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Monotonicity of solutions to quasilinear problems with a first-order term in half-spaces

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Abstract

We consider a quasilinear elliptic equation involving a first-order term, under zero Dirichlet boundary condition in half-spaces. We prove that any positive solution is monotone increasing with respect to the direction orthogonal to the boundary. The main ingredient in the proof is a new comparison principle in unbounded domains. As a consequence of our analysis, we also obtain some new Liouville type theorems.

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1. Introduction and statement of the main results

We consider $C^{1,\alpha}$ weak solutions to the problem

$$\begin{cases} -\operatorname{div}\big(a(u)|\nabla u|^{p-2}\nabla u\big) + b(u)|\nabla u|^q = f(u), & \text{in } \mathbb{R}^N_+, \\ u\big(x', y\big) > 0, & \text{in } \mathbb{R}^N_+, \\ u\big(x', 0\big) = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{1.1}$$

where we assume $N \geqslant 2$, $\alpha \in (0, 1)$ and

(H_1) $1 < p < 2$, $1 < q \leqslant p$;
(H_2) a, b and f are locally Lipschitz continuous functions on $\mathbb{R}$;
(H_3) there exists $\gamma > 0$ such that $a(s) \geqslant \gamma$ for every $s \in \mathbb{R}$.

We denote a generic point in $\mathbb{R}^N_+$ by $x = (x', y)$ with $x' = (x_1, x_2, \dots, x_{N-1})$ and $y = x_N$.

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Note that the $C^{1,\alpha}$ regularity of the solutions follows by the well-known results in [18,30,31,40].

We study monotonicity properties of the solutions, w.r.t. (with respect to) the y-direction, via the Alexandrov–Serrin moving plane method [1,6,27,38]. For the semilinear case, the founding papers on this topic go back to the works [2–5,15,28]. We also refer the readers to [7,9,16,19,20,25,36] for other results concerning the monotonicity of the solutions in half-spaces also in more general settings (always in the uniformly elliptic case).

In the present work, we consider the quasilinear problem (1.1) and we continue the study that we have started in [21,22].

The first main contribution of this paper is the following:

Theorem 1.1. *Let u be a solution to (1.1) and let us assume that $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ and $\nabla u \in L^\infty(\mathbb{R}^N_+)$. Let (H_1), (H_2) and (H_3) be satisfied and assume that $f(s) > 0$ for $s > 0$. Then u is monotone increasing w.r.t. the y-direction, that is*

$$\frac{\partial u}{\partial y} \geqslant 0 \quad \text{in } \mathbb{R}^N_+.$$

For $a(\cdot) = 1$ and $b(\cdot) = 0$ problem (1.1) reduces to

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \mathbb{R}^N_+, \\ u(x', y) > 0, & \text{in } \mathbb{R}^N_+, \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.2}$$

The monotonicity of solutions to (1.2) was first studied in [14] in the two dimensional case and considering positive nonlinearities. Later, in dimension N (always for positive nonlinearities) a first result was obtained in [21]. Both the results hold under the restriction $\frac{2N+2}{N+2} < p < 2$.

As a consequence of Theorem 1.1, we can remove this restriction and get the following:

Corollary 1.2. *Let $1 < p < 2$ and $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ be a solution of problem (1.2) with $|\nabla u| \in L^\infty(\mathbb{R}^N_+)$. Assume that f is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$. Then u is monotone increasing w.r.t. the y-direction.*

Let us point out that in [21] the restriction $\frac{2N+2}{N+2} < p < 2$ is needed because there the strong maximum and comparison principles of [13] are used, which, in turn, are based on the estimates in [12]. A novelty in this paper, even in the special case of the pure p-Laplacian operator, is the fact that we avoid the restriction $\frac{2N+2}{N+2} < p < 2$.

Let us also mention that the case $p = 2$ is well known, as remarked here above, while recently in [22] the monotonicity of solutions to (1.2) is proved in the case $2 < p < 3$, for positive power-like nonlinearities or in the case $p > 2$, for strictly positive nonlinearities.

The technique exploited in the proof of Theorem 1.1 also allows us to improve Theorem 1.8 in [21] allowing the presence of a first-order term in the equation and avoiding also in this case the restriction $p > \frac{2N+2}{N+2}$. Namely we have the following:

Theorem 1.3. *Let (H_1) and (H_2) be satisfied. Assume that $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$, with $u \in W^{1,\infty}(\mathbb{R}^N_+)$, is a solution to*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(u)|\nabla u|^q = f(u), & \text{in } \mathbb{R}^N_+, \\ u(x', y) > 0, & \text{in } \mathbb{R}^N_+, \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases}$$

with $b(u) \geqslant 0$. Suppose that

$$\exists z > 0: \quad 0 < s < z \quad \Rightarrow \quad f(s) > 0 \quad \text{and} \quad s > z \quad \Rightarrow \quad f(s) < 0.$$

Then u is monotone increasing w.r.t. the y-direction.

Let us emphasize some new Liouville type results that complement and improve those we proved in [21, Theorem 1.6]. They follow from our monotonicity results and some techniques used in [21]:

Theorem 1.4. *Assume $1 < p < 2$ and let f be locally Lipschitz continuous. Let $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ be a non-negative solution of*

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \mathbb{R}^N_+, \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.3}$$

Assume that one of the following holds:

(a) $N = 2$ *and* $f(s) > 0$ *for* $s > 0$, *with* $f(0) = 0$,
(b) $N \geqslant 3$, $f(s) > 0$ *for* $s > 0$, $f(0) = 0$ *and* f *is subcritical w.r.t. the Sobolev critical exponent in* $\mathbb{R}^{N-1}$,
(c) $N \geqslant 3$, $f(s) > 0$ *for* $s > 0$, $f(0) = 0$ *and* $f(s) \geqslant \lambda s^{\frac{(N-1)(p-1)}{N-1-p}}$ *in* $[0, \delta]$, *for some* $\lambda, \delta > 0$.

Then $u \equiv 0$.

On the other hand, if $N \geqslant 2$, $f(s) > 0$ *for* $s \geqslant 0$, *then there are no non-negative solutions of* (1.3).

We refer the readers to [14,21,22,42] for other Liouville type theorems for quasilinear elliptic equations in half-spaces.

The proofs of the monotonicity results in half-spaces are generally based on weak comparison principles in narrow unbounded domains. We refer the readers to [3–5,9,15,16,20,25,36].

In our case, the presence of the therm $|\nabla u|^{p-2}$ gives rise to a phenomenon that was first pointed out in [8,11], in the case of bounded domains. Namely, we will prove our monotonicity result via a weak comparison principle in domains that can be decomposed into two parts. A narrow part (w.r.t. the Lebesgue measure of the section) and a part where the gradient of the solution is small.

We will state our comparison principle, which is the second main contribution of the present work, under more general structural assumptions and for more general quasilinear problems (cf. (1.5) below). We also remark that no restriction on the sign of u and v is required. More precisely, we consider continuous functions $a = a(x, u)$, $b = b(x, u)$ and $f = f(x, u)$ defined on $\mathbb{R}^N_+ \times \mathbb{R}$ and satisfying the following conditions:

(H_4) a, b and f are locally Lipschitz continuous functions, uniformly w.r.t. x. Namely, for every $M > 0$, there are positive constants $L_a = L_a(M)$, $L_b = L_b(M)$ and $L_f = L_f(M)$ such that for every $x \in \mathbb{R}^N_+$ and every $u, v \in [-M, M]$ we have

$$\bigl|a(x,u) - a(x,v)\bigr| \leqslant L_a|u - v|, \qquad \bigl|b(x,u) - b(x,v)\bigr| \leqslant L_b|u - v|,$$
$$\bigl|f(x,u) - f(x,v)\bigr| \leqslant L_f|u - v|.$$

For every $M > 0$ there is a constant $K = K(M) > 0$ such that for every $x \in \mathbb{R}^N_+$ and every $s \in [-M, M]$ we have

$$\bigl|a(x,s)\bigr| \leqslant K, \qquad \bigl|b(x,s)\bigr| \leqslant K;$$

(H_5) There is a constant $\gamma > 0$ such that $a(x, s) \geqslant \gamma$ for every $(x, s) \in \mathbb{R}^N_+ \times \mathbb{R}$.

Remark 1.5. (i) When the functions a, b and f depend only on the variable u and are locally Lipschitz continuous on $\mathbb{R}$, assumption (H_4) is automatically satisfied.

(ii) Typical examples of functions $a = a(x, u)$ satisfying both (H_4) and (H_5) are provided by $a(x, u) = (\sin^2(x_1) + 10)(|u| + 1)$ or $a(x, u) = |u| + 2 + \cos^2(|x|)$.

We have the following:

Theorem 1.6. *Let* $1 < p < 2$, $N \geqslant 2$ *and let* (H_1), (H_4), (H_5) *be satisfied. Fix* $\lambda_0 > 0$ *and* $M_0 > 0$. *Consider* $\lambda \in (0, \lambda_0]$, $\tau, \varepsilon > 0$ *and set*

$$\Sigma_{(\lambda, y_0)} := \mathbb{R}^{N-1} \times \left(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\right), \quad y_0 \geqslant \frac{\lambda}{2}. \tag{1.4}$$

Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma_{(\lambda,y_0)}})$ *such that* $\|u\|_\infty + \|\nabla u\|_\infty \leqslant M_0$, $\|v\|_\infty + \|\nabla v\|_\infty \leqslant M_0$ *and*

$$\begin{cases} -\operatorname{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big) + b(x,u)|\nabla u|^q \leqslant f(x,u), & \text{in } \Sigma_{(\lambda,y_0)}, \\ -\operatorname{div}\big(a(x,v)|\nabla v|^{p-2}\nabla v\big) + b(x,v)|\nabla v|^q \geqslant f(x,v), & \text{in } \Sigma_{(\lambda,y_0)}, \\ u \leqslant v, & \text{on } \partial\mathcal{S}_{(\tau,\varepsilon)}, \end{cases} \tag{1.5}$$

where the open set $\mathcal{S}_{(\tau,\varepsilon)} \subseteq \Sigma_{(\lambda,y_0)}$ *is such that*

$$\mathcal{S}_{(\tau,\varepsilon)} = \bigcup_{x' \in \mathbb{R}^{N-1}} I^{\tau,\varepsilon}_{x'},$$

and the open set $I^{\tau,\varepsilon}_{x'} \subseteq \{x'\} \times (y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2})$ *has the form*

$$I^{\tau,\varepsilon}_{x'} = A^\tau_{x'} \cup B^\varepsilon_{x'} \quad \text{with } \big|A^\tau_{x'} \cap B^\varepsilon_{x'}\big| = 0 \tag{1.6}$$

and, for x' *fixed,* $A^\tau_{x'}, B^\varepsilon_{x'} \subset (y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2})$ *are measurable sets such that*

$$\big|A^\tau_{x'}\big| \leqslant \tau \quad \text{and} \quad B^\varepsilon_{x'} \subseteq \big\{y \in \mathbb{R}: \big|\nabla u\big(x', y\big)\big| < \varepsilon, \ \big|\nabla v\big(x', y\big)\big| < \varepsilon\big\}.$$

Then there exist

$$\tau_0 = \tau_0(N, p, q, \lambda_0, M_0, \gamma) > 0$$

and

$$\varepsilon_0 = \varepsilon_0(N, p, q, \lambda_0, M_0, \gamma) > 0$$

such that, if $0 < \tau < \tau_0$ *and* $0 < \varepsilon < \varepsilon_0$, *it follows that*

$$u \leqslant v \quad \text{in } \mathcal{S}_{(\tau,\varepsilon)}.$$

If the functions f, a *and* b *are assumed to be* globally *Lipschitz continuous on* $\mathbb{R}^N_+ \times \mathbb{R}$ *and the functions* a *and* b *are supposed to be* bounded *on* $\mathbb{R}^N_+ \times \mathbb{R}$, *the same conclusion holds true without any assumption on the boundedness of* u *and* v.

Note that, both ε_0 and τ_0 in Theorem 1.6 can be explicitly calculated, cf. Remark 2.3. For later purposes, we also state the following special case of the previous theorem. It corresponds to the case in which $B^\varepsilon_{x'} \equiv \emptyset$ and the set $\mathcal{S}_{(\tau,\varepsilon)}$ is contained in a narrow strip. This result also provides an extension of Theorem 1.1 in [21] to the case of problems involving a first-order term as in (1.1) or in (1.5).

Theorem 1.7. *Let* $1 < p < 2$, $N \geqslant 2$ *and let* (H_1), (H_4), (H_5) *be satisfied. Fix* $M_0 > 0$. *Consider* $\lambda > 0$ *and set*

$$\Sigma_{(\lambda,y_0)} := \mathbb{R}^{N-1} \times \left(y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}\right), \quad y_0 \geqslant \frac{\lambda}{2}.$$

Let $u, v \in C^{1,\alpha}_{loc}(\overline{\Sigma_{(\lambda,y_0)}})$ *such that* $\|u\|_\infty + \|\nabla u\|_\infty \leqslant M_0$, $\|v\|_\infty + \|\nabla v\|_\infty \leqslant M_0$ *and*

$$\begin{cases} -\operatorname{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big) + b(x,u)|\nabla u|^q \leqslant f(x,u), & \text{in } \Sigma_{(\lambda,y_0)}, \\ -\operatorname{div}\big(a(x,v)|\nabla v|^{p-2}\nabla v\big) + b(x,v)|\nabla v|^q \geqslant f(x,v), & \text{in } \Sigma_{(\lambda,y_0)}, \\ u \leqslant v, & \text{on } \partial\mathcal{S}, \end{cases} \tag{1.7}$$

where $\mathcal{S} \subseteq \Sigma_{(\lambda,y_0)}$ *is an open subset.*

Then there exists

$$\bar\lambda = \bar\lambda(N, p, q, M_0, \gamma) > 0$$

such that, if $0 < \lambda < \bar\lambda$, *it follows that*

$$u \leqslant v \quad \text{in } \mathcal{S}.$$

If the functions f, a *and* b *are assumed to be* globally *Lipschitz continuous on* $\mathbb{R}^N_+ \times \mathbb{R}$ *and the functions* a *and* b *are supposed to be* bounded *on* $\mathbb{R}^N_+ \times \mathbb{R}$, *the same conclusion holds true without any assumption on the boundedness of* u *and* v.

We point out that $\bar{\lambda}$ in the above theorem, can be explicitly calculated, cf. Remark 2.3.

Remark 1.8. Theorem 1.6 and Theorem 1.7 are the main ingredients in the proof of our monotonicity result Theorem 1.1. The new geometry of the domain that we consider is crucial to prove our monotonicity result in the case $1 < p < 2$, without the restriction $p > \frac{2N+2}{N+2}$ that appears in [21]. At each x' fixed, the section of the domain is decomposed into two parts: a narrow (w.r.t. the Lebesgue measure) part and a part where the gradients are small.

The qualitative properties of positive solutions to $-\Delta_p u = f(u)$, in various unbounded domains, are also studied in [10,14,17,19,21–24,26,33,37,39,42].

Here below we describe the scheme of the paper:

(i) In Section 2 we prove the general weak comparison principle stated in Theorem 1.6 (and also Theorem 1.7). The proof is carried out exploiting the iteration scheme introduced in [21] (see also [22]), and taking advantage from the geometry of the considered domain.
(ii) In Section 3 we prove a crucial property of local symmetry regions of the solutions. Namely we show that such regions must touch the boundary. This follows by a fine analysis of the limiting profiles of the solution.
(iii) In Section 4 we prove Proposition 4.1, that allows to carry out the moving plane procedure via Theorem 1.6.
(iv) In Section 5 we prove Theorem 1.1.
(v) In Section 6 we prove Theorem 1.3 and Theorem 1.4.

2. The weak comparison principle: Proof of Theorem 1.6

In the sequel we will use the following inequalities: $\forall \eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ there exist positive constants C_1, C_2 depending only on p such that

$$\begin{aligned} &\left[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'\right]\left[\eta - \eta'\right] \geqslant C_1\left(|\eta| + |\eta'|\right)^{p-2}|\eta - \eta'|^2 \quad \text{if } p > 1, \\ &\left||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'\right| \leqslant C_2|\eta - \eta'|^{p-1} \quad \text{if } 1 < p \leqslant 2. \end{aligned} \tag{2.1}$$

Notation. Generic fixed numerical constants will be denoted by C (with subscript in some case) and they will be allowed to vary within a single line or formula. $|S|$ denotes the Lebesgue measure of a set S. f^+ and f^- are the positive part and the negative part of a function f, i.e. $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

We start recalling a lemma, whose proof can be found in [21, Lemma 2.1].

Lemma 2.1. *Let $\theta > 0$ and $\beta > 0$ such that $\theta < 2^{-\beta}$. Moreover let $R_0 > 0$, $c > 0$ and*

$$\mathcal{L} : (R_0, +\infty) \to \mathbb{R}$$

be a non-negative and non-decreasing function such that

$$\begin{cases} \mathcal{L}(R) \leqslant \theta\mathcal{L}(2R) + g(R) & \forall R > R_0, \\ \mathcal{L}(R) \leqslant CR^\beta & \forall R > R_0, \end{cases} \tag{2.2}$$

where $g : (R_0, +\infty) \to \mathbb{R}^+$ is such that $\lim_{R\to+\infty} g(R) = 0$. Then

$$\mathcal{L}(R) = 0.$$

We provide now the proof of a generalized version of the Poincaré inequality in one dimension.

Lemma 2.2 *(Poincaré type inequality). Let I be an open bounded subset of $\mathbb{R}$ and assume that $I = A \cup B$ with $|A \cap B| = 0$, A and B measurable subsets of I. Let $\rho : I \to \mathbb{R} \cup \{\infty\}$ be measurable and such that*

$$\inf_{t\in I} \rho(t) > 0.$$

Then for any $w \in H_0^1(I)$ such that $\int_I \rho(t)|\partial_t w|^2(t)\,dt$ is finite, the following inequality holds:

$$\int_I w^2(t)\,dt \leqslant 2|I| \max\left\{|A| \sup_{t\in A} \frac{1}{\rho(t)}, |B| \sup_{t\in B} \frac{1}{\rho(t)}\right\} \int_I \rho(t)|\partial_t w|^2(t)\,dt. \tag{2.3}$$

Proof. Since w belongs to $H_0^1(I)$, there exists $a \in \bar{I}$ such that $w(x) = \int_a^x \partial_t w(t)\,dt$. Thus we have

$$\begin{aligned}
\left|w(x)\right| &\leqslant \int_a^x \left|\partial_t w(t)\right| dt \leqslant \int_I \left|\partial_t w(t)\right| dt = \int_A \left|\partial_t w(t)\right| dt + \int_B \left|\partial_t w(t)\right| dt \\
&\leqslant |A|^{1/2}\left(\int_A |\partial_t w|^2(t)\,dt\right)^{1/2} + |B|^{1/2}\left(\int_B |\partial_t w|^2(t)\,dt\right)^{1/2} \\
&\leqslant \left(|A| \sup_{t\in A} \frac{1}{\rho(t)}\right)^{1/2}\left(\int_A \rho(t)|\partial_t w|^2(t)\,dt\right)^{1/2} + \left(|B| \sup_{t\in B} \frac{1}{\rho(t)}\right)^{1/2}\left(\int_B \rho(t)|\partial_t w|^2(t)\,dt\right)^{1/2}.
\end{aligned} \tag{2.4}$$

Finally, by using (2.4) we obtain

$$\begin{aligned}
\int_I w^2(t)\,dt &\leqslant |I| \sup_{t\in I} w^2(t) \\
&\leqslant 2|I|\left(|A| \sup_{t\in A} \frac{1}{\rho(t)} \int_A \rho|\partial_t w|^2(t)\,dt + |B| \sup_{t\in B} \frac{1}{\rho(t)} \int_B \rho|\partial_t w|^2(t)\,dt\right),
\end{aligned}$$

from which the thesis immediately follows. □

Proof of Theorem 1.6. In the proof, the quantities $L_a(M_0)$, $L_b(M_0)$, $L_f(M_0)$ and $K(M_0)$ will denote the structural constants appearing in assumption (H_4). Also we denote by $\|\cdot\|_\infty$, the L^∞ norm in $\Sigma_{(\lambda,y_0)}$.

We remark that $(u-v)^+$ belongs to $L^\infty(\Sigma_{(\lambda,y_0)})$ since u and v are bounded in $\Sigma_{(\lambda,y_0)}$. For $\alpha > 1$ we define

$$\psi = \left[(u-v)^+\right]^\alpha \varphi^2, \tag{2.5}$$

where $\varphi(x', y) = \varphi(x') \in C_c^\infty(\mathbb{R}^{N-1})$ is such that

$$\begin{cases}
\varphi \geqslant 0, & \text{in } \mathbb{R}^N_+, \\
\varphi \equiv 1, & \text{in } B'(0,R) \subset \mathbb{R}^{N-1}, \\
\varphi \equiv 0, & \text{in } \mathbb{R}^{N-1} \setminus B'(0,2R), \\
|\nabla\varphi| \leqslant \dfrac{C}{R}, & \text{in } B'(0,2R) \setminus B'(0,R) \subset \mathbb{R}^{N-1},
\end{cases} \tag{2.6}$$

where $B'(0,R) = \{x' \in \mathbb{R}^{N-1}\colon |x'| < R\}$, $R > 1$ and C is a positive constant.

Let $\mathcal{C}(R)$ be defined as

$$\mathcal{C}(R) := \left\{\mathcal{S}_{(\tau,\varepsilon)} \cap \left\{B'(0,R) \times \mathbb{R}\right\}\right\}.$$

The assumptions in (2.6) and the inequality $u \leqslant v$ on $\partial\mathcal{S}_{(\tau,\varepsilon)}$ imply that $\psi \in W_0^{1,p}(\mathcal{C}(2R))$. This allows us to use ψ as a test function in both equations of problem (1.5) and to get (by subtracting)

$$\begin{aligned}
&\int_{\mathcal{C}(2R)} \left(a(x,u)|\nabla u|^{p-2}\nabla u - a(x,v)|\nabla v|^{p-2}\nabla v, \nabla\psi\right) + \int_{\mathcal{C}(2R)} \left(b(x,u)|\nabla u|^q - b(x,v)|\nabla v|^q\right)\psi \\
&\quad\leqslant \int_{\mathcal{C}(2R)} \left(f(x,u) - f(x,v)\right)\psi,
\end{aligned} \tag{2.7}$$

from which we infer

$$\int_{\mathcal{C}(2R)} a(x,u)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\psi\big) + \int_{\mathcal{C}(2R)} \big(a(x,u)-a(x,v)\big)\big(|\nabla v|^{p-2}\nabla v, \nabla\psi\big)$$
$$+ \int_{\mathcal{C}(2R)} b(x,u)\big(|\nabla u|^q - |\nabla v|^q\big)\psi + \int_{\mathcal{C}(2R)} \big(b(x,u)-b(x,v)\big)|\nabla v|^q\psi$$
$$\leqslant \int_{\mathcal{C}(2R)} \big(f(x,u)-f(x,v)\big)\psi \tag{2.8}$$

and hence

$$\int_{\mathcal{C}(2R)} a(x,u)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\psi\big)$$
$$\leqslant \int_{\mathcal{C}(2R)} \big|a(x,u)-a(x,v)\big||\nabla v|^{p-1}|\nabla\psi| + K(M_0)\int_{\mathcal{C}(2R)} \big||\nabla u|^q - |\nabla v|^q\big|\psi$$
$$+ \int_{\mathcal{C}(2R)} \big|b(x,u)-b(x,v)\big||\nabla v|^q\psi + \int_{\mathcal{C}(2R)} \big|f(x,u)-f(x,v)\big|\psi$$
$$\leqslant L_a(M_0)\int_{\mathcal{C}(2R)} |\nabla v|^{p-1}|\nabla\psi||u-v| + K(M_0)\int_{\mathcal{C}(2R)} \big||\nabla u|^q - |\nabla v|^q\big|\psi$$
$$+ L_b(M_0)\int_{\mathcal{C}(2R)} |\nabla v|^q\psi|u-v| + L_f(M_0)\int_{\mathcal{C}(2R)} \psi|u-v|. \tag{2.9}$$

Since ∇u and ∇v belongs to $L^\infty(\Sigma_{(\lambda,y_0)})$, using (2.5) we obtain

$$\alpha\int_{\mathcal{C}(2R)} a(x,u)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+\big)\big[(u-v)^+\big]^{\alpha-1}\varphi^2$$
$$+ \int_{\mathcal{C}(2R)} a(x,u)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla\varphi^2\big)\big[(u-v)^+\big]^{\alpha}$$
$$\leqslant \alpha L_a(M_0)\|\nabla v\|_\infty^{p-1}\int_{\mathcal{C}(2R)} \big[(u-v)^+\big]^{\alpha}\big|\nabla(u-v)^+\big|\varphi^2 + L_a(M_0)\|\nabla v\|_\infty^{p-1}\int_{\mathcal{C}(2R)} \big[(u-v)^+\big]^{\alpha+1}\big|\nabla\varphi^2\big|$$
$$+ \big(L_b(M_0)\|\nabla v\|_\infty^q + L_f(M_0)\big)\int_{\mathcal{C}(2R)} \big[(u-v)^+\big]^{\alpha+1}\varphi^2$$
$$+ qK(M_0){M_0}^{q-1}\int_{\mathcal{C}(2R)} \big|\nabla(u-v)^+\big|\big[(u-v)^+\big]^{\alpha}\varphi^2, \tag{2.10}$$

where, in the last term we used the mean value theorem and the boundedness of ∇u and ∇v to deduce that $||\nabla u|^q - |\nabla v|^q| \leqslant q{M_0}^{q-1}|\nabla(u-v)|$.

Recalling (2.1), from (2.10) we obtain

$$\alpha C_1\gamma\int_{\mathcal{C}(2R)} \big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}\varphi^2$$
$$\leqslant C_2K(M_0)\int_{\mathcal{C}(2R)} \big|\nabla(u-v)^+\big|^{p-1}\big|\nabla\varphi^2\big|\big[(u-v)^+\big]^{\alpha}$$

$$+\alpha L_a(M_0)\|\nabla v\|_\infty^{p-1}\int\limits_{\mathcal{C}(2R)}\left|\nabla(u-v)^+\right|\left[(u-v)^+\right]^{\frac{\alpha-1+\alpha+1}{2}}\varphi^2$$

$$+L_a(M_0)\|\nabla v\|_\infty^{p-1}\int\limits_{\mathcal{C}(2R)}\left[(u-v)^+\right]^{\alpha+1}\left|\nabla\varphi^2\right|$$

$$+\left(L_b(M_0)\|\nabla v\|_\infty^q+L_f(M_0)\right)\int\limits_{\mathcal{C}(2R)}\left[(u-v)^+\right]^{\alpha+1}\varphi^2$$

$$+qK(M_0){M_0}^{q-1}\int\limits_{\mathcal{C}(2R)}\left|\nabla(u-v)^+\right|\left[(u-v)^+\right]^{\frac{\alpha-1+\alpha+1}{2}}\varphi^2.$$

It is easy to resume the computations above as follows:

$$\alpha C_1\gamma\int\limits_{\mathcal{C}(2R)}\left(|\nabla u|+|\nabla v|\right)^{p-2}\left|\nabla(u-v)^+\right|^2\left[(u-v)^+\right]^{\alpha-1}\varphi^2$$
$$\leqslant\left(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0)\right)\int\limits_{\mathcal{C}(2R)}\left|\nabla\varphi^2\right|\left[(u-v)^+\right]^\alpha$$
$$+\alpha\cdot\left(L_a(M_0)M_0^{p-1}+qK(M_0){M_0}^{q-1}\right)\int\limits_{\mathcal{C}(2R)}\left|\nabla(u-v)^+\right|\left[(u-v)^+\right]^{\frac{\alpha-1+\alpha+1}{2}}\varphi^2$$
$$+\left(L_b(M_0)M_0^q+L_f(M_0)\right)\int\limits_{\mathcal{C}(2R)}\left[(u-v)^+\right]^{\alpha+1}\varphi^2. \tag{2.11}$$

Recalling that $p<2$ and using the weighted Young inequality $zy\leqslant\varepsilon z^2+\frac{y^2}{4\varepsilon}$, from

$$\int\limits_{\mathcal{C}(2R)}\left|\nabla(u-v)^+\right|\left[(u-v)^+\right]^{\frac{\alpha-1+\alpha+1}{2}}\varphi^2=\int\limits_{\mathcal{C}(2R)\cap\{|\nabla u|+|\nabla v|>0\}}\left|\nabla(u-v)^+\right|\left[(u-v)^+\right]^{\frac{\alpha-1}{2}}\varphi\cdot\left[(u-v)^+\right]^{\frac{\alpha+1}{2}}\varphi$$

we deduce that, for every $\delta>0$ it holds

$$\alpha C_1\gamma\int\limits_{\mathcal{C}(2R)}\left(|\nabla u|+|\nabla v|\right)^{p-2}\left|\nabla(u-v)^+\right|^2\left[(u-v)^+\right]^{\alpha-1}\varphi^2$$
$$\leqslant\left(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0)\right)\int\limits_{\mathcal{C}(2R)}\left|\nabla\varphi^2\right|\left[(u-v)^+\right]^\alpha$$
$$+\delta\cdot\alpha\cdot\left(L_a(M_0)M_0^{p-1}+qK(M_0){M_0}^{q-1}\right)\int\limits_{\mathcal{C}(2R)}\left|\nabla(u-v)^+\right|^2\left[(u-v)^+\right]^{\alpha-1}\varphi^2$$
$$+\left(\alpha\frac{(L_a(M_0)M_0^{p-1}+qK(M_0){M_0}^{q-1})}{4\delta}+\left(L_b(M_0)M_0^q+L_f(M_0)\right)\right)\int\limits_{\mathcal{C}(2R)}\left[(u-v)^+\right]^{\alpha+1}\varphi^2$$
$$\leqslant\left(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0)\right)\int\limits_{\mathcal{C}(2R)}\left|\nabla\varphi^2\right|\left[(u-v)^+\right]^\alpha$$

$$+\delta\alpha 2^{2-p}\big(L_a(M_0)M_0+qK(M_0){M_0}^{q-(p-1)}\big)\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}\varphi^2$$
$$+\bigg(\alpha\frac{2^{2-p}\cdot(L_a(M_0)M_0+qK(M_0){M_0}^{q-(p-1)})}{4\delta}+\big(L_b(M_0)M_0^q+L_f(M_0)\big)\bigg)\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2. \tag{2.12}$$

Here we are using the fact that $q>1>p-1$, since (H_1) holds.

Take now

$$\delta:=\frac{C_1\gamma}{2\cdot(2^{2-p}L_a(M_0)M_0+2^{2-p}qK(M_0){M_0}^{q-(p-1)})}.$$

Consequently we have

$$\alpha\frac{C_1\gamma}{2}\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}\varphi^2$$
$$\leqslant\big(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0)\big)\int\limits_{\mathcal{C}(2R)}\big|\nabla\varphi^2\big|\big[(u-v)^+\big]^{\alpha}$$
$$+\alpha\frac{(2^{2-p}L_a(M_0)M_0+2^{2-p}qK(M_0){M_0}^{q-(p-1)})^2}{2C_1\gamma}\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2$$
$$+\big(L_b(M_0)M_0^q+L_f(M_0)\big)\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2. \tag{2.13}$$

Since $\alpha>1$ we immediately get that

$$\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}\varphi^2$$
$$\leqslant\frac{2(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0))}{C_1\gamma}\int\limits_{\mathcal{C}(2R)}\big|\nabla\varphi^2\big|\big[(u-v)^+\big]^{\alpha}$$
$$+\frac{(2^{2-p}L_a(M_0)M_0+2^{2-p}qK(M_0){M_0}^{q-(p-1)})^2}{(C_1\gamma)^2}\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2$$
$$+\frac{2(L_b(M_0)M_0^q+L_f(M_0))}{C_1\gamma}\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2. \tag{2.14}$$

Let us define

$$c_1=\frac{2(C_2K(M_0)(2M_0)^{p-1}+2M_0^pL_a(M_0))}{C_1\gamma}, \tag{2.15}$$

$$c_2=\frac{(2^{2-p}L_a(M_0)M_0+2^{2-p}qK(M_0){M_0}^{q-(p-1)})^2}{(C_1\gamma)^2}+\frac{2(L_b(M_0)M_0^q+L_f(M_0))}{C_1\gamma}, \tag{2.16}$$

$$I_1=\int\limits_{\mathcal{C}(2R)}\big|\nabla\varphi^2\big|\big[(u-v)^+\big]^{\alpha},\qquad I_2=\int\limits_{\mathcal{C}(2R)}\big[(u-v)^+\big]^{\alpha+1}\varphi^2$$

and note that both c_1 and c_2 depend only on p, q, γ and M_0, in particular they are independent of $\alpha > 1$. Thus, with the definitions above, we now rewrite (2.14) as follows: for every $\alpha > 1$,

$$\int\limits_{\mathcal{C}(2R)} \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla (u-v)^+\right|^2 \left[(u-v)^+\right]^{\alpha-1} \varphi^2 \leqslant c_1 I_1 + c_2 I_2. \tag{2.17}$$

We also observe that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{N-1}} \left(\int\limits_{I^{\tau,\varepsilon}_{x'}} \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla (u-v)^+\right|^2 \left[(u-v)^+\right]^{\alpha-1} \right) dy\, \varphi^2\left(x'\right) dx' \\
&\quad = \int\limits_{\mathcal{C}(2R)} \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla (u-v)^+\right|^2 \left[(u-v)^+\right]^{\alpha-1} \varphi^2 < +\infty
\end{aligned}$$

since φ depends only on x' and the right-hand side of (2.17) is finite. Hence, for almost every $x' \in \mathbb{R}^{N-1}$ we have that

$$\int\limits_{I^{\tau,\varepsilon}_{x'}} \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla (u-v)^+\right|^2 \left[(u-v)^+\right]^{\alpha-1} dy < +\infty, \tag{2.18}$$

which also entails: for almost every $x' \in \mathbb{R}^{N-1}$

$$\int\limits_{I^{\tau,\varepsilon}_{x'}} \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\partial_y (u-v)^+\right|^2 \left[(u-v)^+\right]^{\alpha-1} dy < +\infty. \tag{2.19}$$

$\hookrightarrow$ *Evaluation of the term* I_1. Let us recall the decomposition stated in (1.6) which gives

$$\mathcal{S}_{(\tau,\varepsilon)} = \bigcup_{x' \in \mathbb{R}^{n-1}} I^{\tau,\varepsilon}_{x'} \quad \text{with } I^{\tau,\varepsilon}_{x'} = A^{\tau}_{x'} \cup B^{\varepsilon}_{x'}.$$

We set

$$\rho_{x'}(t) = \left(\left|\nabla u\left(x',t\right)\right| + \left|\nabla v\left(x',t\right)\right|\right)^{p-2}$$

in order to apply Lemma 2.2 in each $I^{\tau,\varepsilon}_{x'}$, for which (2.19) holds true, with $\rho(t) := \rho_{x'}(t)$, $A := A^{\tau}_{x'}$, $B := B^{\varepsilon}_{x'}$ and $w(t) = [(u-v)^+(x',t)]^{\frac{\alpha+1}{2}}$. Note that the constant in (2.3) in this case is given by

$$C_{\tau,\varepsilon}\left(x'\right) = 2\lambda \max\left\{ \left|A^{\tau}_{x'}\right| \sup_{t \in A^{\tau}_{x'}} \frac{1}{\rho_{x'}(t)}, \left|B^{\varepsilon}_{x'}\right| \sup_{t \in B^{\varepsilon}_{x'}} \frac{1}{\rho_{x'}(t)} \right\}.$$

Therefore, for almost every $x' \in \mathbb{R}^{N-1}$, we have

$$C_{\tau,\varepsilon}\left(x'\right) \leqslant C_{\tau,\varepsilon} := 2\lambda_0 \max\left\{\tau (2M_0)^{2-p}, \lambda_0 (2\varepsilon)^{2-p}\right\}, \tag{2.20}$$

so that, since $1 < p < 2$, $C_{\tau,\varepsilon}$ can be chosen arbitrary small, for τ and ε sufficiently small. Now, recalling that φ depends only on x' and using Young inequality with conjugate exponents $\frac{\alpha+1}{\alpha}$ and $\alpha+1$, we get

$$\begin{aligned}
I_1 &\leqslant 2 \int\limits_{\mathcal{C}(2R)} \left[(u-v)^+\right]^{\alpha} \varphi |\nabla\varphi| = 2 \int\limits_{\mathcal{C}(2R)} \left[(u-v)^+\right]^{\alpha} \varphi |\nabla\varphi|^{\frac{1}{2}} |\nabla\varphi|^{\frac{1}{2}} \\
&\leqslant 2 \int\limits_{\mathcal{C}(2R)} \frac{[(u-v)^+]^{\alpha+1} \varphi^{\frac{\alpha+1}{\alpha}} |\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}}{\frac{\alpha+1}{\alpha}} + 2 \int\limits_{\mathcal{C}(2R)} \frac{|\nabla\varphi|^{\frac{\alpha+1}{2}}}{\alpha+1} \\
&\leqslant 2 \int\limits_{\mathbb{R}^{N-1}} \left(\int\limits_{I^{\tau,\varepsilon}_{x'}} \left(\left[(u-v)^+\right]^{\frac{\alpha+1}{2}}\right)^2 dy \right) \varphi^{\frac{\alpha+1}{\alpha}} |\nabla\varphi|^{\frac{\alpha+1}{2\alpha}} dx' + 2 \int\limits_{\mathcal{C}(2R)} |\nabla\varphi|^{\frac{\alpha+1}{2}}
\end{aligned}$$

and the application of Lemma 2.2 yields

$$\begin{aligned} I_1 &\leqslant C_{\tau,\varepsilon}\frac{(\alpha+1)^2}{2}\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big[(u-v)^+\big]^{\alpha-1}\big|\partial_y(u-v)^+\big|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}+2\int\limits_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}} \\ &\leqslant C_{\tau,\varepsilon}\frac{(\alpha+1)^2}{2}\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big[(u-v)^+\big]^{\alpha-1}\big|\nabla(u-v)^+\big|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}+2\int\limits_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}, \end{aligned} \tag{2.21}$$

where $C_{\tau,\varepsilon}$ has been defined in (2.20).

$\hookrightarrow$ *Evaluation of the term* I_2. We use the same notations as in the evaluation of I_1 and we get

$$\begin{aligned} I_2 &= \int\limits_{\mathcal{C}(2R)}\Big(\big[(u-v)^+\big]^{\frac{\alpha+1}{2}}\Big)^2\varphi^2=\int\limits_{\mathbb{R}^{N-1}}\bigg(\int\limits_{I^{\tau,\varepsilon}_{x'}}\Big(\big[(u-v)^+\big]^{\frac{\alpha+1}{2}}\Big)^2\,dy\bigg)\big(\varphi(x')\big)^2\,dx' \\ &\leqslant C_{\tau,\varepsilon}\Big(\frac{\alpha+1}{2}\Big)^2\int\limits_{\mathbb{R}^{N-1}}\bigg(\int\limits_{I^{\tau,\varepsilon}_{x'}}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big[(u-v)^+\big]^{\alpha-1}\big|\partial_y(u-v)^+\big|^2\,dy\bigg)\big(\varphi(x')\big)^2\,dx' \\ &\leqslant C_{\tau,\varepsilon}\Big(\frac{\alpha+1}{2}\Big)^2\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big[(u-v)^+\big]^{\alpha-1}\big|\nabla(u-v)^+\big|^2\varphi^2. \end{aligned} \tag{2.22}$$

Let us fix

$$\alpha=2N+1>1. \tag{2.23}$$

Recalling that $C_{\tau,\varepsilon}$ tends to 0, as both τ and ε go to zero, we can take $\tau>0$ and $\varepsilon>0$ small enough, such that

$$c_2C_{\tau,\varepsilon}\Big(\frac{\alpha+1}{2}\Big)^2<\frac{1}{2},\qquad c_1C_{\tau,\varepsilon}(\alpha+1)^2<2^{-N} \tag{2.24}$$

so that from (2.17) we have

$$\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}\varphi^2\leqslant 2c_1I_1. \tag{2.25}$$

From (2.6) we infer that

$$\begin{aligned} &\int\limits_{\mathcal{C}(R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2(u-v)^{\alpha-1} \\ &\quad\leqslant\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2(u-v)^{\alpha-1}\varphi^2\leqslant 2c_1I_1 \end{aligned} \tag{2.26}$$

and, using (2.21), we obtain

$$\begin{aligned} &\int\limits_{\mathcal{C}(R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1} \\ &\quad\leqslant c_1C_{\tau,\varepsilon}(\alpha+1)^2\int\limits_{\mathcal{C}(2R)}\big(|\nabla u|+|\nabla v|\big)^{p-2}\big[(u-v)^+\big]^{\alpha-1}\big|\nabla(u-v)^+\big|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}+4c_1\int\limits_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}. \end{aligned} \tag{2.27}$$

Recalling (2.23) one has

$$\int\limits_{\mathcal{C}(R)} \big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2(u-v)^{\alpha-1} \leqslant \theta \int\limits_{\mathcal{C}(2R)} \big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1}+\hat{C}R^{-2}, \tag{2.28}$$

where

$$\theta = c_1 C_{\tau,\varepsilon}(\alpha+1)^2,$$
$$\hat{C} = 4c_1\lambda C^{\frac{\alpha+1}{2}} > 0$$

exploiting also (2.6). Notice that, in view of (2.24), we also have that $\theta < 2^{-N}$. In order to apply Lemma 2.1 we set

$$\mathcal{L}(R) = \int\limits_{\mathcal{C}(R)} \big(|\nabla u|+|\nabla v|\big)^{p-2}\big|\nabla(u-v)^+\big|^2\big[(u-v)^+\big]^{\alpha-1},$$

and

$$g(R) = \hat{C}R^{-2}.$$

Then from (2.28) we have

$$\begin{cases} \mathcal{L}(R) \leqslant \theta\mathcal{L}(2R)+g(R) & \forall R>0,\\ \mathcal{L}(R)\leqslant CR^N & \forall R>0. \end{cases}$$

Applying Lemma 2.1 with $\beta = N$, we get $\mathcal{L}(R)=0$ and consequently the thesis. □

Proof of Theorem 1.7. The desired result is obtained with the same proof as that of Theorem 1.6 with the following slight (but necessary) modifications. Replace $\mathcal{S}_{(\tau,\varepsilon)}$ by $\mathcal{S}$, set $\varepsilon=\tau=\lambda$, $B^{\varepsilon}_{x'}=\emptyset$, $I^{\tau;\varepsilon}_{x'} = A^{\tau}_{x'} = \mathcal{S}\cap\{x'\}\times(y_0-\frac{\lambda}{2}, y_0+\frac{\lambda}{2})$ and observe that (2.20) becomes

$$C_\lambda\big(x'\big) \leqslant C_\lambda := 2\lambda^2(2M_0)^{2-p}, \tag{2.29}$$

and that (2.24) becomes

$$c_2C_\lambda\left(\frac{\alpha+1}{2}\right)^2 < \frac{1}{2}, \qquad c_1C_\lambda(\alpha+1)^2 < 2^{-N}. \tag{2.30}$$

The conclusion then follows by taking λ small enough in the latter one. □

Remark 2.3. In view of (2.15), (2.16), (2.20), (2.23) and of (2.24), it is possible to calculate explicitly the value of ε_0 and τ_0. The same can be done for $\bar{\lambda}$.

3. Properties of the local symmetry regions

Let us start this section recalling a useful change of variable. Under our assumptions on the function $a=a(s)$ set

$$A(s) := \int\limits_0^s a(t)^{\frac{1}{p-1}}\,dt.$$

It follows that A belongs to $C^{1,1}_{loc}([0,+\infty),[0,+\infty))$, is strictly increasing and satisfies $A(0)=0$ and $\lim_{t\to+\infty} A(t) = +\infty$. Set

$$w := A(u) \tag{3.1}$$

then $w \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ and

$$\begin{cases} -\Delta_p w + \tilde{b}(w)|\nabla w|^q = \tilde{f}(w), & \text{in } \mathbb{R}^N_+, \\ w(x', y) > 0, & \text{in } \mathbb{R}^N_+, \\ w(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{3.2}$$

with $\tilde{b}(w) := \frac{b(A^{-1}(w))}{(a(A^{-1}(w)))^{\frac{q}{p-1}}}$ and $\tilde{f}(w) := f(A^{-1}(w))$. It follows that $\tilde{b}$ and $\tilde{f}$ still are locally Lipschitz continuous (here we used the fact that $q > 1 > p - 1$, since (H_1) is in force). We notice that: $f(s) > 0$ for $s > 0$ if and only if $\tilde{f}(s) > 0$ for $s > 0$ and that $f(0) = \tilde{f}(0)$. This shows that, as we will recall later, it is not restrictive to our purposes to assume from now on that

$$a(s) = 1.$$

We also observe that using the mean value theorem in the y-direction, the Dirichlet condition on u and the fact that $\nabla u \in L^\infty(\mathbb{R}^N_+)$, we get that u is bounded on every set of the form $\{0 \leqslant y \leqslant \eta\}$, for any $\eta > 0$. This implies that also w and ∇w are bounded on every set of the form $\{0 \leqslant y \leqslant \eta\}$, for any $\eta > 0$ (the bound might depend on η).

With this in mind, now we are going to prove an important properties concerning the local symmetry regions of the solutions. Let us start with some notations.

We define the strip Σ_λ by

$$\Sigma_\lambda := \{0 < y < \lambda\}$$

and the reflected function $u_\lambda(x)$ by

$$u_\lambda(x) = u_\lambda(x', y) := u(x', 2\lambda - y) \quad \text{in } \Sigma_{2\lambda}.$$

As customary we also define the critical set Z_u by

$$Z_u = \{x \in \mathbb{R}^N_+ : \nabla u(x) = 0\}.$$

We have the following:

Theorem 3.1. *Let $1 < p < 2$ and let $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ be a solution to (1.1) with $a(s) = 1$. Let (H_1), (H_2) be satisfied and assume that $f(s) > 0$ for $s > 0$. Let us assume that both u and ∇u are bounded on every strip Σ_η, $\eta > 0$. Suppose furthermore that u is monotone non-decreasing in Σ_λ, for some $\lambda > 0$. Assume that $\mathcal{U}$ is a (not empty) connected component of $\Sigma_\lambda \setminus Z_u$ such that $u(x) \equiv u_\lambda(x)$ in $\mathcal{U}$, (i.e. a local symmetry region for u). Then necessarily $\mathcal{U}$ touches the boundary of $\mathbb{R}^N_+$, namely*

$$\partial\mathcal{U} \cap \{y = 0\} \neq \emptyset.$$

Proof. Let us start showing the following:

Claim 1. *There exists $\tau = \tau(\mathcal{U}, \lambda) > 0$ such that* $\operatorname{dist}(\mathcal{U}, \{y = 0\}) > \tau$.

By contradiction let us assume that there exists a sequence of points

$$x_n = (x'_n, y_n) \in \mathcal{U},$$

such that

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \{y = 0\}) = \lim_{n\to\infty} y_n = 0. \tag{3.3}$$

We consider the sequence

$$\hat{x}_n = \left(x'_n, \frac{\lambda}{2}\right)$$

and the two different cases

(a) $u(\hat{x}_n)$ is strictly bounded away from zero uniformly on n;
(b) up to subsequences $\lim_{n\to\infty} u(\hat{x}_n) = 0$.

Case (a). Define the sequence

$$u_n(x) = u\big(x' + x'_n, y\big). \tag{3.4}$$

Let $\mathcal{K} \subset \overline{\mathbb{R}^+_N}$ be a compact set. Since both u and ∇u are bounded on every strip Σ_η, $\eta > 0$, we get

$$0 \leqslant u_n(x) \leqslant C(\mathcal{K}), \qquad \big|\nabla u_n(x)\big| \leqslant C(\mathcal{K}) \quad \forall x \in \mathcal{K},$$

for some positive constant $C(\mathcal{K})$.

Therefore $C^{1,\alpha}$ estimates (see the classical results [18,31,40]), Ascoli's Theorem and a standard diagonal process imply that

$$u_n \xrightarrow{C^{1,\alpha'}_{loc}(\overline{\mathbb{R}^N_+})} u_\infty, \tag{3.5}$$

up to subsequences, for $\alpha' < \alpha$. Recalling that $u_n(0, \frac{\lambda}{2}) \geqslant \gamma_0 > 0$, uniform convergence implies that u_∞ is a non-trivial non-negative solution to the equation in (1.1) (with $a(s) = 1$). Actually, by the strong maximum principle [41], we have that

$$u_\infty(x) > 0 \quad \forall x \in \mathbb{R}^N_+. \tag{3.6}$$

By the definition of $\mathcal{U}$ (i.e. $u(x) \equiv u_\lambda(x)$ in $\mathcal{U}$), since by (3.3) (together with the Dirichlet condition and the mean value theorem) we have

$$u\big(x'_n, y_n\big) \leqslant \big\|\nabla u(x)\big\|_\infty \cdot y_n \to 0, \qquad \text{as } n \to +\infty.$$

Then it follows:

$$\lim_{n\to+\infty} u\big(x'_n, 2\lambda - y_n\big) = \lim_{n\to+\infty} u\big(x'_n, y_n\big) = 0. \tag{3.7}$$

From (3.7), using (3.4) and (3.5), we obtain

$$u_\infty(0, 2\lambda) = 0.$$

This is a contradiction with (3.6) and Claim 1 is proved in case (a).

Case (b). Arguing exactly as in the proof of the case (a) here above, it follows that necessarily $u_\infty \equiv 0$. This implies that case (b) occurs only if $f(0) = 0$ because if not, 0 cannot be a solution of our equation.

Recalling (3.4) we define

$$\bar{u}_n\big(x', y\big) \equiv \frac{u_n(x', y)}{u_n(0, \frac{\lambda}{2})} = \frac{u(x' + x'_n, y)}{u(x'_n, \frac{\lambda}{2})}, \tag{3.8}$$

so that

$$\bar{u}_n\left(0, \frac{\lambda}{2}\right) = 1,$$

and u_n uniformly converges to 0 on compact sets of $\overline{\mathbb{R}^N_+}$ by construction. Recalling that we are assuming that $a(s) = 1$, it is easily seen that

$$-\operatorname{div}\big(|\nabla \bar{u}_n|^{p-2}\nabla \bar{u}_n\big) + \left(u_n\left(0, \frac{\lambda}{2}\right)\right)^{q-(p-1)} \cdot b(u_n) \cdot |\nabla \bar{u}_n|^q = \frac{f(u_n)}{u_n^{p-1}} \cdot \bar{u}_n^{p-1},$$

that is

$$-\operatorname{div}\big(|\nabla \bar{u}_n|^{p-2}\nabla \bar{u}_n\big) + \hat{c}_n(x) \cdot |\nabla \bar{u}_n|^{p-1} = \tilde{c}_n(x) \cdot \bar{u}_n^{p-1}, \tag{3.9}$$

where

$$\hat{c}_n(x) = b(u_n)|\nabla u_n|^{q-(p-1)}, \qquad \tilde{c}_n(x) = \frac{f(u_n)}{u_n^{p-1}}.$$

The assumptions on $b(\cdot)$ and $f(\cdot)$ and the fact that $q > 1 > p-1$ imply that $\hat{c}_n$ and $\tilde{c}_n$ are bounded on every strip Σ_η, $\eta > 0$. Indeed, since b and f are locally Lipschitz continuous functions we have

$$\left|\hat{c}_n(x)\right| \leqslant \left\|b(u)\right\|_{\infty,\Sigma_\eta} \|\nabla u\|_{\infty,\Sigma_\eta}^{q-(p-1)}, \tag{3.10}$$

$$0 \leqslant \tilde{c}_n(x) = \frac{f(u_n)}{u_n^{p-1}} \leqslant L_f\big(\|u\|_{\infty,\Sigma_\eta}\big)\big|u_n(x)\big|^{2-p} \leqslant L_f\big(\|u\|_{\infty,\Sigma_\eta}\big)\|u\|_{\infty,\Sigma_\eta}^{2-p}, \tag{3.11}$$

where we used that $f(0) = 0$.

On the other hand we have that $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$, since $f(0) = 0$ as remarked above, $p - 1 < 1$ and f is locally Lipschitz continuous. The latter, together with the fact that both u_n and ∇u_n converge to zero uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ and $q > 1 > p-1$, immediately yields that also $\tilde{c}_n$ and $\hat{c}_n$ converge to zero uniformly on compact sets of $\overline{\mathbb{R}^N_+}$.

Fix $0 < \delta < \frac{\lambda}{2}$ and consider an arbitrary compact set $\mathcal{K}$ of $\overline{\mathbb{R}^N_+}$ containing the point $(0, \frac{\lambda}{2})$. By Harnack inequality (see [35, Theorem 7.2.2]), applied to Eq. (3.9) on the compact set $\mathcal{K} \cap \{y \geqslant \delta\} \subset\subset \mathbb{R}^N_+$, we get that

$$\sup_{\mathcal{K}\cap\{y\geqslant\delta\}} \bar{u}_n \leqslant C_H \inf_{\mathcal{K}\cap\{y\geqslant\delta\}} \bar{u}_n \leqslant C_H. \tag{3.12}$$

Moreover, by the monotonicity of u in Σ_λ we have

$$\sup_{\mathcal{K}\cap\{y\geqslant 0\}} \bar{u}_n \leqslant \sup_{\mathcal{K}\cap\{y\geqslant\delta\}} \bar{u}_n \leqslant C_H. \tag{3.13}$$

Hence we can use (once again) $C^{1,\alpha}$ estimates, Ascoli's Theorem and a diagonal argument to get, up to subsequences, that

$$\bar{u}_n \xrightarrow{C^{1,\alpha'}_{loc}(\overline{\mathbb{R}^N_+})} \bar{u}$$

for $\alpha' < \alpha$. Arguing as above, we see that $\bar{u} \geqslant 0$ in $\overline{\mathbb{R}^N_+}$ and $\bar{u}(0, \frac{\lambda}{2}) = 1$.

Taking into account the properties of $\hat{c}_n$ and of $\tilde{c}_n$, we can pass to the limit in (3.9) obtaining

$$-\Delta_p \bar{u} = 0 \quad \text{in } \mathbb{R}^N_+.$$

By the strong maximum principle [41], we therefore get that $\bar{u} > 0$ since $\bar{u}$ cannot be equal to zero because of the condition $\bar{u}(0, \frac{\lambda}{2}) = 1$. Actually, by construction, we have

$$\begin{cases} -\Delta_p \bar{u} = 0, & \text{in } \mathbb{R}^N_+, \\ \bar{u} > 0, & \text{in } \mathbb{R}^N_+, \\ \bar{u}\big(x', 0\big) = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{3.14}$$

By [29, Theorem 3.1] it follows that $\bar{u}$ is affine linear, that means

$$\bar{u}\big(x', y\big) = ky,$$

for some $k > 0$ by the Dirichlet assumption. This is a contradiction since by assumption u and consequently $\bar{u}$ have a local symmetry region and this concludes the proof of Claim 1.

We show the following:

Claim 2. *There exists $\gamma = \gamma(\mathcal{U}, \lambda) > 0$ such that $u \geqslant \gamma$ in $\mathcal{U}$.*

To show this, assume by contradiction that there exists a sequence of points

$$x_n = \big(x_n', y_n\big) \in \mathcal{U},$$

such that

$$\lim_{n\to\infty} u(x_n) = 0 \tag{3.15}$$

and with y_n converging (up to subsequences) to $y_0 > 0$.

Using definition (3.4) we set

$$\bar{u}_n(x', y) \equiv \frac{u(x' + x'_n, y)}{u(x'_n, y_n)},$$

so that $\bar{u}(0, y_n) = 1$ and u_n uniformly converges to 0 on compact sets of $\overline{\mathbb{R}^N_+}$ by construction (see (3.15)). As above (see (3.9)), it is easy to see that

$$-\operatorname{div}\big(|\nabla \bar{u}_n|^{p-2}\nabla \bar{u}_n\big) + \hat{c}_n(x)\cdot|\nabla \bar{u}_n|^{p-1} = \tilde{c}_n(x)\cdot \bar{u}_n^{p-1}, \tag{3.16}$$

with $\hat{c}_n$ and $\tilde{c}_n$ satisfy (3.10) and (3.11) and, both of them converges to zero uniformly on compact sets of $\overline{\mathbb{R}^N_+}$. Furthermore, we have that

$$\bar{u}_n \xrightarrow{C^{1,\alpha'}_{loc}(\overline{\mathbb{R}^N_+})} \bar{u}$$

up to subsequences, for $\alpha' < \alpha$. Then, arguing as above, we get that $\bar{u} \geqslant 0$ in $\overline{\mathbb{R}^N_+}$ and $\bar{u}(0, y_0) = 1$, with $y_0 > 0$. Moreover

$$-\Delta_p \bar{u} = 0 \quad \text{in } \mathbb{R}^N_+.$$

By the strong maximum principle (see [41]) we get that $\bar{u} > 0$, since $\bar{u}$ cannot be equal to 0 because of the condition $\bar{u}(0, y_0) = 1$. In fact, as in (3.14), by construction and using [29, Theorem 3.1], it follows that $\bar{u}$ is of the form:

$$\bar{u}(x', y) = ky, \tag{3.17}$$

for some $k > 0$.

Since by construction

$$\bar{u}(0, y_0) = \bar{u}_\lambda(0, y_0),$$

if $y_0 < \lambda$, we get a contradiction by (3.17) concluding the proof of Claim 2. If else $y_0 = \lambda$, it follows by construction that $\frac{\partial \bar{u}}{\partial y}(0, y_0) = 0$. This is deduced by observing that $\frac{\partial \bar{u}_n}{\partial y}$ vanishes somewhere on the segment from $(0, y_n)$ to $(0, 2\lambda - y_n)$ (since $(x'_n, y_n) \in \mathcal{U}$) and exploiting the uniform convergence of the gradients. Therefore again we get a contradiction by (3.17) concluding the proof of Claim 2.

Since $f(s) > 0$ for $s > 0$, Claim 2 implies that there exists $\gamma^+ > 0$ such that

$$f(u) \geqslant \gamma^+ \quad \text{in } \mathcal{U}. \tag{3.18}$$

Now we proceed in order to conclude the proof. Let $\varphi_R(x', y) = \varphi_R(x')$ with $\varphi(x') \in C^\infty_c(\mathbb{R}^{N-1})$ defined as in (2.6). For all $\varepsilon > 0$, let $G_\varepsilon : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be defined as

$$G_\varepsilon(t) = \begin{cases} t, & \text{if } t \geqslant 2\varepsilon, \\ 2t - 2\varepsilon, & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0, & \text{if } 0 \leqslant t \leqslant \varepsilon. \end{cases} \tag{3.19}$$

Let χ be the characteristic function of a set. We define

$$\Psi = \Psi_{\varepsilon,R} := e^{-s(u)}\varphi_R \frac{G_\varepsilon(|\nabla u|)}{|\nabla u|}\chi_{(\mathcal{U}\cup\mathcal{U}^\lambda)},$$

where $\mathcal{U}^\lambda$ is the reflected set of $\mathcal{U}$ w.r.t. the hyperplane $T_\lambda = \{y = \lambda\}$ and

$$s(t) = \hat{C}\cdot\int_0^t b^+(t')\,dt', \tag{3.20}$$

with $\hat{C}$ some positive constant to be chosen later. By the definition of $\mathcal{U}$ and taking into account the fact that $\mathcal{U}$ is a local symmetry region of u, we have that $\nabla u = 0$ on $\partial(\mathcal{U} \cup \mathcal{U}^\lambda)$. Moreover $\nabla u \in L^\infty(\mathbb{R}^N_+)$ implies that u is bounded in Σ_λ. Therefore Ψ is well defined and we can use it as a test function in Eq. (1.1) (see also [32, Lemma 5]), getting

$$\int\limits_{\mathcal{U}\cup\mathcal{U}^\lambda} |\nabla u|^{p-2}(\nabla u, \nabla \Psi)\,dx + \int\limits_{\mathcal{U}\cup\mathcal{U}^\lambda} b^+(u)|\nabla u|^q \Psi\,dx = \int\limits_{\mathcal{U}\cup\mathcal{U}^\lambda} b^-(u)|\nabla u|^q \Psi\,dx + \int\limits_{\mathcal{U}\cup\mathcal{U}^\lambda} f(u)\Psi\,dx. \tag{3.21}$$

Since u and Ψ are even w.r.t. the hyperplane $\{y = \lambda\}$, it follows that $(\nabla u, \nabla\Psi)$ is even too. Therefore we infer that

$$\int\limits_{\mathcal{U}} |\nabla u|^{p-2}(\nabla u, \nabla \Psi)\,dx + \int\limits_{\mathcal{U}} b^+(u)|\nabla u|^q \Psi\,dx = \int\limits_{\mathcal{U}} b^-(u)|\nabla u|^q \Psi\,dx + \int\limits_{\mathcal{U}} f(u)\Psi\,dx. \tag{3.22}$$

Let us suppose $1 \leqslant q \leqslant p$. For every $\sigma > 0$ we have

$$x^q \leqslant C(\sigma)\cdot x^p + \sigma, \quad x \geqslant 0, \tag{3.23}$$

say e.g. $C(\sigma) = \sigma^{1-\frac{p}{q}}$. Therefore (3.22) and (3.23) imply

$$\begin{aligned} &\int\limits_{\mathcal{U}} |\nabla u|^{p-2}(\nabla u, \nabla \Psi)\,dx + C(\sigma)\int\limits_{\mathcal{U}} b^+(u)|\nabla u|^p \Psi\,dx + \sigma\int\limits_{\mathcal{U}} b^+(u)\Psi\,dx \\ &\quad \geqslant \int\limits_{\mathcal{U}} b^-(u)|\nabla u|^q \Psi\,dx + \int\limits_{\mathcal{U}} f(u)\Psi\,dx \geqslant \int\limits_{\mathcal{U}} f(u)\Psi\,dx. \end{aligned} \tag{3.24}$$

By (3.18) we can choose σ in (3.23), say $\bar{\sigma}$, small enough such that

$$\gamma^+ - \bar{\sigma}\left\|b^+(u)\right\|_\infty = \tilde{C} > 0,$$

so that

$$\int\limits_{\mathcal{U}} |\nabla u|^{p-2}(\nabla u, \nabla \Psi)\,dx + C(\bar{\sigma})\int\limits_{\mathcal{U}} b^+(u)|\nabla u|^p \Psi\,dx \geqslant \tilde{C}\int\limits_{\mathcal{U}} \Psi\,dx. \tag{3.25}$$

Choosing $\hat{C}$ in (3.20) equal to $C(\bar{\sigma})$ in (3.25) we obtain

$$\begin{aligned} &\int\limits_{\mathcal{U}} e^{-s(u)}\varphi_R|\nabla u|^{p-2}\left(\nabla u, \nabla\frac{G_\varepsilon(|\nabla u|)}{|\nabla u|}\right)dx + \int\limits_{\mathcal{U}} e^{-s(u)}\frac{G_\varepsilon(|\nabla u|)}{|\nabla u|}|\nabla u|^{p-2}(\nabla u, \nabla\varphi_R)\,dx \\ &\quad \geqslant \tilde{C}\int\limits_{\mathcal{U}} e^{-s(u)}\varphi_R\frac{G_\varepsilon(|\nabla u|)}{|\nabla u|}\,dx. \end{aligned} \tag{3.26}$$

We set $h_\varepsilon(t) = \dfrac{G_\varepsilon(t)}{t}$, meaning that $h(t) = 0$ for $0 \leqslant t \leqslant \varepsilon$. We have

$$\begin{aligned} \left|\int\limits_{\mathcal{U}} e^{-s(u)}\varphi_R|\nabla u|^{p-2}\left(\nabla u, \nabla\frac{G_\varepsilon(|\nabla u|)}{|\nabla u|}\right)dx\right| &\leqslant C\int\limits_{\mathcal{U}} |\nabla u|^{p-1}\left|h'_\varepsilon\big(|\nabla u|\big)\right|\left|\nabla\big(|\nabla u|\big)\right|\varphi_R\,dx \\ &\leqslant C\int\limits_{\mathcal{U}} |\nabla u|^{p-2}\big(|\nabla u|h'_\varepsilon\big(|\nabla u|\big)\big)\left\|D^2u\right\|\varphi_R\,dx, \end{aligned} \tag{3.27}$$

where $\|D^2u\|$ denotes the Hessian norm.

Here below, we fix $R > 0$ and let $\varepsilon \to 0$. Later we will let $R \to \infty$. To this aim, let us first show that

(i) $|\nabla u|^{p-2}\|D^2u\|\varphi_R \in L^1(\mathcal{U})\ \forall R > 0$;
(ii) $|\nabla u|h'_\varepsilon(|\nabla u|) \to 0$ a.e. in $\mathcal{U}$ as $\varepsilon \to 0$ and $|\nabla u|h'_\varepsilon(|\nabla u|) \leqslant C$ with C not depending on ε.

Let us prove (i). Defining $\mathcal{D}(R) = \{\mathcal{U} \cap \{B'(0,R) \times \mathbb{R}\}\}$, by Hölder's inequality it follows

$$\begin{aligned}\int_{\mathcal{U}} |\nabla u|^{p-2} \|D^2 u\| \varphi_R\, dx &\leqslant C\big(\mathcal{D}(2R)\big)\Big(\int_{\mathcal{D}(2R)} |\nabla u|^{2(p-2)} \|D^2 u\|^2 \varphi_R^2\, dx\Big)^{\frac{1}{2}} \\ &\leqslant C\Big(\int_{\mathcal{D}(2R)} |\nabla u|^{p-2-\beta} \|D^2 u\|^2 \varphi_R^2 |\nabla u|^{p-2+\beta}\, dx\Big)^{\frac{1}{2}} \\ &\leqslant C\|\nabla u\|_{L^\infty(\mathbb{R}^N_+)}^{(p-2+\beta)/2}\Big(\int_{\mathcal{D}(2R)} |\nabla u|^{p-2-\beta} \|D^2 u\|^2\, dx\Big)^{\frac{1}{2}}, \end{aligned} \tag{3.28}$$

with $0 \leqslant \beta < 1$ and $\varphi_R^2 |\nabla u|^{p-2+\beta}$ consequently bounded. By Claim 1 we have

$$\operatorname{dist}\big(\mathcal{U}, \{y=0\}\big) > 0.$$

Using [34, Proposition 2.1],[3] we infer that

$$\Big(\int_{\mathcal{D}(2R)} |\nabla u|^{p-2-\beta} \|D^2 u\|^2\, dx\Big)^{\frac{1}{2}} \leqslant C.$$

Then by (3.28) we obtain

$$\int_{\mathcal{U}} |\nabla u|^{p-2} \|D^2 u\| \varphi_R\, dx \leqslant C.$$

Let us prove (ii). Recalling (3.19), we obtain

$$h'_\varepsilon(t) = \begin{cases} 0, & \text{if } t \geqslant 2\varepsilon, \\ \frac{2\varepsilon}{t^2}, & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0, & \text{if } 0 \leqslant t \leqslant \varepsilon, \end{cases}$$

and then $|\nabla u| h'_\varepsilon(|\nabla u|)$ tends to 0 almost everywhere in $\mathcal{U}$ as ε goes to 0 and we have $|\nabla u| h'_\varepsilon(|\nabla u|) \leqslant 2$.

Then by (3.26), (3.27) and (i), (ii) above, passing to the limit as $\varepsilon \to 0$, we get

$$\int_{\mathcal{U}} e^{-s(u)} |\nabla u|^{p-2} (\nabla u, \nabla \varphi_R)\, dx \geqslant C \int_{\mathcal{U}} \varphi_R\, dx \quad \forall R > 0.$$

Recalling (2.6), we have that there exists $C = C(\|\nabla u\|_{L^\infty(\mathbb{R}^N_+)})$ (not depending on R) such that

$$\big|\mathcal{U} \cap \operatorname{supp}(\varphi_R)\big| \cdot \frac{1}{R} \geqslant C \cdot \big|\mathcal{U} \cap \operatorname{supp}(\varphi_R)\big|$$

and we get a contradiction for R large, concluding the proof. $\square$

4. Recovering compactness

In this section we prove a crucial result, which allows us to localize the support of $(u - u_{\bar\lambda})^+$, where $\bar\lambda$ is defined in (4.2) below. The localization obtained will enable us to apply the weak comparison principle Theorem 1.6.

With the notations introduced at the beginning of the previous section, we set

$$\Lambda := \big\{\lambda \in \mathbb{R}^+ \colon\ u \leqslant u_\mu \text{in } \Sigma_\mu\ \forall \mu < \lambda\big\} \tag{4.1}$$

[3] Actually in Proposition 2.1 of [34] it is considered the case $q = p$. The same result in the more general case $1 < q \leqslant p$ follows exactly in the same way repeating the same calculations.

and we define

$$\bar{\lambda} := \sup \Lambda. \tag{4.2}$$

We have the following:

Proposition 4.1. *Let $1 < p < 2$ and let $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ be a solution to (1.1) with $a(s) = 1$. Let (H_1), (H_2) and (H_3) be satisfied and assume that $f(s) > 0$ for $s > 0$. Let us assume that both u and ∇u are bounded on every strip Σ_η, $\eta > 0$.*

Assume $0 < \bar{\lambda} < +\infty$ and set

$$W_\varepsilon := (u - u_{\bar{\lambda}+\varepsilon}) \cdot \chi_{\{y \leqslant \bar{\lambda}+\varepsilon\}}$$

where $\varepsilon > 0$.

Given $0 < \delta < \frac{\bar{\lambda}}{2}$ and $\rho > 0$, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \leqslant \varepsilon_0$, it follows

$$\operatorname{Supp} W^+_\varepsilon \subset \{0 \leqslant y \leqslant \delta\} \cup \{\bar{\lambda} - \delta \leqslant y \leqslant \bar{\lambda} + \varepsilon\} \cup \left(\bigcup_{x' \in \mathbb{R}^{N-1}} B^\rho_{x'} \right),$$

where $B^\rho_{x'}$ is such that

$$B^\rho_{x'} \subseteq \left\{ y \in (0, \bar{\lambda} + \varepsilon): \left|\nabla u\left(x', y\right)\right| < \rho, \left|\nabla u_{\bar{\lambda}+\varepsilon}\left(x', y\right)\right| < \rho \right\}. \tag{4.3}$$

Proof. Assume by contradiction that there exists $\delta > 0$, with $0 < \delta < \frac{\bar{\lambda}}{2}$, such that, given any $\varepsilon_0 > 0$, we find $\varepsilon \leqslant \varepsilon_0$ and $x_\varepsilon = (x'_\varepsilon, y_\varepsilon)$ such that:

(i) $u(x'_\varepsilon, y_\varepsilon) \geqslant u_{\bar{\lambda}+\varepsilon}(x'_\varepsilon, y_\varepsilon)$;
(ii) x_ε belongs to the set

$$\left\{ \left(x', y\right) \in \mathbb{R}^N: \delta \leqslant y_\varepsilon \leqslant \bar{\lambda} - \delta \right\}$$

and it holds the alternative: either $|\nabla u(x_\varepsilon)| \geqslant \rho$ or $|\nabla u_{\bar{\lambda}+\varepsilon}(x_\varepsilon)| \geqslant \rho$.

Take now $\varepsilon_0 = \frac{1}{n}$, then there exists $\varepsilon_n \leqslant \frac{1}{n}$ and a sequence

$$x_n = \left(x'_n, y_n\right) = \left(x'_{\varepsilon_n}, y_{\varepsilon_n}\right)$$

such that

$$u\left(x'_n, y_n\right) \geqslant u_{\bar{\lambda}+\varepsilon_n}\left(x'_n, y_n\right)$$

and satisfying conditions (i) and (ii) above. Up to subsequences we may assume that

$$y_n \to y_0 \quad \text{as } n \to +\infty, \text{ with } \delta \leqslant y_0 \leqslant \bar{\lambda} - \delta.$$

Let us define

$$\tilde{u}_n\left(x', y\right) = u\left(x' + x'_n, y\right). \tag{4.4}$$

Since both u and ∇u are bounded on every strip Σ_η, $\eta > 0$, as before, by $C^{1,\alpha}$ estimates, Ascoli's Theorem and a standard diagonal process we get that

$$\tilde{u}_n \xrightarrow{C^{1,\alpha'}_{loc}(\overline{\mathbb{R}^N_+})} \tilde{u} \tag{4.5}$$

(up to subsequences) for $\alpha' < \alpha$.

We claim that

- $\tilde{u} \geqslant 0$ in $\mathbb{R}^N_+$, with $\tilde{u}(x, 0) = 0$ for every $x \in \mathbb{R}^{N-1}$;
- $\tilde{u} \leqslant \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$;

– $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$;
– $|\nabla \tilde{u}(0, y_0)| \geqslant \rho$.

To prove this note that, since each $\tilde{u}_n(x', y)$ is positive and satisfies the homogeneous Dirichlet boundary condition by construction, we have $\tilde{u} \geqslant 0$ in $\mathbb{R}^N_+$ and $\tilde{u}(x, 0) = 0$ for every $x \in \mathbb{R}^{N-1}$. It is also clear that $\tilde{u} \leqslant \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ and $\tilde{u}(0, y_0) \geqslant \tilde{u}_{\bar{\lambda}}(0, y_0)$. Since (as shown above) $\tilde{u} \leqslant \tilde{u}_{\bar{\lambda}}$, actually there holds: $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$. Finally, at $x_0 = (0, y_0)$ (where $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$) we have that $\nabla \tilde{u}(0, y_0) = \nabla \tilde{u}_{\bar{\lambda}}(0, y_0)$, because x_0 is an interior minimum point for the function $w(x) := \tilde{u}_{\bar{\lambda}}(x) - \tilde{u}(x) \geqslant 0$. For all n we have $|\nabla u(x_n)| \geqslant \rho$ or $|\nabla u_{\bar{\lambda}+\varepsilon_n}(x_n)| \geqslant \rho$, and, using the uniform C^1 convergence on compact set, we get $|\nabla \tilde{u}(0, y_0)| \geqslant \rho$.

Recalling that we assumed here $a(s) = 1$, passing to the limit we obtain that $\tilde{u}$ satisfies

$$\int_{\mathbb{R}^N_+} |\nabla \tilde{u}|^{p-2}(\nabla \tilde{u}, \nabla \varphi)\, dx + \int_{\mathbb{R}^N_+} b(\tilde{u})|\nabla \tilde{u}|^q\, dx = \int_{\mathbb{R}^N_+} f(\tilde{u})\varphi\, dx \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N_+).$$

Since $\tilde{u} \geqslant 0$ in $\mathbb{R}^N_+$, by the strong maximum principle [35, Theorem 2.5.1], it follows that $\tilde{u} > 0$ or $\tilde{u} = 0$; by the fact that $|\nabla \tilde{u}(0, y_0)| \geqslant \delta$, the case $\tilde{u} = 0$ is not possible. Hence $\tilde{u} > 0$ on $\mathbb{R}^N_+$. Moreover we have $\tilde{u} \leqslant \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ and $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$. By the strong comparison principle [35, Theorem 2.5.2], we have that $\tilde{u} = \tilde{u}_{\bar{\lambda}}$ in the connected component, say $\mathcal{U}$, of $\Sigma_{\bar{\lambda}} \setminus Z_u$ containing the point $(0, y_0)$. By Theorem 3.1 we have that $\partial \mathcal{U} \cap \partial \mathbb{R}^N_+ \neq \emptyset$. The latter yields the existence of a point $z = (z', 2\bar{\lambda})$ such that $\tilde{u}(z) = 0$, which contradicts $\tilde{u} > 0$ in $\mathbb{R}^N_+$. □

5. Proof of Theorem 1.1

Let us start recalling that, in view of the changing of variable in (3.1), which preserves the monotonicity property, it is not restrictive to prove Theorem 1.1 in the case $a(\cdot) = 1$. As already remarked, the assumption $\nabla u \in L^\infty(\mathbb{R}^N_+)$ implies that w and ∇w (see (3.1)) are bounded on every set of the form $\{0 \leqslant y \leqslant \lambda\}$, for any $\lambda > 0$. Thus, we can use the results demonstrated in Section 3 and Section 4.

The proof is based on the moving planes procedure. By Theorem 1.7 the set Λ defined in (4.1) is not empty and $\bar{\lambda} \in (0, +\infty]$. To conclude the proof we need to show that $\bar{\lambda} = \infty$.

Assume that $\bar{\lambda}$ is finite, set $\lambda_0 = \bar{\lambda} + 2$,

$$M_0 := \|u\|_{L^\infty(\{0 \leqslant y \leqslant 2\bar{\lambda}+10\})} + \|\nabla u\|_{L^\infty(\{0 \leqslant y \leqslant 2\bar{\lambda}+10\})} + 1 > 0$$

and take $\tau_0 = \tau_0(N, p, q, \lambda_0, M_0, \gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(N, p, q, \lambda_0, M_0, \gamma) > 0$ as in Theorem 1.6.

By Proposition 4.1 we have that, given $0 < \delta < \min\{\frac{\bar{\lambda}}{2}, \frac{\tau_0}{4}\}$ and $0 < \rho < \varepsilon_0$, we find $\bar{\varepsilon} > 0$ such that, for any $0 < \varepsilon \leqslant \min\{\bar{\varepsilon}, \frac{\tau_0}{4}, 1\}$, it follows

$$\operatorname{Supp} W^+_\varepsilon \subset \{0 \leqslant y \leqslant \delta\} \cup \{\bar{\lambda} - \delta \leqslant y \leqslant \bar{\lambda} + \varepsilon\} \cup \Bigg(\bigcup_{x' \in \mathbb{R}^{N-1}} B^\rho_{x'} \Bigg),$$

where $W^+_\varepsilon = (u - u_{\bar{\lambda}+\varepsilon})^+ \cdot \chi_{\{y \leqslant \bar{\lambda}+\varepsilon\}}$ and $B^\rho_{x'}$ is defined in (4.3).

We claim that $u \leqslant u_{\bar{\lambda}+\varepsilon}$ in $\Sigma_{\bar{\lambda}+\varepsilon}$, which contradicts the definition of $\bar{\lambda}$ and yields that $\bar{\lambda} = \infty$. This, in turn, implies the desired monotonicity of u, that is $\frac{\partial u}{\partial y}(x', y) \geqslant 0$ in $\mathbb{R}^N_+$.

To this end, we proceed by contradiction. Suppose that the open set

$$\mathcal{S}_{(2\delta+\varepsilon,\rho)} := \big\{x \in \Sigma_{\bar{\lambda}+\varepsilon}\colon u(x) - u_{\bar{\lambda}+\varepsilon}(x) > 0\big\}$$

is not empty, then u and $v = u_{\bar{\lambda}+\varepsilon}$ satisfy (1.5) with $\lambda = y_0 = \bar{\lambda} + \varepsilon (< \lambda_0)$, as well as: $\|u\|_\infty + \|\nabla u\|_\infty \leqslant M_0$, $\|v\|_\infty + \|\nabla v\|_\infty \leqslant M_0$. Since by construction $2\delta + \varepsilon < \tau_0$ and $\rho < \varepsilon_0$, we can apply Theorem 1.6 to conclude that $u \leqslant u_{\bar{\lambda}+\varepsilon}$ on $\mathcal{S}_{(2\delta+\varepsilon,\rho)}$. This clearly contradicts the definition of $\mathcal{S}_{(2\delta+\varepsilon,\rho)}$. Hence $\mathcal{S}_{(2\delta+\varepsilon,\rho)} = \emptyset$, which concludes the proof. □

6. Proof of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. Since we assumed that $b(u) \geqslant 0$ then $-\Delta_p u \leqslant f(u)$ so that Theorem 1.7 in [21] applies and gives that actually

$$0 < u \leqslant z$$

in $\mathbb{R}^N_+$. Note that, once it is proved that $0 < u \leqslant z$, then the strong maximum principle (see [35]) applies and gives that $0 < u < z$. It follows furthermore that u is strictly bounded away from z in Σ_λ for any $\lambda > 0$. In fact, if this is not the case, arguing as in the proof of Theorem 3.1 (see case (a)) we could easily construct a limiting profile u_∞ with $0 < u_\infty \leqslant z$, touching z at some point. This is not possible again by the strong maximum principle [35]. This is enough to repeat the proof of Theorem 1.1 and get the thesis. $\square$

Proof of Theorem 1.4. If u is not identically zero, then it is strictly positive by the strong maximum principle (see [35,41]). Therefore u is monotone increasing w.r.t. the y-direction by Theorem 1.1 and the proof of (b) and of (c) follows by [33,39] exactly in the same way as (b) and (c) in Theorem 1.6 of [21].

To prove (a) let $N = 2$ and denote by (x, y) a point in the plane. Define

$$w(x) := \lim_{y\to\infty} u(x, y). \tag{6.1}$$

We see that (see e.g. the proof of Theorem 8.3 in [21] for details) $w : \mathbb{R} \to \mathbb{R}$ is non-negative and bounded with

$$-\left(\left|w'\right|^{p-2} w'\right)' = f(w) \geqslant 0.$$

A simple O.D.E. analysis shows that w is constant and, by the assumptions on f, it follows that necessarily $w = 0$ that also implies $u = 0$ and the thesis.

To prove the non-existence result when $f(s) > 0$ for $s \geqslant 0$, we first consider the case $N = 2$. By the above argument (which uses only the assumption $f(s) > 0$ for $s > 0$) we infer that $u = 0$ which contradicts $f(0) > 0$. Thus, there are no non-negative solutions. The same argument can be employed to treat the case $N \geqslant 3$. Indeed, in this case the assumption $f(0) > 0$ implies that $f(s) \geqslant \lambda s^{\frac{(N-1)(p-1)}{N-1-p}}$ in $[0, \delta]$, for some $\lambda, \delta > 0$, yielding $u = 0$. Again contradicting $f(0) > 0$. $\square$

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