 0 \\ 0 & h_2 \end{pmatrix}. \tag{3.7}$$

Proof. We can always assume that G_1 and H_1 are as in (3.6)–(3.7). We prove that, in this case, also G_2 is diagonal (For H_2 we argue exactly in the same way.) Set

$$\hat{B} := G_1^{-\frac{1}{2}}, \qquad \hat{G}_1 := \hat{B} G_1 \hat{B} = I, \qquad \hat{G}_2 := \hat{B} G_2 \hat{B},$$

$$\hat{A} := H_1^{-\frac{1}{2}}, \qquad \hat{H}_1 = \hat{A} H_1 \hat{A} = I, \qquad \hat{H}_2 = \hat{A} H_2 \hat{A}.$$

Since $\hat{h}_2 \leqslant h_1 h_2$, $\hat{g}_2 \leqslant g_1 g_2$ and recalling Proposition 3.3 we have

$$\big(K^{\min}\big)^2 = \hat{g}_1 \hat{h}_1 \hat{g}_2 \hat{h}_2 = \hat{g}_2 \hat{h}_2 \leqslant \hat{g}_2 h_1 h_2 \leqslant h_1 h_2 g_1 g_2 = \hat{K}^2,$$

where $\hat{g}_i$ and $\hat{h}_i$ are the largest eigenvalues of $\hat{G}_i$ and $\hat{H}_i$. Since $K^{\min} = \hat{K}$, all the above inequalities are indeed equalities, and in particular $\hat{g}_2 = g_1 g_2$, that implies G_2 diagonal.

We are left to show that e_2 is the eigenvector associated with g_2. Arguing by contradiction, we assume that

$$G_2 = \begin{pmatrix} g_2 & 0 \\ 0 & \frac{1}{g_2} \end{pmatrix}.$$

Without loss of generality we may suppose that $g_1 \leqslant g_2$ and we set

$$\hat{B} := G_1^{-\frac{1}{2}}, \qquad \hat{G}_1 := \hat{B} G_1 \hat{B} = I, \qquad \hat{G}_2 := \hat{B} G_2 \hat{B}.$$

It can be easily checked that $\hat{g}_i < g_i$, that (recall $K^{\min} = \hat{K}$) provides the following contradiction

$$(K^{\min})^2 \leqslant \hat{g}_1 h_1 \hat{g}_2 h_2 < g_1 h_1 g_2 h_2 = \hat{K}^2. \qquad \square$$

4. Two-phase conductivities

In this section we study the gradient summability of σ-harmonic functions corresponding to two-phase conductivities. Let E_1 be a measurable subset of Ω and let $E_2 := \Omega \setminus E_1$. We assume that both E_1 and E_2 have positive measure. Given elliptic matrices $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$, define

$$\sigma := \chi_{E_1}\sigma_1 + \chi_{E_2}\sigma_2. \tag{4.1}$$

Set

$$K^{\min} = K^{\min}(\sigma) := K^{\min}\big(G(\sigma), H(\sigma)\big),$$

where $G(\sigma)$ and $H(\sigma)$ are defined according to (2.4), and $K^{\min}(G, H)$ is defined by (3.2).

4.1. Main results and optimality of the bound (1.8)

Proof of Theorem 1.4. We have to prove that $p^* = p_{K^{\min}}$ satisfies conditions 1) and 2s) of Definition 1.3. Condition 1) follows directly by Proposition 2.9, so we pass to the proof of 2s). By the definition of $K^{\min}$ and by Proposition 3.4, we know that, by means of affine transformations in the corresponding Beltrami equations, the coefficients $G_i(\sigma_i)$ and $H_i(\sigma_i)$, $i = 1, 2$ become diagonal as in (3.6), (3.7), with $g_i h_i = K^{\min}$. A straightforward computation shows that the corresponding transformed σ, defined according to (2.5), takes the form

$$\sigma_1 := \operatorname{diag}\big(S_1, K^{\min}\big), \qquad \sigma_2 := \operatorname{diag}\left(S_2, \frac{1}{K^{\min}}\right), \qquad K^{-1} \leqslant S_i \leqslant K. \tag{4.2}$$

Therefore, without loss of generality, in order to prove 2s) we can assume that σ_1, σ_2 are as in (4.2). The case $S_1 = K^{\min}$, $S_2 = \frac{1}{K^{\min}}$ was studied in [9], where the author exhibits a sequence of solutions with critical gradient integrability, thus proving 2), and in [4] where the authors prove the existence of exact solutions (with boundary condition $u_0(x) = x_1$), thus proving 2s). The general case is proved using the same strategy. The technical details are presented in Theorems A.2 and A.3 in Appendix A. $\square$

We now prove that the bound in (1.8) is achieved by a suitable conductivity $\sigma = \chi_{E_1}\sigma_1 + \chi_{E_2}\sigma_2$ with

$$\sigma_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad \text{with } a = \lambda,\ b = \pm\sqrt{1-\lambda^2}. \tag{4.3}$$

Proof of Theorem 1.5. By (2.4) we have $G_i(\sigma) = I$ for $i = 1, 2$, and

$$H_1 = \frac{1}{\lambda}\begin{pmatrix} 1 & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & 1 \end{pmatrix}, \qquad H_2 = \frac{1}{\lambda}\begin{pmatrix} 1 & -\sqrt{1-\lambda^2} \\ -\sqrt{1-\lambda^2} & 1 \end{pmatrix}.$$

It is easy to check that $K(\sigma) = K^{\min}(\sigma) = \frac{1+\sqrt{1-\lambda^2}}{\lambda} = K_\lambda$. In view of Theorem 1.4 we conclude that $p^*(\sigma_1, \sigma_2) = p_{K_\lambda}$, where p_{K_λ} is given by (2.23).

By an affine transformation we can diagonalize H. A straightforward computation shows that the corresponding conductivity $\hat{\sigma}$ is

$$\hat{\sigma} = K_\lambda I \chi_{E_1} + \frac{1}{K_\lambda} I \chi_{E_2}. \tag{4.4}$$

For this case, the existence of an exact solution was proved in [4] (see also Theorem A.3). $\square$

Next, we fix the ellipticity $\lambda \in (0, 1)$ and we characterize the pairs (σ_1, σ_2) corresponding to a critical integrability exponent. In the following we write $\sigma_i \in \mathcal{M}(\lambda, \Omega)$ with the obvious meaning that the constant function $x \mapsto \sigma_i$ belongs to $\mathcal{M}(\lambda, \Omega)$.

Theorem 4.1. *The following properties hold.*

i) *Let $\sigma_1, \sigma_2 \in \mathcal{M}(\lambda, \Omega)$ be such that $p^*(\sigma_1, \sigma_2) = p_{K_\lambda}$. Then σ_1, σ_2 are as in (4.3).*
ii) *Let $\sigma_1, \sigma_2 \in \mathcal{M}_{sym}(\lambda, \Omega)$ be such that $p^*(\sigma_1, \sigma_2) = p_{K_\lambda^{sym}}$ Then, up to a constant rotation, σ_1 and σ_2 take the following form*

$$\sigma_1 = \operatorname{diag}(S_1, \lambda^{\pm 1}), \qquad \sigma_2 = \operatorname{diag}(S_2, \lambda^{\mp 1}), \quad \textit{with } \lambda \leqslant S_1, S_2 \leqslant \lambda^{-1}.$$

Proof. i) From Proposition 2.2 it follows that $K \leqslant \frac{1+\sqrt{1-\lambda^2}}{\lambda} = K_\lambda$. On the other hand, Theorem 1.4 yields $K^{\min} \geqslant K_\lambda$. Lemma 3.1 implies $K^{\min} = \hat{K} = K_\lambda$, thus yielding $g_i h_i = K^{\min}$ in both phases. Now apply Proposition 2.3 to conclude that i) holds true.

ii) Again from Proposition 2.2, Theorem 1.4 and Lemma 3.1 we deduce that $K^{\min} = \hat{K} = \frac{1}{\lambda}$. Hence Proposition 3.4 implies the thesis. □

We conclude the present section by proving Theorem 1.6 stated in the Introduction.

Proof of Theorem 1.6. Let B be a ball compactly contained in Ω. From Theorem 2.4 it follows that $K(\sigma) = K_\lambda$ on B. Therefore, there exists a subset E_1 of B with positive measure where σ takes one of the values, say σ_1, defined in (4.3). Suppose by contradiction that $\sigma \neq \sigma_2$ a.e. in B. Then it is easy to see that $K^{\min}(\sigma)$ is strictly lower than K_λ on B, which together with Proposition 2.9 yields a contradiction. □

4.2. The explicit formula for $K^{\min}$

Here we give a direct formula for $K^{\min}$ depending on σ_1 and σ_2.

Proposition 4.2. *Let $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ be elliptic. Denote by Σ_1 and Σ_2 the symmetric part of σ_1 and σ_2 respectively, and by d_1 and d_2 their determinant,*

$$\Sigma_i := \sigma_i^S, \qquad d_i := \det \Sigma_i, \quad i = 1, 2.$$

Then,

$$K^{\min} = \left(\frac{m + \sqrt{m^2 - 4}}{2}\right)^{\frac{1}{2}} \left(\frac{n + \sqrt{n^2 - 4}}{2}\right)^{\frac{1}{2}}, \tag{4.5}$$

where

$$\begin{aligned} m :&= \frac{1}{\sqrt{d_1 d_2}} \operatorname{tr}(\Sigma_2 \ \operatorname{Adj} \Sigma_1) \\ &= \frac{1}{\sqrt{d_1 d_2}} \left[(\sigma_2)_{11}(\sigma_1)_{22} + (\sigma_1)_{11}(\sigma_2)_{22} - \frac{1}{2}\big((\sigma_2)_{12} + (\sigma_2)_{21}\big)\big((\sigma_1)_{12} + (\sigma_1)_{21}\big) \right]; \\ n :&= \operatorname{tr}(H_2 \ \operatorname{Adj} H_1) = \frac{1}{\sqrt{d_1 d_2}} \left[\det \sigma_1 + \det \sigma_2 - \frac{1}{2}\big((\sigma_1)_{21} - (\sigma_1)_{12}\big)\big((\sigma_2)_{21} - (\sigma_2)_{12}\big) \right], \end{aligned}$$

and H_1, H_2 are defined via (2.3). If in addition $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}_{sym}$, then (4.5) reduces itself to

$$K^{\min} = \max\left\{ \sqrt{\frac{1}{\lambda_1}}, \sqrt{\lambda_2} \right\},$$

where $\lambda_1 \leqslant \lambda_2$ are the eigenvalues of $\sigma_1^{-1/2} \sigma_2 \sigma_1^{-1/2}$.

Proof. From Proposition 3.3 it follows that $K^{\min} = \sqrt{g_2 h_2}$ where g_2 and h_2 are the maximum eigenvalues of $\widetilde{G}_2 := G_1^{-1/2} G_2 G_1^{-1/2}$ and $\widetilde{H}_2 := H_1^{-1/2} H_2 H_1^{-1/2}$ respectively. Since by (2.4), $G_i = \frac{1}{\sqrt{d_i}} \operatorname{Adj} \Sigma_i$, one has

$$\widetilde{G}_2 = \frac{\sqrt{d_1}}{\sqrt{d_2}} J \Sigma_1^{-1/2} \Sigma_2 \Sigma_1^{-1/2} J^T.$$

The eigenvalues of $\widetilde{G}_2$ are solutions to the following equation in λ

$$\det\left(\sqrt{\frac{d_1}{d_2}} \Sigma_2 - \lambda \Sigma_1\right) = 0.$$

Set $M := \Sigma_2 \ \operatorname{Adj} \Sigma_1$. Since

$$\det\left(\sqrt{\frac{d_1}{d_2}} \Sigma_2 - \lambda \Sigma_1\right) = 0 \quad \Leftrightarrow \quad \lambda^2 - \frac{\operatorname{tr} M}{\sqrt{d_1 d_2}} \lambda + 1 = 0,$$

the maximum eigenvalue g_2 is defined by

$$g_2 = \frac{\frac{\operatorname{tr} M}{\sqrt{d_1 d_2}} + \sqrt{\frac{(\operatorname{tr} M)^2}{d_1 d_2} - 4}}{2}.$$

A straightforward computation shows that

$$\operatorname{tr} M = (\sigma_2)_{11}(\sigma_1)_{22} + (\sigma_1)_{11}(\sigma_2)_{22} - \frac{1}{2}\big((\sigma_2)_{12} + (\sigma_2)_{21}\big)\big((\sigma_1)_{12} + (\sigma_1)_{21}\big) =: m.$$

Similarly, one finds that h_2 is the largest root of the equation

$$\det(H_2 - \lambda H_1) = 0.$$

Therefore

$$h_2 = \frac{\operatorname{tr} N + \sqrt{(\operatorname{tr} N)^2 - 4}}{2},$$

where $N := H_2 \ \operatorname{Adj} H_1$. It is easily checked that

$$\operatorname{tr} N = \frac{1}{\sqrt{d_1 d_2}}\left[\det \sigma_1 + \det \sigma_2 - \frac{1}{2}\big((\sigma_1)_{21} - (\sigma_1)_{12}\big)\big((\sigma_2)_{21} - (\sigma_2)_{12}\big)\right] =: n.$$

Now assume that σ_1, σ_2 are symmetric. By (2.21) we find $g_2 h_2 = \max\{\frac{1}{\lambda_1}, \lambda_2\}$, where $\lambda_1 \leqslant \lambda_2$ are the eigenvalues of

$$\tilde{\sigma}_2 := \frac{1}{(\widetilde{H}_2)_{22}} \widetilde{G}_2^{-1}.$$

Since by (2.4), $G_i = \frac{1}{\sqrt{\det \sigma_i}} \operatorname{Adj} \sigma_i$, one has

$$\begin{aligned}
\tilde{\sigma}_2 &= \frac{1}{(\widetilde{H}_2)_{22}} \frac{\sqrt{\det \sigma_2}}{\sqrt{\det \sigma_1}} J \sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2} J^T \\
&= \frac{1}{(\widetilde{H}_2)_{22}} \frac{1}{\sqrt{\det(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2})}} \operatorname{Adj}\big(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2}\big) \\
&= \frac{1}{(\widetilde{H}_2)_{22}} \sqrt{\det\big(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2}\big)} \big(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2}\big)^{-1}.
\end{aligned}$$

The eigenvalues of $\tilde{\sigma}_2$ are those of $\sigma_1^{-1/2} \sigma_2 \sigma_1^{-1/2}$ as soon as we prove that

$$(\widetilde{H}_2)_{22} = \sqrt{\det\big(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2}\big)} = \sqrt{\frac{\det \sigma_1}{\det \sigma_2}}.$$

This follows from the fact that H_1 and H_2 are diagonal and therefore

$$(\widetilde{H}_2)_{22} = \frac{(H_2)_{22}}{(H_1)_{22}} = \frac{\sqrt{\det \sigma_1}}{\sqrt{\det \sigma_2}}. \qquad \square$$

Remark 4.3. Notice that $m, n \in [2, +\infty)$. Define $f : [2, +\infty) \to [1, +\infty)$ by $f(x) = \sqrt{\frac{x+\sqrt{x^2-4}}{2}}$. With these notations we have $K^{\min} = \sqrt{f(m)f(n)}$. One easily checks that the function f is monotonically increasing and concave in $[2, +\infty)$ and that $K^{\min} = 1 \Leftrightarrow m = n = 2 \Leftrightarrow \sigma_1 = \sigma_2$.

Remark 4.4. Keeping the notation of Proposition 4.2, if $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}_{sym}$ are positive definite, a straightforward computation shows that

$$p_{K^{\min}} = \frac{2}{1 - \min\{\sqrt{\frac{1}{\lambda_1}}, \sqrt{\lambda_2}\}}.$$

5. Some G-closure results revisited

Quasiconformal mappings appear in many branches of mathematics. Only rather recently they have shown their power in the theory of composites. In the composite material literature one of the typical goals is to determine the so-called "G-closure of a set of conductivities". Roughly speaking this means the following. Assume that two matrices, called the conductivity of the "phases" and denoted by $\sigma_1, \sigma_2 \in \mathcal{M}(\lambda, \Omega)$ are given. Consider a two-phase composites, i.e. a conductivity σ of the form $\sigma = \sigma_1 \chi_{E_1} + \chi_{E_2}\sigma_2$ where E_1 and E_2 are a pair of disjoint measurable sets with $E_1 \cup E_2 = \Omega$. The task is to find the set of all possible "effective" tensors σ^* that can be obtained by mixing these two-phases while letting E_1 and E_2 vary in all the admissible ways. To make this concept precise, one needs to define an appropriate concept which is called H-convergence and was invented by Murat and Tartar. This notion was a general framework which was necessary to treat the case non-symmetric conductivity σ which could not be treated by the G-convergence previously introduced by Spagnolo. In both cases one can establish compactness results and a notion of closure. We will continue to call it G-closure according to tradition even if, in this particular case, one really needs to use the H-convergence because the tensor σ is not assumed to be symmetric a priori. We refer to the recent book of Tartar [19] and reference therein for an extensive treatment.

In this context, an extensive use of certain special properties of solutions to (1.1) and therefore to (2.8), has been made. For an accurate review, we refer to [17], see Chapter 4. As a particularly interesting case, we consider Milton's work computing the so-called G-closure of a mixture of two materials with arbitrary volume fractions [16]. In the symmetric case, i.e. when both phases have a symmetric conductivity, the G-closure was found in the eighties. The result has a long history which is reviewed in a very recent work by Francfort and Murat [11]. We refer the reader to the reference therein for more details about the original work.

Milton studied the general case without assuming symmetry. He proved that one can recover the G-closure for this case by first reducing the problem to the study of a two-phase composite in which, in addition, each phase is symmetric, [16] and Chapter 4.3 in [17], and then applying the results for the symmetric case. Milton explained how his work was generalizing previous work by many authors including Keller, Dykhne, Mendelsohn and that, in turn, he was inspired by some work of Francfort and Murat and some unpublished work by Tartar now available in [19], Lemma 20.3: In two dimensions "homogenization commutes with certain Moebius transformations". Without entering into too many details, we want to emphasize here that the basic ingredients behind these transformations have an elegant geometrical counterpart when expressed in terms of the Beltrami equation.

When σ is two-phase, by (2.4), so are the matrices H and G. In particular $H = H_1\chi_1 + H_2\chi_2$. Consider now Eq. (2.1) and make the affine change of variable $f \to F = Af$, then F satisfies a new equation in which the matrix H is replaced by $H_A := A^T HA/(\det A)$. Therefore choosing $A = H^{-\frac{1}{2}} R_2^T$ with $R_2 \in SO(2)$ and such that $R_2^T H_2 R =:$ D_2 is diagonal, one has

$$H_A = I\chi_1 + D_2\chi_2 \tag{5.1}$$

so that H_A is diagonal and thus $(H_A)_{12}$ is identically zero. This in turn implies, by (2.5) that the corresponding conductivity

$$\sigma_A := \frac{G^{-1} + (H_A)_{12} J}{(H_A)_{22}}$$

is symmetric.

We observe, in passing, that applying the same strategy to the domain of f one can independently reduce a two-phase G to the form

$$G_B = I\chi_1 + G_2\chi_2 \tag{5.2}$$

with G_2 a diagonal matrix by a linear transformation $x \to Bx$.

In the work of Milton, the "symmetrization" property for a two-phase composites is obtained as follows. Let $\lambda \in [0, 1)$ and let $\sigma \in \mathcal{M}(\lambda, \Omega)$. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \Sigma_A = (a\sigma + bJ)(cI + dJ\sigma)^{-1} \tag{5.3}$$

and let $U_\sigma = (u^1_\sigma, u^2_\sigma)$ be any solution to Eq. (2.8) i.e. $\sigma\nabla u^1_\sigma = J^T\nabla u^2_\sigma$.

Proposition 5.1. *For any two-phase composites, there exists A as in (5.3) such that the corresponding Σ_A is symmetric and moreover for some $\lambda' \in [0, 1)$ one has $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$.*

To continue the argument Milton needs to prove that the G-closure problem relative to Σ_A is mapped one to one into that relative to σ. He uses the commutation of the linear fractional transformation $\sigma \to \Sigma_A$ with homogenization (see [16] and also [19], Lemma 20.3).

Our perspective is to use the following property.

Proposition 5.2. *For any given A as in (5.3) for which $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$ for some $\lambda' \in [0, 1)$, there exists*

$$A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tag{5.4}$$

such that any solution $U_{\Sigma_A} = (u^1_{\Sigma_A}, u^2_{\Sigma_A})$ to $\Sigma_A\nabla u^1_{\Sigma_A} = J^T\nabla u^2_{\Sigma_A}$ takes the form

$$U_{\Sigma_A} = A'U_\sigma. \tag{5.5}$$

Proof. We need to prove that there exist $\{a', b', c', d'\}$ such that

$$\Sigma_A\big(a'\nabla u^1_\sigma + b'\nabla u^2_\sigma\big) = J^T\big(c'\nabla u^1_\sigma + d'\nabla u^2_\sigma\big),$$

which is equivalent to show that

$$\big(a'\Sigma_A - c'J^T\big)\nabla u^1_\sigma + \big(b'\Sigma_A - d'J^T\big)\nabla u^2_\sigma = 0.$$

We now use the equation $\sigma\nabla u^1_\sigma = J^T\nabla u^2_\sigma$ and write the previous equation as

$$\big[a'\Sigma_A - c'J^T + \big(b'\Sigma_A - d'J^T\big)J\sigma\big]\nabla u^1_\sigma = 0.$$

One possible solution (actually the only one) is found if the matrix in square brackets is zero i.e. if and only if

$$\begin{aligned} a'\Sigma_A - c'J^T + \big(b'\Sigma_A - d'J^T\big)J\sigma = 0 \quad &\Leftrightarrow \quad \Sigma_A\big(a'I + b'J\sigma\big) = c'J^T + d'\sigma \\ &\Leftrightarrow \quad \Sigma_A = \big(c'J^T + d'\sigma\big)\big(a'I + b'J\sigma\big)^{-1} \end{aligned}$$

and the latter is equivalent to make the following choice:

$$A' = \begin{pmatrix} c & d \\ -b & a \end{pmatrix}. \qquad \Box \tag{5.6}$$

Proposition 5.2 is the key property to the commuting rule and it is, indeed, a linear change of variables in the target space of the underlying quasiregular mapping $U = (u, v)$, solution to (2.8).

Finally one may wonder whether (5.6) can be chosen in such a way to have $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$ for some $\lambda' > 0$. To check this we first note that

$$\begin{aligned}\Sigma_A &= (a\sigma + bJ)(cI + dJ\sigma)^{-1} = \frac{a\sigma + bJ}{c^2\det\sigma + d^2}\operatorname{Adj}(cI + dJ\sigma)\\ &= \frac{a\sigma + bJ}{c^2\det\sigma + d^2}\big(cI + dJ^T\big(\sigma^T J^T\big)J\big) = \frac{a\sigma + bJ}{c^2\det\sigma + d^2}\big(cI + dJ^T\sigma^T\big)\\ &= \frac{ac\sigma + bcJ + ad\sigma J^T\sigma^T + bd\sigma^T}{c^2\det\sigma + d^2} = \frac{ac\sigma + bd\sigma^T + bcJ + ad\det\sigma J^T}{c^2\det\sigma + d^2}.\end{aligned}$$

It follows that

$$(\Sigma_A)^S = \frac{\Sigma_A + \Sigma_A^T}{2} = \frac{ac + bd}{c^2\det\sigma + d^2}\sigma^S. \tag{5.7}$$

Therefore, recalling (5.6), the first necessary condition to (1.2) can be expressed as follows

$$c^2 + d^2 > 0, \qquad ac + bd > 0 \quad\Leftrightarrow\quad ac + bd > 0 \quad\Leftrightarrow\quad \det A' > 0. \tag{5.8}$$

Now we need to consider Σ_A^{-1},

$$\begin{aligned}\Sigma_A^{-1} &= (cI + dJ\sigma)(a\sigma + bJ)^{-1} = \frac{cI + dJ\sigma}{a^2\det\sigma + b^2}\operatorname{Adj}(a\sigma + bJ)\\ &= \frac{cI + dJ\sigma}{a^2\det\sigma + b^2}\big(aJ\sigma^T J^T + bJ^T\big) = \frac{(cI + dJ\sigma)(aJ\sigma^T J^T + bJ^T)}{a^2\det\sigma + b^2}\\ &= \frac{acJ\sigma^T J^T + adJ\sigma J\sigma^T J^T + bcJ^T + bdJ\sigma J^T}{a^2\det\sigma + b^2}\\ &= \frac{acJ\sigma^T J^T + bdJ\sigma J^T + adJ^T\det\sigma + bcJ^T}{a^2\det\sigma + b^2}.\end{aligned}$$

It follows that

$$\big(\Sigma_A^{-1}\big)^S = \frac{ac + bd}{a^2\det\sigma + b^2}J\sigma^S J^T. \tag{5.9}$$

Therefore the second necessary condition to (1.3) is expressed as follows

$$a^2 + b^2 > 0, \qquad ac + bd > 0 \quad\Leftrightarrow\quad ac + bd > 0 \quad\Leftrightarrow\quad \det A' > 0. \tag{5.10}$$

Putting (5.8) and (5.10) together we obtain

$$\Sigma_A \in \mathcal{M}\big(\lambda', \Omega\big) \quad \text{for some } \lambda' > 0 \quad\Leftrightarrow\quad \det A' > 0. \tag{5.11}$$

Again, this fact has a clear interpretation in the language of the Beltrami system, recalling that A' represents a linear change of variables in the target space and that ellipticity in this context is measured according to Proposition 2.1.

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Appendix A. Diagonal conductivities with critical integrability properties

In [9,4], the authors exhibit an example of weak solution to (1.1) with critical integrability properties. In their construction the essential range of σ consists of only two isotropic matrices, namely, $\sigma : \Omega \mapsto \{K^{-1}I, KI\}$ with $K > 1$. In this section we show that their arguments work also for non-isotropic matrices, generalizing their construction to the case

$$\sigma_1 := \operatorname{diag}(K, S_1), \qquad \sigma_2 := \operatorname{diag}\big(K^{-1}, S_2\big), \qquad \frac{1}{K} \leqslant S_i \leqslant K, \tag{A.1}$$

thus proving optimality of Astala's theorem for the whole class of matrices above. In Section 4 we have shown that the case of diagonal matrices is indeed representative for any pair of conductivities and that, among diagonal matrices, the class defined in (A.1) cannot be further enlarged.

A.1. Reformulation of (1.1) as a differential inclusion

Recall that u is solution to (1.1) if and only if $u = f_1$ where $f = (f_1, f_2)$ is solution to the associated Beltrami equation. It is easily checked that, for σ of the form (A.1), the latter condition is equivalent to

$$Df \in E := E_1 \cup E_2, \tag{A.2}$$

where

$$E_1 = \left\{ \begin{pmatrix} T \\ J\sigma_1 T \end{pmatrix}, \ T \in \mathbb{R}^2 \right\}, \qquad E_2 = \left\{ \begin{pmatrix} T \\ J\sigma_2 T \end{pmatrix}, \ T \in \mathbb{R}^2 \right\}.$$

The goal is to find solutions $f \in H^1(\Omega; \mathbb{R}^2)$ to the differential inclusion (A.2) with critical gradient integrability, and satisfying in addition the boundary condition $f_1(x) = x_1$ on $\partial\Omega$.

We will need the following definition.

Definition A.1. The family of *laminates of finite order* is the smallest family of probability measures $\mathcal{L}(\mathbb{M}^{2\times 2})$ on $\mathbb{M}^{2\times 2}$ such that

(i) $\mathcal{L}(\mathbb{M}^{2\times 2})$ contains all Dirac masses;
(ii) if $\sum_{i=1}^n \alpha_i \delta_{A_i} \in \mathcal{L}(\mathbb{M}^{2\times 2})$ and $A_1 = \alpha B + (1-\alpha)C$ with $\operatorname{rank}(B - C) = 1$, then the probability measure $\sum_{i=2}^n \alpha_i \delta_{A_i} + \alpha_1(\alpha B + (1-\alpha)C)$ is also contained in $\mathcal{L}(\mathbb{M}^{2\times 2})$.

Given $\nu \in \mathcal{L}(\mathbb{M}^{2\times 2})$, we define the *barycenter* $\bar{\nu}$ of ν as

$$\bar{\nu} := \int_{\mathbb{M}^{2\times 2}} M \, d\nu(M).$$

A.2. Approximate solutions

In [9], it has been proved that for any given $M > 0$, one can find solutions u of (1.1), for $\sigma : \Omega \mapsto \{K^{-1}I, KI\}$, uniformly bounded in H^1 and with $\|\nabla u\|_{p_K} \geqslant M$. This result is a consequence of an explicit construction, called staircase laminates, proving that the identity matrix is the barycenter of a sequence of laminates of finite order $\nu_n \in \mathcal{L}$ such that $\nu_n \overset{*}{\rightharpoonup} \nu$ with

$$\int_{\mathbb{M}^{2\times 2}} |M|^{p_K} \, d\nu(M) = \infty, \tag{A.3}$$

and $\operatorname{supp} \nu_n \subset \partial\mathcal{C}$, where

$$\mathcal{C} := \left\{ t \operatorname{diag}(a, a/K) + (1-t)\operatorname{diag}(a, Ka) \colon \ t \in (0,1), \ a \in \mathbb{R}^+ \right\}. \tag{A.4}$$

Since the set $\partial\mathcal{C}$ is contained in $E_1 \cup E_2$ for every S_1, S_2, we readily see that the construction applies without changes to our non-isotropic case.

For the reader convenience, we restate the main result in [9] in a form which is more convenient for our purposes.

Theorem A.2. *Let σ_1, σ_2 be as in (A.1). Then, p_K is critical for σ_1, σ_2 in the sense of Definition 1.3. More precisely, there exists a two-phase conductivity $\sigma_j : \Omega \to \{\sigma_1, \sigma_2\}$ such that the solutions $u_j \in H^1(\Omega)$ to*

$$\begin{cases} \operatorname{div}(\sigma_j \nabla u_j) = 0 & \text{in } \Omega, \\ u_j(x) = x_1 & \text{on } \partial\Omega \end{cases} \tag{A.5}$$

satisfy

$$\int_B |\nabla u_j|^{p_K} \, dx \to \infty \quad \text{as } j \to \infty \text{ for every ball } B \subset \Omega.$$

Proof. We follow the proof of [9], and in particular the proof suggested in [9, Remark 6.3]. Let $\nu_j \overset{*}{\rightharpoonup} \nu$ be the staircase laminate provided in [9] and described above. Each ν_j is generated by a sequence of so-called pre-laminates, i.e., locally constant maps of the type

$$M_j(x) = \begin{pmatrix} T_j(x) \\ J\sigma_j(x)T_j(x) \end{pmatrix},$$

with $\sigma_j(x) \in \{\sigma_1, \sigma_2\}$ and with barycenter the identity matrix. It is well known that there exist matrices

$$N_j(x) = \begin{pmatrix} G_j(x) \\ H_j(x) \end{pmatrix} \to 0 \quad \text{in } W^{1,\infty}(\Omega)$$

such that $M_j + N_j$ is curl free, and such that the first row of such matrices is the gradient of a function that agrees with x_1 on $\partial\Omega$. Let $F_j := J^{-1}H_j - \sigma_j G_j$. Let v_j be the solution of

$$\operatorname{div}(\sigma_j \nabla v_j) = \operatorname{div} F_j$$

with zero boundary conditions. Notice that $F_j \to 0$ in L^2, and hence $v_j \to 0$ in H^1. By construction we have

$$T_j + G_j + \nabla v_j = \nabla u_j$$

where u_j is the solution of

$$\begin{cases} \operatorname{div}(\sigma_j \nabla u_j) = 0 & \text{in } \Omega, \\ u_j(x) = x_1 & \text{on } \partial\Omega. \end{cases}$$

Moreover, also

$$\hat{M}_j := \begin{pmatrix} \nabla u_j \\ J\sigma_j(x)\nabla u_j \end{pmatrix}$$

generates ν. The conclusion follows by the lower semicontinuity of the p-moment with respect to weak-star convergence of Young measures, see [9, Lemma 6.1]. □

A.3. Exact solutions

The results of [9] have been improved in [4, Theorem 3.13] which establishes the existence of exact solutions by an application of the Baire category method. While the methods used to construct approximate solutions can be easily extended to our cases, as done in the previous paragraph, the methods developed in [4] cannot be applied straightforwardly in our non-isotropic setting. The following theorem improves Theorem A.2, establishing the existence of exact solutions for the case (A.1).

Theorem A.3. *Let σ_1, σ_2 be as in (A.1). There exists a measurable matrix field $\sigma : \Omega \to \{\sigma_1, \sigma_2\}$ such that the solution $u \in H^1(\Omega)$ to*

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u(x) = x_1 & \text{on } \partial\Omega \end{cases} \tag{A.6}$$

satisfies for every ball $B \subset \Omega$,

$$\int_B |\nabla u|^{p_K}\, dx = \infty.$$

The proof is split into two parts.

1. *The case $S_1 \neq S_2$.*

The proof follows the strategy in [4, Theorem 3.13], where the result is proved for $\sigma_1 = KI$, $\sigma_2 = K^{-1}I$. Here the main difference is that we work with coefficients that are not isotropic. For the reader's convenience we shortly reproduce the arguments of [4], without providing a self contained proof, but only pointing out the essential modifications needed in our case.

First, we define a setting where to apply the Baire category method. Fix $\delta > 0$ such that

$$\delta < \left(\frac{(1-1/K)(K-1)}{4\max\{S_1, S_2\}K^2}\right)^{\frac{1}{2}}, \tag{A.7}$$

and let

$$\tilde{E} := E \cap \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}^{2\times 2}\colon\ |a_{12}| < \delta a_{11}\right\}. \tag{A.8}$$

Notice that the introduction of the small parameter δ enforces the solutions to (A.6) to have gradient pointing in a direction relatively close to $(1, 0)$. This property hides the anisotropy of the coefficients σ_i, and allows us to follow the strategy of [4].

Define $\mathcal{U}$ as the interior of the quasiconvex hull of $\tilde{E}$ (defined as the set of range of weak limits in L^2 of solutions to (A.2)). As in [4] it can be proved that $\mathcal{U}$ is not empty, containing for instance the identity matrix. We stress that it is at this point that we use the assumption $S_1 \neq S_2$. Indeed, if $S = S_1 = S_2$, then any anti-diagonal matrix M in the quasiconvex hull of $\tilde{E}$ satisfies $M_{2,1} = -SM_{1,2}$, and then lies in a one dimensional line; in this case, the quasiconvex hull of $\tilde{E}$ has not full dimension, so that its interior $\mathcal{U}$ is empty.

The following characterization of $\overline{\mathcal{U}}$ holds

$$\tilde{E}^{lc,1} = \overline{\mathcal{U}} = \tilde{E}^{pc},$$

where $\tilde{E}^{lc,1}$ and $\tilde{E}^{pc}$ denote the first lamination hull and the polyconvex hull of $\tilde{E}$, respectively. We refer to [4, Lemma 3.5] for the proof of the identity above and for the notion of first lamination hull and polyconvex hull. Set

$$X_0 = \left\{f \in W^{1,\infty}\left(\overline{\Omega}; \mathbb{R}^2\right)\colon\ f \text{ piecewise affine},\ Df \in \mathcal{U} \text{ a.e.},\ f|_{\partial\Omega} = x\right\},$$

let X be its closure in the weak topology of H^1, and denote by (X, w) the set X endowed with the weak topology w of H^1.

The existence of solutions to the differential inclusion is proved by an application of the Baire category method, and is based on the fact that the gradient operator $D : X \mapsto L^2(\Omega; \mathbb{M}^{2\times 2})$ is a Baire-1 mapping, i.e., the pointwise limit of continuous mappings. We refer to [12, page 57] and references therein for further clarifications on this subject. The existence result is stated in the next theorem. We refer to [4, Lemma 3.7] for its proof.

Theorem A.4. *The space (X, w) is compact and metrizable. Each $f \in X$ satisfies $f \in \overline{\mathcal{U}}$ and $f|_{\partial\Omega} = x$. The metric d on X is equivalent to the metrics induced by the L^2 and L^∞ norms. Moreover, the points of continuity of the map $D : (X, w) \to L^2(\Omega; \mathbb{M}^{2\times 2})$ form a residual set in (X, w). Finally, any point of continuity $f \in X$ of D satisfies $Df \in E_1 \cup E_2$.*

We deduce that the set of solutions to the differential inclusion (A.2) is residual in (X, w). The proof of Theorem A.2 is then a consequence of Theorem A.4 and of the following theorem.

Theorem A.5. *The set*

$$\left\{f \in X\colon \int_B |Df|^{p_K} = +\infty \text{ for all balls } B \subset \Omega\right\}$$

is residual in X.

Theorem A.5 is proved following the same strategy of the proof of Corollary 3.12 in [4], and is a direct consequence of the following lemma.

Lemma A.6. *Every $A \in \mathcal{U}$ is the barycenter of a sequence of laminates of finite order $\nu_n \in \mathcal{L}$ such that* $\operatorname{supp}\nu_n \subset \mathcal{U}$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{M}^{2\times 2}} |M|^{p_K}\, d\nu_n(M) = \infty. \tag{A.9}$$

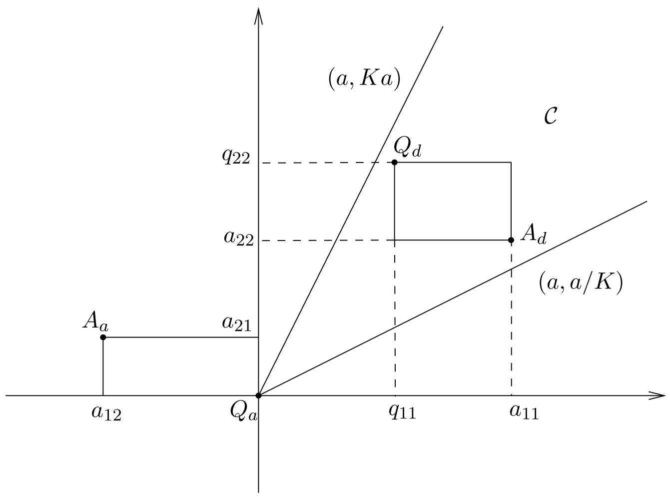

Fig. 1. The rank-one connected points A and Q.

Proof. The proof of Lemma A.6 follows the strategy of the proof of [4, Proposition 3.10], where the particular case of $S_1 = K$ and $S_2 = 1/K$ is considered. In [4] it is first showed that the identity matrix is the barycenter of a sequence of laminates of finite order satisfying (A.9) and with support on $\partial\mathcal{C}$, where $\mathcal{C}$ is the cone defined by (A.4). The proof is based on the construction of the so-called staircase laminates, which was originally made in [9]. Then, they extend the result to all other matrices by using the conformal invariance of the quasiconvex hull. In our case $\mathcal{U}$ does not enjoy conformal invariance, due to the anisotropy of the coefficients σ_i. Therefore, we have to proceed in a different way.

For this purpose it is convenient to introduce some notation. Given a matrix $A = (a_{ij}) \in \mathbb{M}^{2\times 2}$, we denote by A_d and A_a its diagonal and anti-diagonal part, namely

$$A_d := \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \qquad A_a := \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}. \tag{A.10}$$

Moreover we will identify A_d and A_a with points of $\mathbb{R}^2$: $A_d = (a_{11}, a_{22})$, $A_a = (a_{12}, a_{21})$.

By slightly modifying the staircase construction in [9,4] (in fact only a finite number of steps at the beginning of the staircase) one can easily show that each point in $\mathcal{C}$ can be obtained as the barycenter of a sequence of laminates of finite order, satisfying (A.9) and with support on $\partial\mathcal{C}$. Moreover, by a suitable shift of the support, one can obtain that these measures have support in the interior of the cone $\mathcal{C}$.

Now let $A = (a_{ij}) \in \mathcal{U}$. We claim that A is rank-one connected to a diagonal matrix $Q = (q_{ij}) = Q_d \in \mathcal{C}$ and we conclude the proof. It is easy to show that $Q \in \mathcal{U}$ (that is to say, Q belongs to the interior of the quasiconvex hull), and that A belongs to a suitable segment $[P, Q]$ still contained in $\mathcal{U}$, i.e., $A = \tau P + (1-\tau)Q$ for some $\tau \in (0, 1)$. Since $Q \in \mathcal{C}$, Q is the barycenter of a sequence of laminates $\nu_n = \sum \lambda_j \delta_{A_j}$ supported in $\mathcal{U}$ and satisfying (A.9). The required laminates can then be defined as

$$\tilde{\nu}_n = \tau\delta_P + (1-\tau)\sum \lambda_j \delta_{A_j}.$$

We conclude by proving the claim. From (A.8) it follows that $A_d \in \mathcal{C}$. The condition of rank-one connectedness reads as

$$(a_{11} - q_{11})(a_{22} - q_{22}) = a_{21}a_{12}. \tag{A.11}$$

This is equivalent to the fact that the two rectangles with sides parallel to the axis and diagonal Q_dA_d and Q_aA_a have the same signed area (see Fig. 1). Notice that the sign of the areas is given by the sign of the slope of Q_dA_d and of Q_aA_a, respectively. Define $t(A_d, Q_d)$ as the signed area of the corresponding rectangle and remark that it is a continuous function. Given A, the problem is to find Q_d such that

$$t(A_d, Q_d) = a_{21}a_{12}. \tag{A.12}$$

Notice that

$$\{t(A_d, Q_d), Q_d \in \bar{\mathcal{C}}\} = [m, \infty)$$

for a suitable negative $m < 0$ depending on A_d. Therefore, if $a_{21}a_{12} > 0$ we can always solve (A.12). Assume instead that $a_{21}a_{12} < 0$ like in Fig. 1. Let $\tilde{t}(A_d)$ be the infimum of h over Q_d. For a fixed a_{11}, it is easy to see that $\tilde{t}$ attains its maximum for $a_{22} = a_{11}$. In this case, the optimal Q_d is given by

$$Q_d = \frac{1}{2}\left(a_{11} + \frac{a_{11}}{K}, K a_{11} + a_{11}\right),$$

and

$$\max_{a_{22}} \tilde{t}(A_d) = -\frac{a_{11}^2}{4}(1 - 1/K)(K - 1).$$

Therefore (A.12) has a solution whenever

$$-\frac{a_{11}^2}{4}(1 - 1/K)(K - 1) < a_{21}a_{12}. \tag{A.13}$$

From (A.8) it follows that

$$|a_{21}| < \max\{S_1, S_2\} K\delta a_{22},$$

and hence

$$|a_{12}a_{21}| < \max\{S_1, S_2\} K\delta a_{22}|a_{12}| < \max\{S_1, S_2\} K\delta^2 a_{11}a_{22} < \max\{S_1, S_2\} K\delta^2 K a_{11}^2.$$

By the very definition (A.7) of δ we deduce that

$$\frac{a_{11}^2}{4}(1 - 1/K)(K - 1) > \max\{S_1, S_2\} K^2\delta^2 a_{11}^2,$$

so that (A.13) holds, and the proof is completed.

2. *The case $S_1 = S_2$.*

By an affine change of variables one can reduce to the case when $S_1 = S_2 = 1$. (In fact one takes $A = B = \operatorname{diag}(S_1^{-1/4}, S_1^{1/4})$ in (2.24).) The existence of solutions in this case follows from the following theorem which was proved in [4] in the context of studying equations in non-divergence form.

Theorem A.7. *There exists a measurable set $E \subset \Omega$ such that the solution $v \in W^{2,2}(\Omega; \mathbb{R}^2)$ to*

$$\begin{cases} \operatorname{Tr}(M D^2 v) = 0 & \text{in } \Omega, \\ v(x) = \frac{|x|^2}{2} & \text{on } \partial\Omega \end{cases} \tag{A.14}$$

with

$$M = \chi_E \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix} + (1 - \chi_E) \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix},$$

satisfies

$$v(x) = x_1 \quad \text{on } \partial\Omega, \quad \text{and} \quad \int_B |D^2 v|^{p_K}\, dx = \infty \quad \text{for every ball } B \subset \Omega.$$

Indeed, one can easily check that if v is the solution to (A.14) provided by Theorem A.7, then the function $u := \partial_{x_1} v$ satisfies (A.6) for $\sigma = \chi\sigma_1 + (1 - \chi)\sigma_2$. □

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